On a Homogeneous Algorithm for the Monotone Complementarity Problem

Erling D. Andersen *

Yinyu Ye †

January 1995 (Revised November 1996, April 1997)

Abstract

We present a generalization of a homogeneous self-dual linear programming (LP) algorithm to solving the monotone complementarity problem (MCP). The algorithm does not need to use any "big-M" parameter or two-phase method, and it generates either a solution converging towards feasibility and complementarity simultaneously or a certificate proving infeasibility. Moreover, if the MCP is polynomially solvable with an interior feasible starting point, then it can be polynomially solved without using or knowing such information at all. To our knowledge, this is the first interior-point and infeasible-starting algorithm for solving the MCP that possesses these desired features. Preliminary computational results are presented.

Key words: Monotone complementarity problem, homogeneous and self-dual, infeasible-starting algorithm.

Running head: A homogeneous algorithm for MCP.

*Department of Management, Odense University, Campusvej 55, DK-5230 Odense M, Denmark, email: eda@busieco.ou.dk.

†Department of Management Sciences, The University of Iowa, Iowa City, Iowa 52242, USA, email: yinyu-ye@uiowa.edu.
This author is supported in part by NSF Grants DDM-9207347 and DMI-9522507.

1 Introduction

Consider the monotone complementarity problem (MCP) in the standard form:

$$\begin{array}{lll} (MCP) & \text{minimize} & x^T s \\ & \text{subject to} & s = f(x), \ (x, s) \geq 0, \end{array}$$

where $f(x)$ is a continuous *monotone* mapping from $R^n_+ := \{x \in R^n : \ x \geq 0\}$ to R^n, $x, s \in R^n$, and ^{T} denotes transpose. In other words, for every $x^1, x^2 \in R^n_+$, we have

$$(x^1 - x^2)^T (f(x^1) - f(x^2)) \geq 0.$$

Denote by ∇f the Jacobian matrix of f, which is positive semi-definite in R^n_+ (see Cottle et al. [2]).

(MCP) is said to be (asymptotically) feasible if and only if there is a *bounded* sequence $\{(x^t, s^t)\} \subset R^{2n}_{++}$, $t = 1, 2, ...,$ such that

$$\lim_{t \to \infty} \quad s^t - f(x^t) \to 0,$$

where any limit point $(\hat{x}, \hat{s})$ of the sequence is called an (asymptotically) feasible point for (MCP). (MCP) has an interior feasible point if it has an (asymptotically) feasible point $(\hat{x} > 0, \hat{s} > 0)$. (MCP) is said to be (asymptotically) solvable if there is an (asymptotically) feasible point $(\hat{x}, \hat{s})$ such that $\hat{x}^T \hat{s} = 0$, where $(\hat{x}, \hat{s})$ is called the "optimal" or "complementary" solution for (MCP). (MCP) is (strongly) infeasible if and only if there is no sequence $\{(x^t, s^t)\} \subset R^{2n}_{++}$, $t = 1, 2, ...,$ such that

$$\lim_{t \to \infty} \quad s^t - f(x^t) \to 0.$$

Denote the feasible set of (MCP) by $\mathcal{F}$ and the solution set by $\mathcal{S}$. Note that (MCP) is feasible does not imply that (MCP) has a solution. If (MCP) has a solution, then it has a maximal solution (x^*, s^*) where the number of positive components in (x^*, s^*) is maximal. Note that the indices of those positive components are invariant among all maximal solutions for (MCP) (see Güler [6]).

Consider a class of (MCP) where f satisfies the following condition. Let

$$\upsilon : (0, 1) \to (1, \infty)$$

be a monotone increasing function such that

$$\|X(f(x + d_x) - f(x) - \nabla f(x) d_x)\|_1 \leq \upsilon(\alpha) d_x^T \nabla f(x) d_x \tag{1}$$

whenever

$$d_x \in R^n, \quad x \in R^n_{++} := \{x \in R^n : \ x > 0\}, \quad \left\| X^{-1} d_x \right\|_\infty \leq \alpha < 1.$$

Then, f is said to be scaled Lipschitz in R^n_{++}. Such a condition for linearly constrained convex optimization was first proposed by Monteiro and Adler [21] and later generalized by Zhu [36]. Finally, Potra and Ye [25] extended it for the monotone complementary problem. This condition is included in a more general condition analyzed by Nesterov and Nemirovskii [24] and Jarre [10]. Given $x^0 > 0$ and $s^0 = f(x^0) > 0$ one can develop an interior-point algorithm that generates a maximal complementary solution of the scaled Lipschitz (MCP) in $O(\sqrt{n}\log(1/\epsilon))$ interior-point iterations, where ϵ is the complementarity error.

However, the initial point x^0 is generally unknown. In fact, we do not even know whether such a point exists or not, that is, (MCP) might be infeasible or feasible but have no positive feasible point. To overcome this difficulty, Ye, Todd, and Mizuno [35] recently developed a homogeneous linear programming (LP) algorithm based on the construction of a homogeneous and self-dual LP model (see Xu et al. [32] for a simplification of the model). More recently, Ye [34] extended the model to solving the monotone linear complementarity problem, where f is an affine mapping. However, unlike the original LP homogeneous model, the model constructed in [34] may not be a one-phase model, that is, it may have to find a feasible point first and then search for a solution. This two-phase approach is undesirable in practical implementation.

In this paper, we present a *one-phase* homogeneous model to solving the *general* monotone complementarity problem. To our knowledge, the result is the first MCP algorithm possessing the following desired features:

- It achieves $O(\sqrt{n}\log(1/\epsilon))$-iteration complexity if f satisfies the scaled Lipschitz condition.

- It solves the problem without any regularity assumption concerning the existence of optimal, feasible, or interior feasible points.

- It can start at a positive point, feasible or infeasible, near the central ray of the positive orthant (cone), and it does not need to use any big-M penalty parameter or lower bound.

- If (MCP) has a solution, the algorithm generates a sequence that approaches feasibility and optimality *simultaneously*; if the problem is (strongly) infeasible, the algorithm generates a sequence that converges to a certificate proving infeasibility.

Related interior-point infeasible-starting algorithms for solving (MCP) or nonlinear programming problems has been suggested for example by den Hertog et al. [3], El-Bakry et al. [4], Kojima et al. [13], Kortanek et al. [15], Mizuno et al. [19], Monteiro [20], Monteiro and Wright [23], Tanabe [27], Vial [30], and Wang et al. [31].

2 A homogeneous MCP model

Consider an augmented homogeneous model related to (MCP):

$$(HMCP) \quad \text{minimize} \quad x^T s + \tau\kappa$$

$$\text{subject to} \quad \begin{pmatrix} s \\ \kappa \end{pmatrix} = \begin{pmatrix} \tau f(x/\tau) \\ -x^T f(x/\tau) \end{pmatrix}, \ (x, \tau, s, \kappa) \geq 0.$$

A similar augmented transformation was discussed in Ye [33] and it is closely related to the recession function in convex analysis of Rockafellar [26]. Let

$$\psi(x,\tau) = \begin{pmatrix} \tau f(x/\tau) \\ -x^T f(x/\tau) \end{pmatrix} : \ R^{n+1}_{++} \to R^{n+1}. \tag{2}$$

Then, it is easy to verify that $\nabla\psi$ is positive semi-definite as shown in the following lemma.

Lemma 1 . *Let ∇f be positive semi-definite in R^n_+. Then $\nabla\psi$ is positive semi-definite in R^{n+1}_{++}, i.e. given $(x;\tau) > 0$*

$$(d_x; d_\tau)^T \nabla\psi(x,\tau)(d_x; d_\tau) \geq 0$$

for any $(d_x; d_\tau) \in R^{n+1}$, where

$$\nabla\psi(x,\tau) = \begin{pmatrix} \nabla f(x/\tau) & f(x/\tau) - \nabla f(x/\tau)(x/\tau) \\ -f(x/\tau)^T - (x/\tau)^T \nabla f(x/\tau) & (x/\tau)^T \nabla f(x/\tau)(x/\tau) \end{pmatrix}. \tag{3}$$

Proof:

$$\begin{aligned} (d_x; d_\tau)^T \nabla\psi(x,\tau)(d_x; d_\tau) = \ & d_x^T \nabla f(x/\tau) d_x - d_x^T \nabla f(x/\tau) x (d_\tau/\tau) \\ & -(d_\tau/\tau) x^T \nabla f(x/\tau) d_x + d_\tau^2 x^T \nabla f(x/\tau) x/\tau^2 \\ = \ & (d_x - d_\tau x/\tau)^T \nabla f(x/\tau)(d_x - d_\tau x/\tau) \end{aligned} \tag{4}$$

□

Furthermore, we have the following theorem.

Theorem 2 . *Let ψ be given by (2). Then,*

i. *ψ is a continuous homogeneous function in R^{n+1}_{++} with degree 1 and for any $(x;\tau) \in R^{n+1}_{++}$*

$$(x;\tau)^T \psi(x,\tau) = 0$$

and

$$(x;\tau)^T \nabla\psi(x,\tau) = -\psi(x,\tau)^T$$

ii. *If f is a continuous monotone mapping from R^n_+ to R^n, then ψ is a continuous monotone mapping from R^{n+1}_{++} to R^{n+1}.*

iii. *If f is scaled Lipschitz with $\upsilon = \upsilon_f$, then ψ is scaled Lipschitz, that is, it satisfies condition (1) with*

$$\upsilon = \upsilon_\psi(\alpha) = \left(1 + \frac{2\upsilon_f(2\alpha/(1+\alpha))}{1-\alpha}\right)\left(\frac{1}{1-\alpha}\right). \tag{5}$$

iv. *$(HMCP)$ is (asymptotically) feasible and every (asymptotically) feasible point is an (asymptotically) complementary solution.*

Now, let $(x^*, \tau^*, s^*, \kappa^*)$ be a maximal complementary solution for $(HMCP)$. Then

v. *(MCP) has a solution if and only if $\tau^* > 0$. In this case, $(x^*/\tau^*, s^*/\tau^*)$ is a complementary solution for (MCP).*

vi. *(MCP) is (strongly) infeasible if and only if $\kappa^* > 0$. In this case, $(x^*/\kappa^*, s^*/\kappa^*)$ is a certificate to prove (strong) infeasibility.*

Proof: The proof of (i) is straightforward.

