Online Learning via Linear Programming

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(Joint work with many...)

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Offline and Online Linear Programming

$$
\begin{array}{ll}
\text{maximize}_{\mathbf{x}} & \sum_{t=1}^{n} r_t x_t \\
\text{subject to} & \sum_{t=1}^{n} \mathbf{a}_t x_t \leq \mathbf{b}, \\
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r_t: reward/revenue offered by the t-th customer/order

$\mathbf{a}_t \in R^m$: the bundle of resources requested by the t-th order

x_t: acceptance or rejection decision to the t-th order

$\mathbf{b} \in R^m$: initially available budget/resource amounts

The objective $\sum_{t=1}^{n} r_t x_t$: the total collected revenue.

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- an irrevocable decision must be made as soon as an order arrives (without knowing the future data).
- Conform to resource capacity constraints at the end.

Primal and Dual Offline LPs

$$
P: \begin{array}{ll} \max & \mathbf{r}^\top \mathbf{x} \\ \text{s.t.} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{0} \le \mathbf{x} \le \mathbf{e} \end{array}
\qquad
D: \begin{array}{ll} \min & \mathbf{b}^\top \mathbf{p} + \mathbf{e}^\top \mathbf{s} \\ \text{s.t.} & A^\top \mathbf{p} + \mathbf{s} \ge \mathbf{r} \\ & \mathbf{p} \ge \mathbf{0}, \mathbf{s} \ge \mathbf{0} \end{array}
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where the decision variables are $\mathbf{x} \in R^n$, $\mathbf{p} \in R^m$, $\mathbf{s} \in R^n$ ($\mathbf{e}$ vector of all ones).

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Denote the primal/dual optimal solution as $\mathbf{x}^*$, $\mathbf{p}^*$, $\mathbf{s}^*$, then LP duality/complementarity theory tells that for $t = 1, ..., n$,

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x_t^* = \begin{cases} 1, & r_t > \mathbf{a}_t^\top \mathbf{p}^* \\ 0, & r_t < \mathbf{a}_t^\top \mathbf{p}^* \end{cases}
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(x_t^* may take non-integer value when $r_t = \mathbf{a}_t^\top \mathbf{p}^*$).

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(x_t^* may take non-integer value when $r_t = \mathbf{a}_t^\top \mathbf{p}^*$).

Most online LP algorithms are based on learning $\mathbf{p}^*$ by dynamically solving small sample-sized LPs based on revealed data.

Data/Model Assumptions for Analyses

Stochastic Input (i.i.d) Model:

(a) $(r_t, \mathbf{a}_t)$'s are i.i.d. from an unknown distribution

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Both assume boundedness:

(b) $|r_t| \leq \bar{r}$ and $\|\mathbf{a}_t\|_\infty \leq \bar{a}$ for all t

(c) The right-hand-side $\mathbf{b} = n \cdot \mathbf{d}(> \mathbf{0})$.

All early works also assume $r_t \geq 0, \mathbf{a}_t \geq 0$ (one-sited market).

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- What are the necessary and sufficient assumptions on the right-hand-side $\mathbf{b}$ to achieve $(1-\epsilon)$-competitive ratio of the expected online reward over the optimal offline reword?

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- What are the necessary and sufficient assumptions on the right-hand-side $\mathbf{b}$ to achieve $(1-\epsilon)$-competitive ratio of the expected online reward over the optimal offline reword?
- If the right-hand-side $\mathbf{b}$ (such as $\mathbf{b} = O(n)$), what is the best achievable gap or regret between the two?

Competitive Ratio Summary of One-Sited Market

The journey to design $(1-\epsilon)$-competitive online algorithms against benchmark OPT-Optimal Offline Objective Value where $B = \min_i b_i$:

	Sufficient Condition
Kleinberg (2005)	$B \geq \frac{1}{\epsilon^2}$, for $m = 1$
Devanur et al (2009)	$OPT \geq \frac{m^2 \log n}{\epsilon^3}$
Feldman et al (2010)	$B \geq \frac{m \log n}{\epsilon^3}$ and $OPT \geq \frac{m \log n}{\epsilon}$
Agrawal/Wang/Y (2010,14)	$B \geq \frac{m \log n}{\epsilon^2}$ or $OPT \geq \frac{m^2 \log n}{\epsilon^2}$
Molinaro/Ravi (2013)	$B \geq \frac{m^2 \log m}{\epsilon^2}$
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	Necessary Condition
Agrawal/Wang/Y (2010,14)	$B \geq \frac{\log m}{\epsilon^2}$

Remarks

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- The key difference between OLP and Online Convex Optimization with Constraints (OCOwC):
  - Online LP problem employs a stronger benchmark where the decision variables are allowed to take different values at each time period
  - OCOwC (Mahdavi et al., 2012; Yu et al., 2017; Yuan and Lamperski, 2018) and OCO problems usually considers a stationary benchmark where the the decision variables are required to be the same at each time period.

