A FPTAS for Computing a Symmetric Leontief Competitive Economy Equilibrium*

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Abstract

We consider a linear complementarity problem (LCP) arisen from the Arrow-Debreu-Leontief competitive economy equilibrium where the LCP coefficient matrix is symmetric. We prove that the decision problem, to decide whether or not there exists a complementary solution, is NP-complete. Under certain conditions, an LCP solution is guaranteed to exist and we present a fully polynomial-time approximation scheme (FPTAS) for computing such a solution, although the LCP solution set can be non-convex or non-connected. Our method is based on solving a quadratic social utility optimization problem (QP) and showing that a certain KKT point of the QP problem is an LCP solution. Then, we further show that such a KKT point can be approximated with running time $\mathcal{O}((\frac{1}{\epsilon})\log(\frac{1}{\epsilon})\log(\log(\frac{1}{\epsilon}))$ in accuracy $\epsilon \in (0,1)$ and a polynomial in problem dimensions. We also report preliminary computational results which show that the method is highly effective.

1 Introduction

Given a real n by n matrix A, consider the linear complementarity problem (LCP) to find u and v such that

$$A^T u + v = e,\ u^T v = 0,\ (u \neq 0, v) \geq 0, \tag{1}$$

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where e is the vector of all ones. Note that $u^T v = 0$ implies that $u_i v_i = 0$ for all $i = 1, \cdots, n$. Also, $u = 0$ and $v = e$ is a trivial complementary solution. But we look for a non-trivial solution where $u \neq 0$ (see Cottle at al. [5] for more literature on linear complementarity problems).

In this note, we focus on the case that A is symmetric. We first prove that the decision problem, to decide whether or not there exists such a complementary solution, is NP-complete. Under certain conditions, for example, that all entries of A is non-negative, an LCP solution is guaranteed to exist. Then, we present a fully polynomial-time approximation scheme (FPTAS) for computing a solution, although the LCP solution set can be non-convex or non-connected.

Our method is based on solving a quadratic social utility optimization problem (QP) and showing that a certain KKT point of the QP problem is an LCP solution. Then, we further show that such a KKT point can be approximated with running time $\mathcal{O}((\frac{1}{\epsilon})\log(\frac{1}{\epsilon})\log(\log(\frac{1}{\epsilon}))$ in accuracy $\epsilon \in (0,1)$ and a polynomial in problem dimensions. We also report preliminary computational results which show that the method is highly effective in comparison with other well known LCP solvers.

2 Connection to Competitive Market and Bimatrix Game Equilibria

The LCP (1) rises from the Arrow-Debreu exchange competitive economy equilibrium problem where it was first formulated by Léon Walras in 1874. In this equilibrium problem everyone in a population of m traders has an initial endowment of a divisible goods and a utility function for consuming all goods—their own and others'. Every trader sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's goods such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that, under mild conditions, such equilibrium would exist if the utility functions were concave. In general, it is unknown whether or not an equilibrium can be computed efficiently.

Consider a special class of Arrow-Debreu's problems, where each of the n traders has exactly one unit of a divisible and distinctive good for trade, and let trader i, $i = 1, ..., n$, bring good i, which class of problems is called the *pairing class* [13]. For given prices p_j on good j, consumer i's maximization problem is

$$\begin{array}{ll} \text{maximize} & u_i(x_{i1}, ..., x_{in}) \\ \text{subject to} & \sum_j p_j x_{ij} \leq p_i, \\ & x_{ij} \geq 0, \quad \forall j. \end{array} \tag{2}$$

Let x_i^* denote a maximal solution vector of (2). Then, vector p is called the Arrow-Debreu price equilibrium if there exists an x_i^* for consumer i, $i = 1, ..., n$, such that

$$\sum_i x_i^* = e$$

where e represents available amount of goods on the exchange market.