We now prove (ii). First, ψ is continuous in R^{n+1}_{++} if f is in R^n_{++}. Let $(x^1;\tau^1)$ and $(x^2;\tau^2)$ both in R^{n+1}_{++}. Then, since both x^1/τ^1 and x^2/τ^2 in R^n_{++} and (i), we have

$$\begin{aligned}
& (x^1 - x^2; \tau^1 - \tau^2)^T(\psi(x^1,\tau^1) - \psi(x^2,\tau^2)) \\
= \; & -(x^1;\tau^1)^T\psi(x^2,\tau^2) - (x^2;\tau^2)^T\psi(x^1,\tau^1) \\
= \; & -(\tau^2(x^1)^T f(x^2/\tau^2) - \tau^1(x^2)^T f(x^2/\tau^2) + \tau^1(x^2)^T f(x^1/\tau^1) - \tau^2(x^1)^T f(x^1/\tau^1)) \\
= \; & \tau^1\tau^2(x^1/\tau^1 - x^2/\tau^2)^T(f(x^1/\tau^1) - f(x^2/\tau^2)) \\
\geq \; & 0.
\end{aligned}$$

We now prove (iii). Assume $(x;\tau) \in R^{n+1}_{++}$ and let $(d_x; d_\tau)$ be given such that $\left\|(X^{-1}d_x; \tau^{-1}d_\tau)\right\|_\infty \leq \alpha < 1$. To prove ψ is scaled Lipschitz we must bound

$$\left\| \begin{pmatrix} X & 0 \\ 0 & \tau \end{pmatrix} \left(\psi(x+d_x, \tau+d_\tau) - \psi(x,\tau) - \nabla\psi(x,\tau) \begin{pmatrix} d_x \\ d_\tau \end{pmatrix} \right) \right\|_1 . \tag{6}$$

From (2) and (3), the upper part in (6) is identical to

$$\begin{aligned}
& X\left(f(y+d_y)(\tau+d_\tau) - f(y)\tau - (\nabla f(y)d_x + f(y)d_\tau - \nabla f(y)xd_\tau/\tau)\right) \\
= \; & (\tau + d_\tau)X\left(f(y+d_y) - f(y) - \nabla f(y)d_y\right) \\
= \; & \tau(\tau+d_\tau)Y\left(f(y+d_y) - f(y) - \nabla f(y)d_y\right)
\end{aligned} \tag{7}$$

where

$$y = x/\tau \quad \text{and} \quad y + d_y = \frac{x + d_x}{\tau + d_\tau}, \tag{8}$$

that is,

$$d_y = \frac{\tau d_x - x d_\tau}{\tau(\tau + d_\tau)} = \frac{d_x - (d_\tau/\tau)x}{\tau + d_\tau}. \tag{9}$$

Note

$$\begin{aligned} \left\|Y^{-1}d_y\right\|_\infty &= \left\|\tau X^{-1}(\tau d_x - d_\tau x)/(\tau(\tau + d_\tau))\right\|_\infty \\ &= \left\|(\tau X^{-1} d_x - d_\tau e)/(\tau + d_\tau)\right\|_\infty \\ &\leq (\left\|X^{-1}d_x\right\|_\infty + \alpha)/(1-\alpha) \\ &\leq 2\alpha/(1-\alpha). \end{aligned} \tag{10}$$

By assumption that f is scaled Lipschitz with $\upsilon = \upsilon_f$. It follows for $\alpha \in [0, 1)$ that

$$\begin{aligned} & \|\tau(\tau + d_\tau)Y(f(y + d_y) - f(y) - \nabla f(y) d_y)\|_1 \\ \leq\ & \tau(\tau + d_\tau)\upsilon_f(2\alpha/(1-\alpha)) d_y^T \nabla f(y) d_y \\ =\ & \tfrac{\tau \upsilon_f(2\alpha/(1-\alpha))}{\tau + d_\tau}(d_x - x d_\tau/\tau)^T \nabla f(y)(d_x - x d_\tau/\tau) \\ =\ & \tfrac{\upsilon_f(2\alpha/(1-\alpha))}{1 + d_\tau/\tau}(d_x; d_\tau)^T \nabla\psi(x,\tau)(d_x; d_\tau) \\ \leq\ & \tfrac{\upsilon_f(2\alpha/(1-\alpha))}{1-\alpha}(d_x; d_\tau)^T \nabla\psi(x,\tau)(d_x; d_\tau) \end{aligned} \tag{11}$$

Next we bound the lower part of (6). This part is equal to

$$\begin{aligned} & \tau\left(-f(y+d_y)^T(x+d_x) - (-f(y)^T x) - [-f(y)^T d_x - x^T \nabla f(y) d_x/\tau + x^T \nabla f(y) x d_\tau/\tau^2]\right) \\ =\ & \tau\left((x+d_x)^T(-f(y+d_y) + f(y) + \nabla f(y) d_y) - (x+d_x)^T \nabla f(y) d_y + (x/\tau)^T \nabla f(y) d_y (\tau + d_\tau)\right) \\ =\ & \tau\left((x+d_x)^T(-f(y+d_y) + f(y) + \nabla f(y) d_y) - (d_x - d_\tau x/\tau)^T \nabla f(y) d_y\right) \\ =\ & \tau^2(e + X^{-1}d_x)^T Y\left(-f(y+d_y) + f(y) + \nabla f(y) d_y\right) - \tau(\tau + d_\tau) d_y^T \nabla f(y) d_y. \end{aligned}$$

Thus, using (4) and (10)

$$\begin{aligned} & |\tau\left(-f(y+d_y)^T(x+d_x) - (-f(y)^T x) - [-f(y)^T d_x - x^T \nabla f(y) d_x/\tau + x^T \nabla f(y) x d_\tau/\tau^2]\right)| \\ \leq\ & \tau^2\left\|e + X^{-1}d_x\right\|_\infty \|-Y(f(y+d_y) - (f(y) + \nabla f(y) d_y))\|_1 + |\tau(\tau + d_\tau)| d_y^T \nabla f(y) d_y \\ \leq\ & \left(\tau^2(1+\alpha)\upsilon_f(2\alpha/(1-\alpha)) + \tau(\tau + d_\tau)\right) d_y^T \nabla f(y) d_y \\ =\ & \tfrac{\tau^2(1+\alpha)\upsilon_f(2\alpha/(1-\alpha)) + \tau(\tau+d_\tau)}{(\tau+d_\tau)^2}(d_x^T - d_\tau x/\tau)^T \nabla f(y)(d_x - d_\tau x/\tau) \\ =\ & \tfrac{(1+\alpha)\upsilon_f(2\alpha/(1-\alpha)) + (1+d_\tau/\tau)}{(1+d_\tau/\tau)^2}(d_x; d_\tau)^T \nabla\psi(x,\tau)(d_x; d_\tau) \\ \leq\ & \left(\tfrac{(1+\alpha)\upsilon_f(2\alpha/(1-\alpha))}{(1-\alpha)^2} + \tfrac{1}{1-\alpha}\right)(d_x; d_\tau)^T \nabla\psi(x,\tau)(d_x; d_\tau). \end{aligned} \tag{12}$$

The sum of (11) and (12) is equal to

$$\upsilon_\psi(\alpha)(d_x; d_\tau)^T \nabla\psi(x,\tau)(d_x; d_\tau)$$

and it bounds the term in (6) leading to the desired result.

We now prove (iv). Take $x^t = (1/2)^t e$, $\tau^t = (1/2)^t$, $s^t = (1/2)^t e$, and $\kappa^t = (1/2)^t$. Then we see, as $t \to \infty$,

$$s^t - \tau^t f(x^t/\tau^t) = (1/2)^t (e - f(e)) \to 0$$

and

$$\kappa^t + (x^t)^T f(x^t/\tau^t) = (1/2)^t (1 + e^T f(e)) \to 0.$$

Thus, the system is (asymptotically) feasible. Let $(\hat{x}, \hat{\tau}, \hat{s}, \hat{\kappa}) \geq 0$ be any (asymptotically) feasible point for $(HMCP)$, then we clearly see from (i)

$$\hat{x}^T \hat{s} + \hat{\tau}\hat{\kappa} = (\hat{x}; \hat{\tau})^T \psi(\hat{x}, \hat{\tau}) = 0.$$

We now prove (v). If $(x^*, \tau^*, s^*, \kappa^*)$ is a solution for $(HMCP)$ and $\tau^* > 0$. Then we have

$$s^*/\tau^* = f(x^*/\tau^*) \quad \text{and} \quad (x^*)^T s^*/(\tau^*)^2 = 0,$$

that is, $(x^*/\tau^*, s^*/\tau^*)$ is a solution for (MCP). Let $(\hat{x}, \hat{s})$ be a solution to (MCP). Then for any $\tau = 1$, $x = \hat{x}$, $s = \hat{s}$, $\kappa = 0$ is a solution for $(HMCP)$. Thus, every maximal solution of $(HMCP)$ must have $\tau^* > 0$.