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- Recent focuses are on dealing with two-sited markets/platforms, regret analyses, simple and fast algorithms, interior-point online algorithm, extension to bandit models, ...

Today's Talk: Recent Developments

- Part (I): Fast algorithms for online linear programming
  - Setup: First observe $(r_t, \mathbf{a}_t)$ then decide x_t
- Part (II): A Fairer online interior-point LP algorithm
  - Setup: A "fair" online decision-making mechanism
- Part (III): Bandits with knapsacks
  - Setup: First choose "x_t" (the arm/customer), then observe $(r_t, \mathbf{a}_t)$

Other recent works on OLP: papers by Balseiro, Lu, and Mirrokni (2020,21), etc.

Regret Analysis and Model

Let "offline" optimal solution be $\mathbf{x}^*$ and "online" solution of n orders be $\mathbf{x}_n$, and

$$R_n^* = \sum_{j=1}^{n} r_j x_j^*, \quad R_n = \sum_{j=1}^{n} r_j x_j.$$

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Then define

$$\Delta_n = \sup \mathbb{E}\left[R_n^* - R_n\right], \quad v(\mathbf{x}) = \sup \mathbb{E}\left[\|\left(A\mathbf{x} - \mathbf{b}\right)^+\|_2\right]$$

where the expectation is taken with respect to i.i.d distribution or random permutation, and the sup operator is over all permissible distributions and admissible data.

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Remark: A bi-criteria performance measure, but one can easily modify the algorithms such that the constraints are always satisfied at the end.

Part (I): Equivalent Form of the Dual Problem

Recall the dual problem

$$\min \ \mathbf{b}^\top \mathbf{p} + \sum_{t=1}^{n} s_t \quad \text{s.t. } s_t \geq r_t - \mathbf{a}_t^\top \mathbf{p}, \forall t; \quad \mathbf{p}, \mathbf{s} \geq \mathbf{0}$$

can be rewritten as

$$\min \ \mathbf{b}^\top \mathbf{p} + \sum_{t=1}^{n} \left(r_t - \mathbf{a}_t^\top \mathbf{p} \right)^+ \quad \text{s.t. } \ \mathbf{p} \geq \mathbf{0}$$

where $(\cdot)^+$ is the positive-part or ReLU function.

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After normalizing the objective, it becomes

$$\min_{\mathbf{p} \geq \mathbf{0}} \mathbf{d}^\top \mathbf{p} + \frac{1}{n} \sum_{t=1}^{n} \left(r_t - \mathbf{a}_t^\top \mathbf{p} \right)^+$$

which can be viewed as a simple-sample-average (SSA) (with n sample points) of a stochastic optimization problem under an i.i.d distribution.

Convergence of $\mathbf{p}_n^*$

Theorem (Li & Y (2019))

Denote the n-sample SSA optimal solution by $\mathbf{p}_n^*$. *Then, for the stochastic input model under moderate conditions that guarantees a local strong convexity of the underlying stochastic program* $f(p)$ *around its optimal solution* $\mathbf{p}^*$, *there exists a constant* C *such that*

$$\mathbb{E}\|\mathbf{p}_n^* - \mathbf{p}^*\|_2^2 \leq \frac{Cm \log \log n}{n}$$

holds for all $n > m$.

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holds for all $n > m$.

This is L_2 convergence for the dual optimal solution. Heuristically,

$$\mathbf{p}_n^* \approx \mathbf{p}^* + \frac{1}{\sqrt{n}} \cdot \textbf{Noise}$$

Fast Online Algorithm for Online and Binary LP

Input: $d = \mathbf{b}/n$

Initialize $\mathbf{p}_1 = \mathbf{0}$

For $t = 1, 2, ..., n$

$$x_t = \begin{cases} 1, & \text{if } r_t > \mathbf{a}_t^\top \mathbf{p}_t \\ 0, & \text{if } r_t \le \mathbf{a}_t^\top \mathbf{p}_t \end{cases}$$

Compute

$$\mathbf{p}_{t+1} = \mathbf{p}_t + \gamma_t \left(\mathbf{a}_t x_t - \mathbf{d} \right)$$
$$\mathbf{p}_{t+1} = \mathbf{p}_{t+1} \vee \mathbf{0}$$

$\mathbf{x} = (x_1, ..., x_n)$

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5: Compute

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6: $\mathbf{x} = (x_1, ..., x_n)$

Line 5 performs (projected) stochastic gradient descent in the dual.