The Leontief exchange economy problem is the Arrow-Debreu equilibrium when the utility functions are in the Leontief form:

$$u_i(x_i) = \min_{j:\ a_{ij}>0} \left\{ \frac{x_{ij}}{a_{ij}} \right\},$$

where the Leontief coefficient matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}.$$

It was proved that

Theorem 1 *(Ye [13]) Let $B \subset \{1, 2, ..., n\}$, $N = \{1, 2, ..., n\} \setminus B$, A_{BB} be irreducible, and u_B satisfy the linear system*

$$A_{BB}^T u_B = e, \quad A_{BN}^T u_B \le e, \quad \text{and} \quad u_B > 0.$$

Then the (right) Perron-Frobenius eigen-vector p_B of $U_B H_{BB}$ together with $p_N = 0$ will be a Leontief economy equilibrium. And the converse is also true.

Theorem 1 has thus established a combinatorial algorithm to compute a Leontief economy equilibrium by finding a right block $B \neq \emptyset$, which is precisely a (non-trivial) complementary solution to the LCP problem (1).

The LCP (1) is also connected to the bimatrix game equilibrium problem specified by a pair of $n \times m$ pay-off matrices C and R, with positive entries, one can construct a Leontief exchange economy with $n+m$ traders and $n+m$ goods as follows.

Theorem 2 *(Codenotti et al. [4]) Let (C, R) denote an arbitrary bimatrix game, where assume, w.l.o.g., that the entries of the matrices C and R are all positive. Let*

$$A^T = \begin{pmatrix} 0 & C \\ R^T & 0 \end{pmatrix}$$

describe the Leontief utility coefficient matrix of the traders in a Leontief economy. There is a one-to-one correspondence between the Nash equilibria of the game (C, R) and the market equilibria A of the Leontief economy.

Therefore, computing a bimatrix game equilibrium is also equivalent to computing a complementary solution of LCP (1). The reader may want to read Brainard and Scarf [2], Gilboa and Zemel [8], Chen, Deng and Teng [3], Daskalakis, Goldberg ans Papadimitriou [7], and Tsaknakis and Spirakis [11] on hardness and approximation results of computing a bimatrix game equilibrium.

3 Decision of the Existence of an LCP Solution

In general, it's difficult to decide if LCP (1) has a complementary solution or not, even when A is symmetric.

Theorem 3 *Let A be a real symmetric matrix. Then, it is NP-complete to decide whether or not LCP (1) has a complementary solution such that $u \neq 0$.*

Proof Given a symmetric matrix A, it's NP-complete (see Murty and Kabadi [10]) to decide if

$$\exists u \geq 0 \quad \text{such that} \quad u^T Au > 0? \tag{3}$$

The complement problem is to decide if or not for all $u \geq 0$ one has $u^T Au \leq 0$, or $-A$ is co-positive plus.

We now prove that the decision problem (3) is equivalent to the problem that if or not LCP (1) has a complementary solution $u \neq 0$.

If (1) has a complementary solution $u \neq 0$, then

$$0 = u^T(e - Au) = e^T u - u^T Au.$$

Since $u \geq 0$ and $u \neq 0$, we have $u^T Au = e^T u > 0$.

On the other hand, if the answer to the decision problem (3) is "yes", then the maximal value of the following bounded quadratic problem:

$$\begin{array}{lll} (QP) & \text{maximize} & u^T Au \\ & \text{subject to} & e^T u = 1, \ u \geq 0, \end{array} \tag{4}$$

is strictly positive. Let u^* be the global maximizer of the problem. Then, u^* must satisfy the Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned} -2Au + \lambda e &= v \\ u^T v &= 0, \\ e^T u &= 1, \\ (u, v) &\geq 0, \lambda \text{ free.} \end{aligned} \tag{5}$$

The first two equations in (5) imply that $\lambda = \frac{2(u^*)^T A u^*}{e^T u^*} = 2(u^*)^T A u^* > 0$. Thus, $\bar{u} = \frac{2u^*}{\lambda} \geq 0$ is complementary solution of LCP (1) and $\bar{u} \neq 0$. ∎