Finally, we prove (vi). Consider the set

$$R_{++} = \{s - f(x) \in R^n : \quad (x, s) > 0\}.$$

As proved by Güler [6] and Kojima et al. [14], R_{++} is an open convex set. If (MCP) is strongly infeasible, then we must have $0 \notin \bar{R}_{++}$ where $\bar{R}_{++}$ represents the closure of R_{++}. Thus, there is a hyperplane that separates 0 and $\bar{R}_{++}$, that is, there is a vector $a \in R^n$ with $\|a\| = 1$ and a positive number ξ such that

$$a^T(s - f(x)) \geq \xi > 0 \quad \forall\, x \geq 0,\; s \geq 0. \tag{13}$$

For $j = 1, 2, ..., n$, set s_j sufficiently large, but fix x and the rest of s, it must be true $a_j \geq 0$. Thus,

$$a \geq 0, \quad \text{or} \quad a \in R^n_+.$$

On the other hand, for any fixed x, we set $s = 0$ and see that

$$-a^T f(x) \geq \xi > 0 \quad \forall\, x \geq 0. \tag{14}$$

In particular,

$$-a^T f(ta) \geq \xi > 0 \quad \forall\, t \geq 0. \tag{15}$$

From the monotonicity of f, for every $x \in R^n_+$ and any $t \geq 0$ we have

$$(tx - x)^T (f(tx) - f(x)) \geq 0.$$

Thus for all $t \geq 1$,

$$x^T f(tx) \geq x^T f(x) \tag{16}$$

and

$$\lim_{t\to\infty} x^T f(tx)/t \geq 0. \tag{17}$$

Thus, from (15) and (17)

$$\lim_{t\to\infty} a^T f(ta)/t = 0.$$

For an $x \in R^n_+$, denote

$$f^\infty(x) := \lim_{t\to\infty} f(tx)/t,$$

where $f^\infty(x)$ represents the limit of any subsequence and its values may include ∞ or $-\infty$.

We now prove $f^\infty(a) \geq 0$. Suppose that $f^\infty(a)_j < -\delta$. Then consider the vector $x = a + \epsilon e_j$ where e_j is the vector with the jth component being 1 and zeros everywhere. Then, for ϵ sufficiently small and t sufficiently large we have

$$\begin{aligned}
x^T f(tx)/t &= (a+\epsilon e_j)^T f(t(a+\epsilon e_j))/t \\
&= a^T f(t(a+\epsilon e_j))/t + \epsilon e_j^T f(t(a+\epsilon e_j))/t \\
&< \epsilon e_j^T f(t(a+\epsilon e_j))/t \quad \text{(from (14))} \\
&= \epsilon \frac{f(t(a+\epsilon e_j))_j - f(ta)_j}{t} + \epsilon \frac{f(ta)_j}{t} \\
&\leq \epsilon \left(O(\epsilon) + \frac{f(ta)_j}{t} \right) \quad \text{(from continuity of } f\text{)} \\
&\leq \epsilon(O(\epsilon) - \delta/2) \\
&\leq -\epsilon\delta/4.
\end{aligned}$$

But this contradicts to relation (17). Thus, we must have

$$f^\infty(a) \geq 0.$$

We now further prove that $f^\infty(a)$ is bounded. Consider

$$\begin{aligned}
0 &\leq (ta-e)^T (f(ta) - f(e))/t \\
&= a^T f(ta) - e^T f(ta)/t - a^T f(e) + e^T f(e)/t \\
&< -e^T f(ta)/t - a^T f(e) + e^T f(e)/t.
\end{aligned}$$

Take a limit as $t \to \infty$ from both sides, we have

$$e^T f^\infty(a) \leq -a^T f(e).$$

$\tau^* \setminus \kappa^*$	=0	>0
$=0$	all other cases	infeasible (a finite certificate exists)
>0	solvable (a finite solution exists)	N/A

Figure 1: Four possible combinations of the optimal τ^* and κ^*.

Thus, $f^\infty(a) \geq 0$ is bounded. Again, we have $a^T f(ta) \leq -\xi$ from (15) and $a^T f(ta) \geq a^T f(a)$ from (16). Thus, $\lim a^T f(ta)$ is bounded. To summarize, $(HMCP)$ has an asymptotical solution $(x^* = a, \tau^* = 0, s^* = f^\infty(a), \kappa^* = \lim -a^T f(ta) \geq \xi)$.

Conversely, if there is a bounded sequence $(x^k > 0, \tau^k > 0, s^k > 0, \kappa^k > 0)$

$$\lim s^k = \lim \tau^k f(x^k/\tau^k) \geq 0, \quad \lim \kappa^k = \lim -(x^k)^T f(x^k/\tau^k) \geq \xi > 0.$$

Then, we claim that there is no feasible point $(x \geq 0, s \geq 0)$ such that $s - f(x) = 0$. We prove this fact by contradiction. If there is one. Then,

$$\begin{aligned} 0 &\leq ((x^k;\tau^k) - (x;1))^T(\psi(x^k,\tau^k) - \psi(x,1)) \\ &= (x^k - x)^T(\tau^k f(x^k/\tau^k) - f(x)) + (\tau^k - 1)^T(xf(x) - (x^k)^T f(x^k/\tau^k)). \end{aligned}$$

Therefore,

$$(x^k)^T f(x^k/\tau^k) \geq (x^k)^T f(x) + \tau^k x^T f(x^k/\tau^k) - \tau^k x^T f(x).$$

Now the first term at the right-hand side is nonnegative and $\lim \tau^k = 0$, because $\lim \kappa^k \geq \xi > 0$. Therefore, we have

$$\lim (x^k)^T f(x^k/\tau^k) \geq 0,$$

which is a contradiction to $\kappa^k = -(x^k)^T f(x^k/\tau^k) \geq \xi > 0$. Also, any limit of x^k is a separating hyperplane, i.e., a certificate proving infeasibility. □

Assume $(x^*, \tau^*, y^*, s^*, \kappa^*)$ is a maximal complementarity solution, then in Figure 1 the four possible combinations of τ^* and κ^* are shown. If τ^* and κ^* are both zero, then neither a finite solution nor a finite certificate proving infeasibility exists. This cannot happen if f is an affine and monotone function.

3 Central path of the homogeneous model

Due to Theorem 2, we can solve (MCP) by finding a maximal complementary solution of $(HMCP)$. Select $x^0 > 0$, $s^0 > 0$, $\tau^0 > 0$ and $\kappa^0 > 0$, and let the residual vectors

$$r^0 = s^0 - \tau^0 f(x^0/\tau^0), \quad z^0 = \kappa^0 + (x^0)^T f(x^0/\tau^0).$$

Also let

$$\bar{n} = (r^0)^T x^0 + z^0 \tau^0 = (x^0)^T s^0 + \tau^0 \kappa^0.$$

For simplicity, we set

$$x^0 = e,\ \tau^0 = 1,\ s^0 = e,\ \kappa^0 = 1,\ \theta^0 = 1,$$

then

$$X^0 s^0 = e \quad \text{and} \quad \tau^0 \kappa^0 = 1,$$

where X^0 is the diagonal matrix of x^0. Note that $\bar{n} = n + 1$ in this setting.

We present the next theorem.

Theorem 3 . *Consider* $(HMCP)$.

i. *For any* $0 < \theta \leq 1$, *there exists a strictly positive point* $(x > 0, \tau > 0, s > 0, \kappa > 0)$ *such that*

$$\begin{pmatrix} s \\ \kappa \end{pmatrix} - \psi(x, \tau) = \begin{pmatrix} s - \tau f(x/\tau) \\ \kappa + x^T f(x/\tau) \end{pmatrix} = \theta \begin{pmatrix} r^0 \\ z^0 \end{pmatrix}. \tag{18}$$

ii. *Starting from* $(x^0 = e, \tau^0 = 1, s^0 = e, \kappa^0 = 1)$, *for any* $0 < \theta \leq 1$ *there is a unique strictly positive point* $(x(\theta), \tau(\theta), s(\theta), \kappa(\theta))$ *that satisfies equation (18) and*

$$\begin{pmatrix} Xs \\ \tau\kappa \end{pmatrix} = \theta e. \tag{19}$$

iii. *For any* $0 < \theta \leq 1$, *the solution* $(x(\theta), \tau(\theta), s(\theta), \kappa(\theta))$ *in [ii] is bounded. Thus,*

$$\mathcal{C}(\theta) := \left\{ (x, \tau, s, \kappa) : \begin{pmatrix} s \\ \kappa \end{pmatrix} - \psi(x, \tau) = \theta \begin{pmatrix} r^0 \\ z^0 \end{pmatrix}, \quad \begin{pmatrix} Xs \\ \tau\kappa \end{pmatrix} = \theta e, \quad 0 < \theta \leq 1 \right\} \tag{20}$$

is a continuous bounded trajectory.

iv. *Any limit point* $(x(0), \tau(0), s(0), \kappa(0))$ *is a maximal complementary solution for* $(HMCP)$.

Proof. We prove [i]. Again, the set

$$H_{++} := \left\{ \begin{pmatrix} s \\ \kappa \end{pmatrix} - \psi(x, \tau) : \quad (x, \tau, s, \kappa) > 0. \right\}$$

is open and convex. (Note that ψ is a continuous monotone mapping from R^{n+1}_{++}, rather than R^{n+1}_{+}, to R^{n+1}. Nevertheless, the proof is a straightforward extension of Kojima et al. [14].) We have $(r^0; z^0) \in H_{++}$ by construction. On the other hand, $0 \in \bar{H}_{++}$ from Theorem 2. Thus,

$$\theta \begin{pmatrix} r^0 \\ z^0 \end{pmatrix} \in H_{++}.$$

The proof of [ii] is due to McLinden [17], Güler [6] and Kojima et al. [14].

We now prove [iii]. Again, the existence is due to Güler [6] and Kojima et al. [14]. We prove the boundedness. Assume $(x,\tau,s,\kappa) \in \mathcal{C}(\theta)$ then

$$
\begin{aligned}
&(x;\tau)^T(r^0;z^0) \\
&= (x;\tau)^T(s^0;\kappa^0) - (x;\tau)^T\psi(x^0;\tau^0) \\
&= (x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) - (s;\kappa)^T(x^0;\tau^0) - (x;\tau)^T\psi(x^0;\tau^0) \\
&= (x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) - (x^0;\tau^0)^T(\theta(r^0;z^0) + \psi(x,\tau)) - (x;\tau)^T\psi(x^0;\tau^0) \\
&= (x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) - \theta(x^0;\tau^0)^T(r^0;z^0) - (x^0;\tau^0)^T\psi(x,\tau) - (x;\tau)^T\psi(x^0;\tau^0) \\
&\geq (x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) - \theta(x^0;\tau^0)^T(r^0;z^0) - (x;\tau)^T\psi(x,\tau) - (x^0;\tau^0)^T\psi(x^0;\tau^0) \\
&= (x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) - \theta(x^0;\tau^0)^T(r^0;z^0) \\
&= (x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) - \theta(x^0;\tau^0)^T((s^0;\kappa^0) - \psi(x^0,\tau^0)) \\
&= (x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) - \theta(x^0;\tau^0)^T(s^0;\kappa^0).
\end{aligned}
$$

Also for $0 < \theta \leq 1$,

$$\theta(x;\tau)^T(r^0;z^0) = (x;\tau)^T((s;\kappa) - \psi(x,\tau)) = (x;\tau)^T(s;\kappa) = \theta(n+1) = \theta(x^0;\tau^0)^T(s^0;\kappa^0).$$

From the above two relations, we have

$$(x;\tau)^T(s^0;\kappa^0) + (s;\kappa)^T(x^0;\tau^0) \leq (1+\theta)(x^0;\tau^0)^T(s^0;\kappa^0).$$

Thus, $(x;\tau;s;\kappa)$ is bounded.

The proof of (iv) is well-known, see McLinden [16, 17] and Kojima, Mizuno, and Noma [12]. We include a version here for completeness. Let $(x^*,\tau^*,s^*,\kappa^*)$ be any maximal complementarity solution for $(HMCP)$ such that

$$(s^*;\kappa^*) = \psi(x^*;\tau^*) \quad \text{and} \quad (x^*)^T s^* + \tau^*\kappa^* = 0,$$

and it is normalized by

$$(r^0;z^0)^T(x^*;\tau^*) = (r^0;z^0)^T(x^0;\tau^0) = (s^0;\kappa^0)^T(x^0;\tau^0) = (n+1).$$

For any $0 < \theta \leq 1$, let (x,τ,s,κ) be the solution on the path. Then, we have

$$
\begin{aligned}
&((x;\tau) - (x^*;\tau^*))^T((s;\kappa) - (s^*;\kappa^*)) \\
&= ((x;\tau) - (x^*;\tau^*))^T(\psi(x;\tau) - \psi(x^*;\tau^*)) + \theta(r^0;z^0)^T((x;\tau) - (x^*;\tau^*)) \\
&\geq \theta(r^0;z^0)^T((x;\tau) - (x^*;\tau^*)).
\end{aligned}
$$

Therefore,

$$\begin{aligned}(x;\tau)^T(s^*;\kappa^*)+(s;\kappa)^T(x^*;\tau^*) &\leq (x;\tau)^T(s;\kappa)-\theta(r^0;z^0)^T((x;\tau)-(x^*;\tau^*))\\ &= (x;\tau)^T(s;\kappa)-(x;\tau)^T(s;\kappa)+\theta(r^0;z^0)^T(x^*;\tau^*)\\ &= \theta(r^0;z^0)^T(x^*;\tau^*)\\ &= \theta(n+1).\end{aligned}$$

Using $x_js_j=\theta$ and $\tau\kappa=\theta$ we obtain,

$$\begin{aligned}(x;\tau)^T(s^*;\kappa^*)+(s;\kappa)^T(x^*;\tau^*) &= \theta(\sum\frac{s_j^*}{s_j}+\frac{\kappa^*}{\kappa}+\sum\frac{x_j^*}{x_j}+\frac{\tau^*}{\tau})\\ &\leq \theta(n+1).\end{aligned}$$

Thus, we have

$$\frac{s_j^*}{s_j}\leq(n+1), \quad \text{and} \quad \frac{\kappa^*}{\kappa}\leq(n+1)$$

and

$$\frac{x_j^*}{x_j}\leq(n+1), \quad \text{and} \quad \frac{\tau^*}{\tau}\leq(n+1).$$

Thus, any limit point, $(x(0),\tau(0),s(0),\kappa(0))$, is a maximal complementarity solution for $(HMCP)$.

□

4 The homogeneous interior-point algorithm

We now present an interior-point algorithm that generates iterates within a neighborhood of $\mathcal{C}(\theta)$. For simplicity, in what follows we let $x:=(x;\tau)\in R^{n+1}$, $s:=(s;\kappa)\in R^{n+1}$, and $r^0:=(r^0;z^0)$. Recall that, for any $x,s>0$

$$x^T\psi(x)=0 \quad \text{and} \quad x^T\nabla\psi(x)=-\psi(x)^T. \tag{21}$$

Furthermore, ψ is monotone and satisfies the scaled Lipschitz condition. We will use these facts frequently in our analyses.

At iteration k with iterate $(x^k,s^k)>0$, the algorithm solves a system of linear equations for direction (d_x,d_s) from

$$d_s-\nabla\psi(x^k)d_x \quad = \quad -\eta r^k \tag{22}$$

and

$$X^kd_s+S^kd_x \quad = \quad \gamma\mu^ke-X^ks^k, \tag{23}$$

where η and γ are proper given parameters between 0 and 1, and

$$r^k = s^k - \psi(x^k) \quad \text{and} \quad \mu^k = \frac{(x^k)^T s^k}{n+1}.$$

First we prove the following lemma.

Lemma 4 . *The direction* (d_x, d_s) *satisfies*

$$d_x^T d_s = d_x^T \nabla\psi(x^k) d_x + \eta(1-\eta-\gamma)(n+1)\mu^k.$$

Proof: Premultiplying each side of (22) by d_x^T gives

$$d_x^T d_s - d_x^T \nabla\psi(x^k) d_x = -\eta d_x^T (s^k - \psi(x^k)) \tag{24}$$

Multiplying each side of (22) by x^k and using (21) give

$$\begin{aligned} (x^k)^T d_s + \psi(x^k) d_x &= -\eta(x^k)^T r^k \\ &= -\eta(x^k)^T (s^k - \psi(x^k)) \\ &= -\eta(x^k)^T s^k \\ &= -\eta(n+1)\mu^k \end{aligned} \tag{25}$$

These two equalities in combination with (23) implies

$$\begin{aligned} d_x^T d_s &= d_x^T \nabla\psi(x^k) d_x - \eta(d_x^T s^k + d_s^T x^k + \eta(n+1)\mu^k) \\ &= d_x^T \nabla\psi(x^k) d_x - \eta(-(n+1)\mu^k + \gamma(n+1)\mu^k + \eta(n+1)\mu^k) \\ &= d_x^T \nabla\psi(x^k) d_x + \eta(1-\gamma-\eta)(n+1)\mu^k \end{aligned}$$

□

For a step-size $\alpha > 0$, let the new iterate

$$x^+ := x^k + \alpha d_x > 0, \tag{26}$$

and

$$\begin{aligned} s^+ &:= s^k + \alpha d_s + \psi(x^+) - \psi(x^k) - \alpha\nabla\psi(x^k) d_x \\ &= \psi(x^+) + (s^k - \psi(x^k)) + \alpha(d_s - \nabla\psi(x^k) d_x) \\ &= \psi(x^+) + (s^k - \psi(x^k)) - \alpha\eta(s^k - \psi(x^k)) \\ &= \psi(x^+) + (1-\alpha\eta) r^k. \end{aligned} \tag{27}$$

The update (27) is a generalization of the corresponding update for the feasible case suggested in [21]. The last two equalities come from (22) and the definition of r^k. Also let

$$r^+ = s^+ - \psi(x^+).$$

Then, we have

Lemma 5 . *Consider the new iterate* (x^+, s^+) *given by (26) and (27).*

a). $r^+ = (1-\alpha\eta) r^k$

b). $(x^+)^T s^+ = (x^k)^T s^k (1-\alpha(1-\gamma)) + \alpha^2 \eta (1-\eta-\gamma)(n+1)\mu^k$

Proof: From (27)

$$\begin{aligned} r^+ &= s^+ - \psi(x^+) \\ &= (1-\alpha\eta) r^k. \end{aligned}$$

Next we prove b). Using (21), (23), and Lemma 4, we have

$$\begin{aligned} (x^+)^T s^+ &= (x^+)^T (s^k + \alpha d_s + \psi(x^+) - \psi(x^k) - \alpha \nabla\psi(x^k) d_x) \\ &= (x^+)^T (s^k + \alpha d_s) - (x^+)^T (\psi(x^k) + \alpha \nabla\psi(x^k) d_x) \\ &= (x^+)^T (s^k + \alpha d_s) - (x^k + \alpha d_x)^T (\psi(x^k) + \alpha \nabla\psi(x^k) d_x) \\ &= (x^+)^T (s^k + \alpha d_s) - \alpha (x^k)^T \nabla\psi(x^k) d_x - \alpha d_x^T \psi(x^k) - \alpha^2 d_x^T \nabla\psi(x^k) d_x \\ &= (x^+)^T (s^k + \alpha d_s) - \alpha^2 d_x^T \nabla\psi(x^k) d_x \\ &= (x^k + \alpha d_x)^T (s^k + \alpha d_s) - \alpha^2 d_x^T \nabla\psi(x^k) d_x \\ &= (x^k)^T s^k + \alpha (d_x^T s^k + d_s^T x^k) + \alpha^2 (d_x^T d_s - d_x^T \nabla\psi(x^k) d_x) \\ &= (x^k)^T s^k + \alpha (d_x^T s^k + d_s^T x^k) + \alpha^2 \eta (1-\eta-\gamma)(n+1)\mu^k \\ &= (1-\alpha(1-\gamma))(x^k)^T s^k + \alpha^2 \eta (1-\eta-\gamma)(n+1)\mu^k. \end{aligned}$$

□

This lemma shows that, for setting $\eta = 1-\gamma$, the infeasibility residual and the complementarity gap are reduced at exactly the same rate, which is the case in the homogeneous linear programming algorithm [32]. Now we prove the following result.