The question remains: given symmetric A, is it easy to compute one if LCP (1) is known to have a complementary solution? Note that, the complementary solution set of (1), even non-empty, is not convex nor even connected. For example, let

$$A^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then, there are three isolated non-trivial complementary solutions.

$$u^1 = (1/2;\ 0), \quad u^2 = (0;\ 1/2), \quad u^3 = (1/3;\ 1/3).$$

In the next section, however, we develop a fully polynomial-time approximation scheme (FPTAS) to compute ϵ-approximate complementary solution for LCP (1) when A is symmetric and $\sum_{i,j} a_{ij} > 0$, that is, the sum of all entries of A is positive. Here, an ϵ-approximate complementary solution is a pair $(u \neq 0, v)$ such that

$$A^T u + v = e,\ (u \neq 0, v) \geq 0,\ \frac{u^T v}{\bar{a}} \leq \epsilon,$$

where $\bar{a}$ is the largest entry in A:

$$\bar{a} = \max_{i,j}\{a_{ij}\}\ (> 0). \tag{6}$$

In most applications, one can scale A such that $\bar{a} = 1$.

4 A Social Optimization and FPTAS

We consider a quadratic "social" utility function $u^T A u$, which we like to maximize over the simplex $\{u :\ e^T u = 1,\ u \geq 0\}$. This can be written as the quadratic programming problem of QP (4) in the previous section.

Since $e^T A e > 0$ so that LCP (1) has at least one non-trivial complementary solution. Further more, the maximal value of QP (4) is strictly greater than 0 but bounded above by $\bar{a}$ (recall that $\bar{a}$ is the largest entry of A). These facts, together with the proof of Theorem 3, lead to

Lemma 1 *Let A be symmetric. Then, every KKT point u of problem (4), with $u^T A u > 0$, is a (non-trivial) complementary solution for LCP (1).*

In [14], an interior-point potential reduction algorithm was proved to be a FPTAS for computing an ϵ-approximate KKT point of general quadratic programming with bounded feasible region. It can be adapted in solving QP (4) in a running time bounded by $\mathcal{O}((\frac{n^4}{\epsilon}\log\frac{1}{\epsilon}+n^4\log n)(\log\frac{1}{\epsilon}+\log n))$ arithmetic operations. The algorithm reduces the potential function

$$P(u)=\rho\log\left(\bar{a}-u^TAu\right)-\sum_{j=1}^{n}\log(u_j),$$

where $\rho=(2n+\sqrt{n})/\epsilon$, by a constant each iteration from the initial point $u^0=\frac{1}{n}e$, till u becomes an ϵ-approximate KKT point.

Note that

$$P(u^0)=\rho\log\left(\bar{a}-\frac{1}{n^2}e^TAe\right)+n\log(n),$$

and for any $u\in\{u:\ e^Tu=1,\ u>0\}$,

$$-\sum_{j=1}^{n}\log(u_j)\geq n\log(n).$$

Thus, $P(u)<P(u^0)$ implies that

$$\rho\log\left(\bar{a}-u^TAu\right)<\rho\log\left(\bar{a}-\frac{1}{n^2}e^TAe\right)$$

or

$$u^TAu>\frac{1}{n^2}e^TAe>0,$$

that is, any KKT point u generated by the algorithm must have $u^TAu>0$. To conclude, we have

Theorem 4 *There is a FPTAS to compute an ϵ-approximate non-trivial complementary solution of LCP (1) when A is symmetric and $e^TAe>0$. Moreover, such a solution is an ϵ-approximate equilibrium of the symmetric Leontief economy when all entries of A are positive.*

5 Preliminary Computational Results

Here, we computationally compare three type methods to solve the complementarity problem of (1): 1) the QP-based potential reduction algorithm (referred as QP) presented in this paper; 2) a homotopy-based path-following algorithm method (referred as HOMOTOPY) developed in Dang at al. [6]; 3) Mixed Complementarity Problem (MCP) general solvers PATH (Ferris and Munson, http://www.gams.com/dd/docs/solvers/path.pdf) and MILES (Rutherford http://www.gams.com/dd/docs/solvers/miles.pdf), where both solvers use a Lemke type algorithm that is based on a sequence of pivots similar to those generated by the simplex method for linear programming; see Lemke [9].