Theorem 6 . *Assume that* ψ *is scaled Lipschitz with* $\upsilon = \upsilon_\psi$ *defined in (5) and at iteration k*

$$\|X^k s^k - \mu^k e\| \le \beta \mu^k, \quad \mu^k = \frac{(x^k)^T s^k}{n+1}$$

where

$$\beta = \frac{1}{3 + 4\upsilon_\psi(\sqrt{2}/2)} \le 1/3.$$

Furthermore, let $\eta = \beta/\sqrt{n+1}$, $\gamma = 1-\eta$, *and* $\alpha = 1$ *in the algorithm. Then, the new iterate*

$$x^+ > 0, \quad s^+ = \psi(x^+) + (1-\eta) r^k > 0,$$

and

$$\left\|X^+s^+ - \mu^+e\right\| \leq \beta\mu^+, \quad \mu^+ = \frac{(x^+)^T s^+}{n+1}$$

Proof: Since $\alpha = 1$ it follows from Lemma 5 that $\mu^+ = \gamma\mu^k$. From (23) we further have

$$S^k d_x + X^k d_s = -X^k s^k + \mu^+ e.$$

Hence,

$$D^{-1}d_x + Dd_s = -(X^kS^k)^{-1/2}(X^ks^k - \mu^+e)$$

where $D = (X^k)^{1/2}(S^k)^{-1/2}$. Note

$$d_x^T d_s = d_x^T \nabla\psi(x^k) d_x \geq 0$$

from Lemma 4 and $\gamma = 1 - \eta$. This together with the assumption of the theorem imply

$$\|D^{-1}d_x\|^2 + \|Dd_s\|^2 \leq \|(X^kS^k)^{-1/2}(X^ks^k - \mu^+e)\|^2 \leq \frac{\left\|X^ks^k - \mu^+e\right\|^2}{(1-\beta)\mu^k}.$$

Also note

$$\begin{aligned}
\left\|X^ks^k - \mu^+e\right\|^2 &= \left\|X^ks^k - \mu^ke + (1-\gamma)\mu^ke\right\|^2 \\
&= \left\|X^ks - \mu^ke\right\|^2 + ((1-\gamma)\mu^k)^2\left\|e\right\|^2 \\
&\leq (\beta^2 + \eta^2(n+1))(\mu^k)^2 \\
&= 2\beta^2(\mu^k)^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\|(X^k)^{-1}d_x\right\| = \left\|(X^kS^k)^{-1/2}D^{-1}d_x\right\| &\leq \frac{\left\|D^{-1}d_x\right\|}{\sqrt{(1-\beta)\mu^k}} \\
&\leq \frac{\sqrt{2}\beta\mu^k}{(1-\beta)\mu^k} = \frac{\sqrt{2}\beta}{1-\beta} \leq \frac{\sqrt{2}}{2},
\end{aligned}$$

since $\beta \leq 1/3$. This implies that $x^+ = x^k + d_x > 0$. Furthermore, we have

$$\begin{aligned}
\|D_xd_s\| &= \left\|D^{-1}D_xDd_s\right\| \\
&\leq \left\|D^{-1}d_x\right\|\|Dd_s\| \\
&\leq (\left\|D^{-1}d_x\right\|^2 + \|Dd_s\|^2)/2 \\
&\leq \left\|(X^kS^k)^{-1/2}(X^ks^k - \mu^+e)\right\|^2/2 \\
&\leq \frac{\left\|X^ks^k - \mu^+e\right\|^2}{2(1-\beta)\mu^k} \\
&\leq \frac{2\beta^2(\mu^k)^2}{2(1-\beta)\mu^k} \\
&= \frac{\beta^2\mu^k}{1-\beta},
\end{aligned}$$

and

$$d_x^T d_s = d_x^T D^{-1} D d_s \leq \left\| D^{-1} d_x \right\| \| D d_s \| \leq \frac{\beta^2 \mu^k}{1-\beta}.$$

Consider

$$\begin{aligned} X^+ s^+ - \mu^+ e &= X^+ (s^k + d_s + \psi(x^+) - \psi(x^k) - \nabla \psi(x^k) d_x) - \mu^+ e \\ &= (X^k + D_x)(s^k + d_s) - \mu^+ e + X^+ (\psi(x^+) - \psi(x^k) - \nabla \psi(x^k) d_x) \\ &= D_x d_s + X^+ (\psi(x^+) - \psi(x^k) - \nabla \psi(x^k) d_x). \end{aligned}$$

Using that ψ is scaled Lipschitz, $d_x^T \nabla \psi(x^k) d_x = d_x^T d_s$ and the above four relations we obtain

$$\begin{aligned} \left\| X^+ s^+ - \mu^+ e \right\| &= \left\| D_x d_s + X^+ (\psi(x^+) - \psi(x^k) - \nabla \psi(x^k) d_x) \right\| \\ &= \left\| D_x d_s + (X^k)^{-1} X^+ X^k (\psi(x^+) - \psi(x^k) - \nabla \psi(x^k) d_x) \right\| \\ &\leq \| D_x d_s \| + \left\| (X^k)^{-1} X^+ \right\|_\infty \left\| X^k (\psi(x^+) - \psi(x^k) - \nabla \psi(x^k) d_x) \right\|_1 \\ &\leq \| D_x d_s \| + 2 \left\| X^k (\psi(x^+) - \psi(x^k) - \nabla \psi(x^k) d_x) \right\|_1 \\ &\leq \| D_x d_s \| + 2 \upsilon_\psi(\sqrt{2}/2) d_x^T \nabla \psi(x^k) d_x \\ &= \| D_x d_s \| + 2 \upsilon_\psi(\sqrt{2}/2) d_x^T d_s \\ &\leq \frac{\beta^2 \mu^k}{1-\beta} + 2 \upsilon_\psi(\sqrt{2}/2) \frac{\beta^2 \mu^k}{1-\beta} \\ &\leq \frac{(1 + 2 \upsilon_\psi(\sqrt{2}/2)) \beta^2 \mu^k}{1-\beta}. \end{aligned}$$

Finally, $\beta = 1/(3 + 4\upsilon_\psi(\sqrt{2}/2))$ implies that

$$\frac{(1 + 2 \upsilon_\psi(\sqrt{2}/2)) \beta^2 \mu^k}{1-\beta} \leq \frac{\beta}{2} \mu^k$$

and

$$\left\| X^+ s^+ - \mu^+ e \right\| \leq \beta \mu^k / 2 < \beta \gamma \mu^k = \beta \mu^+.$$

It is easy to verify that $x^+ > 0$ and $\|X^+ s^+ - \mu^+ e\| < \beta \mu^+$ implies $s^+ > 0$. □

The above theorem shows that the homogeneous algorithm will generate a sequence $(x^k, s^k) > 0$ with $(x^{k+1}, s^{k+1}) := (x^+, s^+)$ such that $s^k = \psi(x^k) + r^k$ and $\|X^k s^k - \mu^k e\| \leq \beta \mu^k$, where both $\left\| r^k \right\|$ and $(x^k)^T s^k$ converge to zero at a global rate $\gamma = 1 - \beta/\sqrt{n+1}$. We see that if $\upsilon_\psi(\sqrt{2}/2)$ is a constant, or $\upsilon_f(2/(1+\sqrt{2}))$ is a constant in (MCP) due to (iii) of Theorem 2, then it results in an $O(\sqrt{n} \log(1/\epsilon))$ iteration algorithm with error ϵ. It generates a maximal solution for $(HMCP)$, which is either a solution or a certificate proving infeasibility for (MCP), due to (v) and (vi) of Theorem 2.

One more comment is that our results should hold for the case that $f(x)$ is a continuous *monotone* mapping from R^n_{++} to R^n. In other words, $f(x)$ may not be defined at the boundary of R^n_+ but asymptotically exist. In general, many convex optimization problems can be solved, either obtaining a solution or proving infeasibility or unboundedness, by solving their Karush-Kuhn-Tucker (KKT) system which is a monotone complementarity problem.

5 A preliminary implementation

In this section we discuss a preliminary implementation of the homogeneous algorithm proposed in the previous sections. We restrict our implementation at this moment to the linearly constrained convex programming problem which has the form

$$\begin{array}{ll} \text{minimize} & c(x) \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \tag{28}$$

where $A \in R^{m\times n}$ and $b \in R^m$. Without loss of generality we will assume $n > m$ and A is of full row rank. We assume c is convex and at least twice differentiable on R^n_{++}. (Others have proposed algorithms for solving this class of problems, e.g. [21, 20, 30, 15].)

The Wolfe dual to (28) is

$$\begin{array}{ll} \text{maximize} & c(x) - \nabla c(x)x + b^T y \\ \text{subject to} & \nabla c(x)^T - A^T y - s = 0, \\ & x, s \geq 0. \end{array} \tag{29}$$

The KKT optimality conditions to (28) are

$$\begin{array}{rcll} \nabla c(x)^T - A^T y - s & = & 0, & s \geq 0, \\ Ax & = & b, & x \geq 0, \\ Xs & = & 0, & \end{array} \tag{30}$$

where y and s are the Lagrange multipliers corresponding to the equality and inequality constraints in (28), respectively. Now the system (30) is an (MCP) problem with free variables y. Introducing the homogenization variable τ and slack variable κ into (30) we obtain an $(HMCP)$ problem

$$\begin{array}{rcll} \tau\nabla c(x/\tau)^T - A^T y - s & = & 0, & s \geq 0,\ y \text{ free} \\ Ax - b\tau & = & 0, & x \geq 0,\ \tau \geq 0 \\ -x^T\nabla c(x/\tau)^T + b^T y - \kappa & = & 0, & \kappa \geq 0, \\ Xs & = & 0, & \\ \tau\kappa & = & 0. & \end{array} \tag{31}$$

It should be noticed that all previous $(HMCP)$ results hold for system (31), although it has free variables y.