If one applies Lemke's algorithm directly to solving LCP (1), then it will return the trivial solution $u = 0,\ v = e$. To exclude it, we rewrite LCP (1) into an equivalent homogeneous LCP as follows:

$$Mz + q = w,\ z^T w = 0,\ (z, w) \geq 0, \tag{7}$$

where $z, w \in R^{n+1}$,

$$M = \begin{pmatrix} -A^T & e \\ e^T & 0 \end{pmatrix} \in M^{n+1}, q = \begin{pmatrix} 0_n \\ -1 \end{pmatrix}.$$

Then, we can obtain a solution for LCP (1) from a complementary solution of LCP (7). However, the standard Lemke algorithm may not be able to solve LCP (7) either, since it may terminate at the second iteration with a non-complementary "secondary-ray" solution. Thus, as shown below, commonly used LCP solver PATH or MILES seems cannot successfully solve LCPs (7) most of times.

Both QP and HOMOTOPY are coded in MATLAB script files, and all solvers are run in the MATLAB environment on a desktop PC (2.8GHz CPU). For the QP-based potential reduction algorithm, we set $\epsilon = 1.e - 8$. After the termination, we use the support of u, $\{i :\ u_i \geq 1.e - 5\}$, to recalibrate an "exact" solution (to the machine accuracy) for LCP (1).

For different size n ($n = 20 : 20 : 100, 100 : 100 : 1000, 1500 : 500 : 3000$), we randomly generate 15 symmetric and sparse matrices A of two different types (uniform in $[0, 1]$ or binary $\{0, 1\}$) and solve them by the three methods. In the following tables, "mean_sup" the average support size of u and "max_sup" the maximum support size of u in the 15 problems, "mean_iter" the average number of iterations of QP and Homotopy algorithms (each iteration solves a system of linear equations), and "mean_time" the average computing CPU time in seconds.

From our preliminary computational results, we can draw few conclusions. First, LCP (1), although the matrix A is symmetric, seems not an easy problem to solve. Secondly, the QP-based FPTAS algorithm lives up with its theoretical expectation and it is numerically effective. Thirdly, the homotopy-based algorithm seems able to solve sizable problems, although its computational complexity is not proven to be a PTAS. Finally, as mentioned earlier, the general LCP solvers, PATH and MILES, may terminate with a "secondary-ray" solution at the second Lemke pivot, therefore fail to solve LCP (7). As a result, in our numerical experiments MILES can solve none of our test problems, and PATH can only solve a small number of test problems with size no more than 50. (PATH use an alternative default pivoting rule

n	mean_sup	mean_iter	mean_time	max_sup
20	4.1	39.5	0.1	5
40	4.5	46.0	0.1	5
60	4.5	47.9	0.1	5
80	4.9	47.5	0.2	6
100	5.3	48.2	0.3	7
200	5.5	53.5	1.2	6
300	5.6	59.3	3.4	8
400	5.7	55.1	5.9	7
500	5.9	62.5	11.3	7
600	5.7	58.8	16.0	7
700	5.8	58.8	23.4	7
800	5.8	62.6	33.8	8
900	5.7	65.1	47.3	7
1000	6.3	65.0	60.2	7
1500	6.1	71.5	187.2	8
2000	5.9	73.5	411.9	7
2500	6.4	74.6	774.5	8
3000	6.2	78.7	1404.2	8