5.1 The search direction

The search direction used in our algorithm is the same as that proposed in Section 4. In other words, we solve a perturbed version of (31) using Newton's method. Let $(x^k > 0; \tau^k > 0; y^k; s^k > 0; \kappa^k > 0)$ be the current point on iteration k. Now define the residual vectors

$$
\begin{array}{rcl}
r_D^k & = & s^k + A^T y^k - \tau^k \nabla c(x^k/\tau^k)^T, \\
r_G^k & = & \kappa^k + (x^k)^T \nabla c(x^k/\tau^k)^T - b^T y^k, \\
r_P^k & = & b\tau^k - Ax^k,
\end{array}
$$

and let

$$
H^k = \nabla^2 c(x^k/\tau^k), \quad g^k = \nabla c(x^k/\tau^k), \quad \text{and} \quad \xi^k = x^k/\tau^k.
$$

Then, the Newton search direction is given by

$$
\begin{bmatrix} H^k & (g^k)^T - H^k \xi^k & -A^T \\ -g^k - (\xi^k)^T H^k & (\xi^k)^T H^k \xi^k & b^T \\ A & -b & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_\tau \\ d_y \end{bmatrix} - \begin{bmatrix} d_s \\ d_\kappa \\ 0 \end{bmatrix} = \eta \begin{bmatrix} r_D^k \\ r_G^k \\ r_P^k \end{bmatrix} \tag{32}
$$

and

$$
\begin{array}{rcl}
S^k d_x + X^k d_s & = & -X^k s^k + \gamma \mu^k e, \\
\kappa^k d_\tau + \tau^k d_\kappa & = & -\tau^k \kappa^k + \gamma \mu^k,
\end{array} \tag{33}
$$

where $\mu^k = ((x^k)^T s^k + \tau^k \kappa^k)/(n+1)$, and η and γ are parameters to be chosen lately. If we use (33) to eliminate $(d_s; d_\kappa)$ we obtain the reduced Newton equation system

$$
\begin{array}{c}
\begin{bmatrix} H^k + (X^k)^{-1} S^k & (g^k)^T - H^k \xi^k & -A^T \\ -g^k - (\xi^k)^T H^k & (\xi^k)^T H^k \xi^k + (\tau^k)^{-1} \kappa^k & b^T \\ A & -b & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_\tau \\ d_y \end{bmatrix} \\
= \begin{bmatrix} \eta r_D^k + (X^k)^{-1}(-X^k s^k + \gamma \mu^k e) \\ \eta r_G^k + (\tau^k)^{-1}(-\tau^k \kappa^k + \gamma \mu^k) \\ \eta r_P^k \end{bmatrix}
\end{array} \tag{34}
$$

The coefficient matrix in (34) is in general not symmetric and cannot be made symmetric by a simple row or column scaling. (The matrix is skew-symmetric if c is a linear function.)

Let

$$
K = \begin{bmatrix} H^k + (X^k)^{-1} S^k & -A^T \\ A & 0 \end{bmatrix}. \tag{35}
$$

It is easy to verify that if A is of full row rank, then K is a non-singular matrix (see Lemma 7 below). Now define h^k and p^k as solutions to the linear systems

$$Kh^k = \begin{bmatrix} (g^k)^T - H^k\xi^k \\ -b \end{bmatrix} \quad \text{and} \quad Kp^k = \begin{bmatrix} \eta r_D^k + (X^k)^{-1}(-X^k s^k + \gamma\mu^k e) \\ \eta r_P^k \end{bmatrix}$$

then $(d_x; d_\tau)$ can be computed as follows

$$d_\tau = \frac{\eta r_G^k + (\tau^k)^{-1}(-\tau^k\kappa^k + \gamma\mu^k e) + (g^k + (\xi^k)^T H^k; -b^T)p^k}{(\tau^k)^{-1}\kappa^k + (\xi^k)^T H^k \xi^k + (g^k + (\xi^k)^T H^k; -b^T)h^k}$$

and

$$(d_x; d_y) = p^k - h^k d_\tau.$$

Given $(d_x; d_\tau)$ then (d_s, d_κ) can easily be computed from relation (33). In the following lemma we will show that d_τ is always well defined.

Lemma 7 . *Let K be defined by (35) then*

$$K^{-1} = \begin{bmatrix} Q^{-1} - Q^{-1}A^T(AQ^{-1}A^T)^{-1}AQ^{-1} & Q^{-1}A^T(AQ^{-1}A^T)^{-1} \\ -(AQ^{-1}A^T)^{-1}AQ^{-1} & (AQ^{-1}A^T)^{-1} \end{bmatrix},$$

where $Q = H^k + (X^k)^{-1}S^k$. Moreover,

$$(\xi^k)^T H^k \xi^k + (g^k + (\xi^k)^T H^k; -b^T)h^k \geq 0.$$

Proof: It is easy to verify that $KK^{-1} = I$. Note that $Q^{-1} - Q^{-1}A^T(AQ^{-1}A^T)^{-1}AQ^{-1}$ is symmetric and positive semi-definite. By definition

$$h^k = K^{-1}((g^k)^T - H^k\xi^k; -b).$$

In the rest of the proof we use the definition $B := (AQ^{-1}A^T)^{-1}$. Thus, we have by definition that

$$h^k = \begin{bmatrix} Q^{-1} - Q^{-1}A^T BAQ^{-1} & Q^{-1}A^T B \\ -BAQ^{-1} & B \end{bmatrix} \begin{bmatrix} (g^k)^T - H^k\xi^k \\ -b \end{bmatrix}.$$

Hence

$$\begin{aligned}
& (g^k + (\xi^k)^T H^k; -b^T)h^k \\
= \; & (g^k + (\xi^k)^T H^k)(Q^{-1} - Q^{-1}A^T BAQ^{-1})((g^k)^T - H^k\xi^k) \\
& + b^T BAQ^{-1}((g^k)^T - H^k\xi^k) - (g^k + (\xi^k)^T H^k)Q^{-1}A^T Bb + b^T Bb \\
= \; & g^k(Q^{-1} - Q^{-1}A^T BAQ^{-1})(g^k)^T - (\xi^k)^T H^k(Q^{-1} - Q^{-1}A^T BAQ^{-1})H^k\xi^k \\
& + (b - AQ^{-1}H^k\xi^k)^T B(b - AQ^{-1}H^k\xi^k) - (\xi^k)^T H^k Q^{-1}A^T BAQ^{-1}H^k\xi^k \\
\geq \; & -(\xi^k)^T H^k Q^{-1} H^k \xi^k.
\end{aligned}$$

Since H^k is positive semi-definite and $Q = H^k + (X^k)^{-1}S^k$,

$$(\xi^k)^T H^k \xi^k - (\xi^k)^T H^k Q^{-1} H^k \xi^k \geq 0,$$

which gives the desired result. $\square$

It should be observed that h^k needs to be computed only once in each iteration, even if (32) and (33) are solved for different right-hand-side vectors. Thus, the homogeneous algorithm needs exact one more solve per iteration than the "conventional" primal-dual algorithm with an interior feasible starting point.

The main work in computing the search direction lies in computing h^k and p^k. Both h^k and p^k are solutions to a linear system of the form

$$K(u_1; u_2) = \bar{K}(u_1; -u_2) = (v_1; v_2) \tag{36}$$

where

$$\bar{K} = \begin{bmatrix} H^k + (X^k)^{-1}S^k & A^T \\ A & 0 \end{bmatrix}.$$

$\bar{K}$ is a symmetric indefinite matrix and its inverse is given by

$$\bar{K}^{-1} = \begin{bmatrix} Q^{-1} - Q^{-1}A^T(AQ^{-1}A^T)AQ^{-1} & Q^{-1}A^T(AQ^{-1}A^T)^{-1} \\ (AQ^{-1}A^T)^{-1}AQ^{-1} & -(AQ^{-1}A^T)^{-1} \end{bmatrix}$$

where $Q = H^k + (X^k)^{-1}S^k$. Clearly Q is a positive definite matrix, because c is convex.

If Choleski factorizations of the positive definite matrices Q and $(AQ^{-1}A^T)$ are computed and known, then it is easy to solve system (36). Especially, if Q is a diagonal matrix, then the Choleski factorization method might be desired. Indeed most interior-point codes for LP use this approach.

Another approach to solve (36) is to compute the factorization

$$\bar{K} = LDL^T \tag{37}$$

using the Bunch-Parlett strategy, where L is a lower triangular matrix and D is a non-singular block diagonal matrix. This approach has been pioneered for LP by Fourer and Mehrotra [5] and Turner [28]. The main advantage of this approach is that it has good numerical properties and it is not hampered by a few dense columns in A.

Vanderbei [29] proposes an alternative factorization based on the observation that the matrix $\bar{K}$ is quasi-definite. It follows that there exists a factorization of $\bar{K}$ such that D is a diagonal matrix. The main advantage of Vanderbei's approach is a sparsity preserving ordering can be chosen once and for all. This leads to a very efficient implementation. However, a factorization based on the quasi-definite property might not be as numerically stable as the more general Bunch-Parlett factorization.

In our preliminary implementation we use a factorization based on Vanderbei's ideas. The ordering is obtained with the minimum degree heuristic. In the future we plan to do some experiments with an iterative solver and the Bunch-Parlett factorization.

5.2 Choice of parameters and high-order

In each iteration we must choose the parameters η and γ related by $\gamma = 1 - \eta$. In Theorem 6 we have shown that the choice $\gamma = 1 - O(1/\sqrt{n})$ leads to polynomial complexity. Unfortunately this choice also leads to a very slow linear convergence. Therefore in our implementation we use a more aggressive approach. It is based on a heuristic originally proposed by Mehrotra [18] for solving LP problems. The idea is first to compute the affine scaling direction which is a solution to (32) and (33) for $\gamma = 0$ and $\eta = 1$. Let $(d_x^a; d_\tau^a; d_y^a; d_s^a)$ be the affine scaling direction then compute

$$\begin{array}{rcll} \bar{\alpha}^a & = & \text{minimize} & ((x;\tau) + \alpha(d_x^a; d_\tau^a))^T((s^a;\kappa^a)^T + \alpha(d_s^a; d_\kappa^a)) \\ & & \text{subject to} & (x;\tau;s;\kappa) + \alpha(d_x^a; d_\tau^a; d_s^a; d_\kappa^a) \geq 0, \\ & & & \alpha \geq 0, \end{array} \tag{38}$$

and

$$r = \frac{((x;\tau) + \bar{\alpha}^a(d_x^a; d_\tau^a))^T((s;\kappa)^T + \bar{\alpha}^a(d_s^a; d_\kappa^a))}{(x;\tau)^T(s;\kappa)}. \tag{39}$$

Next we let

$$\gamma = \max(\min(r^2, r/10.0), 1.0e-6). \tag{40}$$

The idea is to reduce the barrier parameter γ fast whenever good progress can be made along the affine scaling direction.

Afterwards we recompute the search direction using this choice of γ and $\eta = 1 - \gamma$. The new point is given by

$$x^+ = x + \alpha d_x, \quad \tau^+ = \tau + \alpha d_\tau,$$

and

$$y^+ = y + \alpha d_y, \quad s^+ = s + \alpha d_s, \quad \kappa^+ = \kappa + \alpha d_\kappa.$$

Note that in our preliminary implementation we do not take different step size for the primal and dual variables. Moreover, we do not use the nonlinear update (27) of the dual variables due to complication of finding a feasible step size. However we choose the step size α such that the new point satisfies

$$\min(x_j^+ s_j^+, \tau^+\kappa^+) \geq \beta_l \frac{(x^+;\tau^+)^T(s^+;\kappa^+)}{n+1} \tag{41}$$

where β_l is a constant. This prevents the iterates from prematurely to converge to the boundary (e.g., see Güler and Ye [7]).

It is common to choose the step size α such that the iterates remain in a certain neighborhood or a merit function is decreased to secure global convergence, see Kojima et.al. [11] and Anstreicher and Vial [1]. We have not used this safeguard yet.

It is well-known that high-order corrections can be used to improve the practical efficiency of the primal-dual barrier algorithm for LP significantly. Especially ideas along the lines of those proposed by Mehrotra [18] seem to work well. A theoretical result for interior-point algorithms using high-order information was first obtained by Monteiro, Adler, and Resende [22]. Furthermore, Hung and Ye [9] have recently obtained more theoretical results for a high-order version of the homogeneous algorithm for LP. In our preliminary implementation we have used a high-order algorithm similar to Mehrotra's. In our preliminary implementation we only do the correction on the complementarity equation.

5.3 Simple bounds

Many practical problems contains a lot of simple bounds of the type

$$l \leq x \leq u$$

Therefore we solve the problem

$$\begin{array}{ll} \text{minimize} & c(x' + l) \\ \text{subject to} & Ax' = b - Al \\ & x' + z = u - l \\ & x' \geq 0,\ z \geq 0, \end{array}$$

where z is an additional slack variable. We only introduce z variable for each finite upper bound, and our linear algebra handles the upper bounds explicitly. The function $c(x' + l)$ is seen as a new composite function.

5.4 Starting point and stopping criteria

Our starting point is

$$(x^0, \tau^0, y^0, s^0, \kappa^0) = (e, 1, 0, e, 1).$$

Define

$$pfeas_1^k = \frac{\left\|Ax^k - \tau^k b\right\|_1}{\tau^k + \|x^k\|_1} \quad \text{and} \quad dfeas_1^k = \frac{\left\|\tau^k \nabla c(x^k/\tau^k) - A^T y^k - s^k\right\|_1}{\tau^k + \|s^k\|_1}.$$

and

$$rgap^k = \frac{\nabla c(x^k/\tau^k)x^k - b^T y^k}{\tau^k + |\tau^k c(x^k/\tau^k) - \nabla c(x^k/\tau^k)x^k + b^T y^k|}.$$

$rgap^k$ is equal to the absolute duality gap divided by the dual objective value. We terminate the algorithm in the case we have

$$pfeas_1^k \leq tolpfeas_1, \quad dfeas_1^k \leq toldfeas_1, \quad \text{and} \quad rgap^k \leq tolrgap,$$

where $tolpfeas_1, toldfeas_1$, and $tolrgap$ are user-specified tolerances. In this case we say the solution is feasible and we report the solution

$$(x', y', s') = (x^k/\tau^k, y^k/\tau^k, s^k/\tau^k).$$

We also terminate the algorithm if

$$\frac{\mu^k}{\mu^0} \leq tolured.$$

If

$$\frac{\tau^k \kappa^0}{\kappa^k \tau^0} \leq tolinf.$$

we report the problem is infeasible. $tolured$ and $tolinf$ are also user-specified tolerances. Our termination criteria are a slight generalization of those proposed by Xu et al. [32].

6 Computational results

In this section we will report our computational results to demonstrate effectiveness of the homogeneous algorithm for linearly constraint convex programming problems. All our code is written in ANSI C. The test is performed on a HP 715/75 UNIX workstation. All timing results are obtained using the C procedure "clock". We use the tolerances $tolpfeas_1 = toldfeas_1 = 1.0e-6,\ tolrgap = 1.0e-8,\ tolured = 1.0e-14$ and $tolinf = 1.0e-12$.

In the first part of the test we use quadratic problems as test problems. The test problem is formed by adding a quadratic term $0.5x^TQx$ to the objective function of some of the NETLIB problems. The Q matrix has the form

$$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & 2 & -1 \\ & & -1 & 2 \end{bmatrix}.$$

Q is clearly positive definite because it is strictly diagonal dominant. Moreover, the Q matrix is block diagonal. We select the feasible problems 25FV47, DEGEN3, GREENBEA, PILOT and WOOD1P and two (primal) infeasible problems KLEIN2 and VOL1 from the NETLIB LP problems.

Title	rows	columns	status	itera-tions	time	rgap	pfeas1	dfeas1
25FV47	821	1876	OPTIMAL	18	14.8	4.9E-10	2.0E-13	1.0E-15
DEGEN3	1503	2604	OPTIMAL	14	86.1	8.7E-09	1.0E-09	1.0E-10
GREENBEA	2392	5495	OPTIMAL	27	49.0	6.1E-09	1.0E-09	7.0E-13
KLEIN2	477	531	INFEASIBLE	21	21.8	1.0E+00	2.0E-02	6.0E-15
B1M100T1	0	10000	OPTIMAL	10	19.4	2.4E-10	4.0E-16	2.0E-13
B1M200T1	0	40000	OPTIMAL	10	107.9	9.1E-10	4.0E-16	5.0E-13
B1M300T1	0	90000	OPTIMAL	11	346.4	3.3E-10	7.0E-16	2.0E-12
B2M100T1	0	10000	OPTIMAL	13	20.9	5.9E-09	0.0E+00	5.0E-14
B2M200T1	0	40000	OPTIMAL	12	113.5	8.6E-13	0.0E+00	6.0E-13
B2M300T1	0	90000	OPTIMAL	10	287.8	9.9E-10	0.0E+00	2.0E-14
B3M100T20	0	10000	OPTIMAL	11	21.9	7.7E-11	6.0E-16	1.0E-11
B3M200T20	0	40000	OPTIMAL	10	108.0	4.2E-09	2.0E-16	3.0E-09
B3M300T20	0	90000	OPTIMAL	14	440.1	2.6E-09	8.0E-17	5.0E-09
PILOT	1441	4693	OPTIMAL	33	254.9	9.3E-09	2.0E-08	4.0E-11
VOL1	323	459	INFEASIBLE	29	2.6	1.0E+00	3.0E-01	5.0E-12
WOOD1P	244	2595	OPTIMAL	22	58.0	2.9E-10	9.0E-10	2.0E-15

Table 1: Quadratic test problems.

In addition we use some box-constrained QP problems arising from the so-called obstacle and elastic-plastic torsion problems. These problems are taken from the paper Han et al. [8]. The name of these problems all start with the letter B. The B2 problems are slightly modified, because the original problems contain some very large upper bounds on the variables. These bounds are always redundant and therefore we have removed them in B2, which leads to a reduction in the number of iterations by approximately 50%. Our results are presented in Table 1.

The columns in Table 1 shows the number constraints, the number variables (including slack variables), the solution status, the number of iterations, the solution time in seconds, $rgap$, $pfeas_1$ and $dfeas_1$. We see that all of the feasible problems are solved to the required precision and two infeasible problems are correctly detected. As expected, the algorithm uses very few iterations to solve these problems. In summary our algorithm seems to work well for solving quadratic problems.

In the second part of the test we solve some entropy problems. These problems is formed by adding an entropy term, $\sum x_j \ln(x_j)$, to the objective function of some of the NETLIB LP problems: 25FV47,

BNL1, FIT1D, KLEIN2, MONDOU2, SCSD8, SHIP12L, and WOOD1P. Except for problems KLEIN2 and MONDOU2, these problems are feasible, but many of them have no interior similar to the following example

$$\begin{array}{ll} \text{minimize} & x\ln(x) \\ \text{subject to} & x = 0, \\ & x \geq 0. \end{array}$$

Its dual problem is

$$\begin{array}{ll} \text{maximize} & -x \\ \text{subject to} & 1 + \ln(x) - y - s = 0, \\ & x, s \geq 0. \end{array}$$

In any feasible primal-dual pair the dual solution y must be equal to $-\infty$. Therefore if the entropy problem has an optimal solution with some of the primal variables equal to zero, unbounded dual solutions, and numerical difficulties are expected.