Table 1: QP for solving uniform symmetric matrix LCP

n	mean_sup	mean_iter	mean_time	max_sup
20	4.1	37.7	0.2	5
40	4.4	52.7	0.4	5
60	4.4	58.3	0.8	6
80	4.6	68.2	1.4	6
100	5.3	72.6	2.2	7
200	4.9	108.9	14.0	6
300	5.5	127.7	49.3	8
400	5.5	160.5	111.9	7
500	5.7	159.7	181.6	7
600	5.5	182.5	317.0	6
700	5.9	202.9	515.6	7
800	5.5	208.9	706.3	6
900	5.7	231.7	1039.2	7
1000	5.9	267.2	1644.0	7
1500	5.9	305.5	4726.4	7
2000	5.7	307.1	10105.2	6

Table 2: HOMOTOPY for solving uniform symmetric matrix LCP

n	mean_sup	mean_time	max_sup
20	8.7	0.1004	12
40	13.8	0.3406	23
n≥60		fail to solve	

Table 3: PATH for solving uniform symmetric matrix LCP

n	mean_sup	mean_iter	mean_time	max_sup
20	11.8	35.2	0.1	13
40	16.6	43.3	0.1	20
60	21.1	44.4	0.2	23
80	22.1	46.9	0.3	25
100	23.9	53.3	0.5	27
200	30.0	54.5	1.7	34
300	32.5	66.9	5.2	35
400	34.1	65.1	9.5	38
500	35.4	67.1	16.1	39
600	36.0	82.9	31.4	39
700	37.9	68.0	35.4	42
800	37.8	74.9	55.4	41
900	37.8	78.1	76.5	43
1000	38.7	82.1	106.6	42
1500	40.0	84.9	305.3	43
2000	42.4	91.4	702.2	45
2500	42.9	94.7	1382.8	47
3000	43.9	99.5	1959.4	48

Table 4: QP for solving binary symmetric matrix LCP

n	mean_sup	mean_iter	mean_time	max_sup
20	11.7	48.6	0.2	14
40	16.2	68.3	0.5	21
60	20.6	75.3	0.9	24
80	22.9	84.0	1.7	26
100	24.3	92.9	2.9	27
200	31.3	111.1	14.6	39
300	32.3	130.4	51.1	39
400	32.4	108.2	79.9	34
500	34.8	153.6	263.7	41
600	34.4	144.8	451.3	37
700	35.6	184.0	572.3	38
800	36.5	208.0	1628.1	37
900	37.2	261.2	4733.4	41
1000	37.2	502.8	5370.1	38

Table 5: HOMOTOPY for solving binary symmetric matrix LCP

n	mean_sup	mean_time	max_sup
20	8.2	0.0445	12
40	10.2	0.3229	17
n≥60		fail to solve	

Table 6: PATH for solving binary symmetric matrix LCP

and it switches to original Lemke's pivot rule only when the default rule fails or the users force to do so.)

6 Further Remarks

We make few final remarks and open questions.

First, is symmetric LCP (1) in the PPAD class described by [3] and [7]?

Secondly, by restricting A being symmetric for bimatrix game setting described in Section 2, we must have $R = C$, that is, the two payoff matrices are identical. But in this case, a trivial, pure-strategic, and Pareto-optimal bimatrix game equilibrium is to simply play the largest entry in C. Thus, it remains to be seen if the QP-based approach offer a PTAS for computing a bimatrix equilibrium with a larger support. Note that the constant-approximation result of Tsaknakis and Spirakis [11] was indeed based on computing a KKT point of a social QP problem.

Thirdly, an important direction is to study the LCP problem (1) where A is not necessarily symmetric. In this case, even all entries of A being nonnegative may not guarantee the existence of a (non-trivial) complementary solution; see example:

$$A^T = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}.$$

Finally, the computational results based on randomly generated data show that the support of u is relative small. Is there a theoretical justification for this fact or observation?

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