In solving these entropy minimization problems we have used the nonlinear update

$$s^+ = \nabla f(x + \alpha d_x) - A^T(y + \alpha d_y) + (1 - \alpha\eta)r_D. \tag{42}$$

of the dual slacks proposed in Section 4, see (27). If this update is used the primal and dual infeasibility is reduced at the same rate. We have implemented the update as follows. First the maximum step-size $\bar{\alpha}$ is determined subject to $(x + \bar{\alpha}d_x, \tau + \bar{\alpha}d_\tau, s + \bar{\alpha}d_s, \kappa + \bar{\alpha}d_\kappa) \geq 0$. Next the first t in $\{1, 2, 4, 16, \ldots\}$ is chosen with $\alpha = 0.9^t\bar{\alpha}$ such that s^+ in (42) is positive and (41) is satisfied. We also use a less aggressive choice of γ, see (40). γ is computed using $\gamma = \min(0.9, \max(r^2, 1.0e-6))$ where r is defined by (39). The step-size $\bar{\alpha}^a$ used in (39) is a approximation for the maximum step-size to the boundary using the nonlinear update of the dual slacks. In Table 2 we have reported the results for solving these problems. We see from Table 2 that a good accuracy is achieved and all the problems are solved satisfactorily. As expected the iterations count is worse especially for the problem BNL1. The iteration count is increased because for these nonlinear problems we must follow the central path closely. We expect that a more sophisticated line search in the dual update will make the implementation more efficient.

7 Conclusion

In this paper we have presented a generalization of the homogeneous model for LP to solving the monotone complementarity problem. The advantages of the model are: 1) it can detect the infeasible status by generating a certificate; 2) it can start from any point in the positive orthant; 3) if the problem is polynomially solvable with an interior feasible starting point, then the problem is polynomially solvable, either find a solution or prove infeasibility, without using or knowing such information at all.

Title	rows	columns	status	itera-tions	time	rgap	pfeas1	dfeas1
25FV47	821	1876	OPTIMAL	32	15.0	4.2E-09	5.0E-09	7.0E-10
BNL1	643	1586	OPTIMAL	98	22.5	3.4E-09	4.0E-10	1.0E-13
FIT1D	24	1049	OPTIMAL	21	4.0	3.2E-09	4.0E-09	3.0E-10
KLEIN2	477	531	INFEASIBLE	28	25.6	1.0E+00	3.0E-02	1.0E-16
MONDOU2	312	604	INFEASIBLE	13	0.9	1.0E+00	7.0E+05	6.0E-15
SCSD8	397	2750	OPTIMAL	14	4.5	1.5E-11	1.0E-15	9.0E-15
SHIP12L	1151	5533	OPTIMAL	39	28.5	6.0E-09	9.0E-08	2.0E-09
WOOD1P	244	2595	OPTIMAL	43	49.7	2.9E-08	2.0E-07	1.0E-17

Table 2: Results for the entropy problems.

We have specialized the proposed algorithm to solving the linearly constraint convex program, and implemented the proposed algorithm on solving some convex test-problems. Our preliminary computational results are promising, although some issues, such as to handle unbounded asymptotical optimal solutions, need some further research work.

References

[1] K. M. Anstreicher and J. -P. Vial, On the convergence of an infeasible primal-dual interior-point method for convex programming, *Optimization Methods and Software* 3 (1994) 285–316.

[2] R. W. Cottle, J. -S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, (Computer Science and Scientific Computing, Academic Press, San Diego, 1992).

[3] D. den Hertog, J. Kaliski, C. Roos, and T. Terlaky, A logarithmic barrier cutting plane method for convex programming, in: I. Maros and G. Mitra, editors, *Annals of Operations Research* 58 (Baltzer Science Publishers, Amsterdam, The Netherlands, 1995) 69–98.

[4] A. S. El-Bakry and R. A. Tapia and T. Tsuchiya and Y. Zhang, On the formulation and theory of the primal-dual Newton interior-point method for nonlinear programming, *J. Optim. Theory Appl.* 89 (1996) 507–541.

[5] R. Fourer and S. Mehrotra, Solving symmetric indefinite systems in an interior point method for linear programming, *Math. Programming* 62 (1993) 15–40.

[6] O. Güler, Existence of interior points and interior paths in nonlinear monotone complementarity problems, *Math. Oper. Res.* 18 (1993) 148–162.

[7] O. Güler and Y. Ye, Convergence behavior of interior-point algorithms, *Math. Programming* 60 (1993) 215–228.

[8] C.-G. Han, P. M. Pardalos, and Y. Ye, Computational aspects of an interior point algorithm for quadratic programming problems with box constraints, in: T. F. Coleman and Y. Li, editors, *Large-Scale Numerical Optimization* (SIAM, Philadelphia, 1990) 92–112.

[9] P. -F. Hung and Y. Ye, An asymptotical $O(\sqrt{n}L)$-iteration path-following linear programming algorithm that uses wide neighborhoods, *SIAM J. on Optim.* 6 (1996) 570–586.

[10] F. Jarre, Interior-point methods for convex programming, *Applied Mathematics and Optimization* 26 (1992) 287–311.

[11] M. Kojima, N. Megiddo, and S. Mizuno, A primal-dual infeasible-interior-point algorithm for linear programming, *Math. Programming* 61 (1993) 263–280.

[12] M. Kojima, S. Mizuno, and T. Noma, Limiting behavior of trajectories by continuation method for monotone complementarity problems, *Math. Oper. Res.* 15 (1990) 662–675.

[13] M. Kojima, S. Mizuno, and A. Yoshise, An $O(\sqrt{n}L)$ iteration potential reduction algorithm for linear complementarity problems, *Math. Programming* 50 (1991) 331–342.

[14] M. Kojima, S. Mizuno, and A. Yoshise, A convex property of monotone complementarity problems, Technical Report B-267, Department of Information Sciences, Tokyo Institute of Technology, Tokyo, Japan (1993).

[15] K. O. Kortanek, X. Xu, and Y. Ye, An interior-point algorithm for solving primal and dual geometric programs, Technical Report, Department of Management Sciences, The University of Iowa (1994) (to appear in Math. Programming).

[16] L. McLinden, An analogue of Moreau's proximation theorem, with applications to nonlinear complementarity problems, *Pacific J. Math.* 88 (1980) 101–161.

[17] L. McLinden, The complementarity problem for maximal monotone multifunctions, in F. Giannessi and J. -L Lions, editors, *Variational inequalities and complementarity problems* (John Wiley and Sons, New York, 1980) 251–270.

[18] S. Mehrotra, On the implementation of a primal-dual interior point method, *SIAM J. on Optim.* 2 (1992) 575–601.

[19] S. Mizuno, F. Jarre, and J. Stoer, A unified approach to infeasible-interior-point algorithms via geometrical linear complementarity problems, *Appl. Math. Optim.* 33 (1996) 315–341.

[20] R. D. C. Monteiro, A globally convergent primal-dual interior point algorithm for convex programming, *Math. Programming* 64 (1994) 123–147.

[21] R. D. C. Monteiro and I. Adler, An extension of Karmarkar type algorithm to a class of convex separable programming problems with global convergence, *Math. Oper. Res.* 15 (1990) 408–422.

[22] R. D. C. Monteiro, I. Adler, and M. G. C. Resende, A polynomial-time primal-dual affine scaling for linear and convex quadratic programming and its power series extension, *Math. Oper. Res.* 15 (1990) 191–214.

[23] R. D. C. Monteiro and S. Wright, Local convergence of interior-point algorithms for degenerate monotone LCP, Technical Report MCS-P357-0493, Mathematics and Computer Science Division, Argonne National Laboratory, Chicago, IL (1993).

[24] Y. Nesterov and A. Nemirovskii, *Interior-Point Polynomial Algorithms in Convex Programming* (SIAM, Philadelphia, 1994).

[25] F. Potra and Y. Ye, Interior-point methods for nonlinear complementarity problem, *J. Optim. Theory Appl.* 68 (1996) 617–642.

[26] R. T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, New Jersey, 1970).

[27] K. Tanabe, Complementarity-enforcing centered Newton method for mathematical programming: global method, *The Symposium of New Methods for Linear Programming* (The Institute of Statistical Mathematics, Tokyo, Japan 106, 1987).

[28] K. Turner, Computing projections for Karmarkar algorithm, *Linear Algebra Appl.* 152 (1991) 141–154.

[29] R. J. Vanderbei, Symmetric quasi-definite matrices, *SIAM J. on Optim.* 5 (1995) 100–113.

[30] J. -P. Vial, Computational experience with a primal-dual algorithm for smooth convex programming, *Optimization Methods and Software* 3 (1994) 273–283.

[31] T. Wang, R. D. C. Monteiro, and J. -S. Pang, An interior point potential reduction algorithm for constrained equations, *Math. Programming* 74 (1996) 159–195.

[32] X. Xu, P. -F. Hung, and Y. Ye, A simplified homogeneous and self-dual linear programming algorithm and its implementation, *Annals of Operations Research* 62 (1996) 151–171.

[33] Y. Ye, *Interior algorithms for linear, quadratic, and linearly constrained convex programming* (PhD thesis, Stanford University, Stanford, CA, 1987).

[34] Y. Ye, On homogeneous and self-dual algorithm for LCP. Technical Report, College of Business Administration, The University of Iowa, Iowa City, IA (1994) (To appear in Math Programming).

[35] Y. Ye, M. J. Todd, and S. Mizuno, An $O(\sqrt{n}L)$ - iteration homogeneous and self-dual linear programming algorithm, *Math. Oper. Res.* 19 (1994) 53–67.

[36] J. Zhu, A path following algorithm for a class of convex programming problems, *Zeitschrift für Operations Research* 36 (1992) 359–377.