Performance Analysis

Theorem (Li, Sun & Y (2020))

With step size $\gamma_t = 1/\sqrt{n}$, the regret and expected constraint violation of the algorithm satisfy

$$\mathbb{E}[R_n^* - R_n] \leq \tilde{O}(m\sqrt{n}), \quad \mathbb{E}\left[v(\mathbf{x})\right] \leq \tilde{O}(m\sqrt{n}).$$

under both the stochastic input and the random permutation models.

- $\tilde{O}$ omits the logarithm terms and the constants related to $(\bar{a}, \bar{r})$, but the algorithm does not require any prior knowledge on the constants.
- The optimal offline value is in the range $O(mn)$.
- The algorithms runs in *nm* times - the time to read in the data.

Fast Online LP Algorithm for Solving Offline LPs?

A crucial assumption is that the right-hand-side $\mathbf{b} = n\mathbf{d}$ scales linearly with n. Is there a remedy for this case where we do not want to compromise the computational efficiency of simple online algorithm?

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A crucial assumption is that the right-hand-side $\mathbf{b} = n\mathbf{d}$ scales linearly with n. Is there a remedy for this case where we do not want to compromise the computational efficiency of simple online algorithm?

Consider a "Replicated" LP from the original LP

$$\begin{aligned} \max \quad & \sum_{t=1}^{n} \sum_{h=1}^{k} r_t x_{th} \\ \text{s.t.} \quad & \sum_{t=1}^{n} \sum_{h=1}^{k} \mathbf{a}_t x_{th} \leq k\mathbf{b}, \;\; 0 \leq x_t \leq 1, \quad t = 1, \ldots, n. \end{aligned}$$

Algorithm: Solve the new LP with Simple Online Algorithm and use $x_t = \frac{1}{k}(x_{t1} + \ldots + x_{tk})$ as the solution to the original LP.

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The algorithm runs in $O(kmn)$ times.

Performance of the Variable-Replicating Algorithm

Proposition (Gao, Sun, Ye & Y (2020))

Under the random permutation model, the variable-replicating algorithm finds a solution for the original LP that achieves at least $(1-\mathcal{O}(\varepsilon))OPT$ *with the constraint violation bounded by* $(1+\mathcal{O}(\varepsilon))B$ *where* $B = \min_{i=1,\ldots,m} b_i$*, if* $\sqrt{k}B^2 \geq \frac{n^{3/2}\log kn}{\varepsilon}$ *and* $\sqrt{k}B \geq \frac{m\sqrt{n}}{\varepsilon}$ *for any* $\varepsilon > 0$*. Moreover, if* $kn \geq m$*, the relative constraint violation can be bounded by* $(1+\mathcal{O}(\frac{\varepsilon}{\sqrt{m}}))$*.*

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Takeaway: k times more computation cost for a $\sqrt{k}$ factor improvement in regret performance.

Multi-knapsack Problem Instances - Binary LP

Benchmark dataset of Chu & Beasley implementation

		V.R. Alg.	Gurobi
$m = 5, n = 500, k = 50$	Time	0.000	0.211
	Cmpt. Ratio	88.2%	95.3%
$m = 5, n = 500, k = 1000$	Time	0.007	0.211
	Cmpt. Ratio	89.2%	95.3%
$m = 8, n = 10^3, k = 50$	Time	0.004	3.800
	Cmpt. Ratio	89.9%	99.0%
$m = 8, n = 10^3, k = 1000$	Time	0.077	3.800
	Cmpt. Ratio	95.6%	99.0%
$m = 64, n = 10^4, k = 50$	Time	0.013	> 60
	Cmpt. Ratio	90.3%	98.7%
$m = 64, n = 10^4, k = 1000$	Time	0.223	> 60
	Cmpt. Ratio	96.4%	98.7%

Fast Online Algorithm as Pre-Classifier for LP

The key combinatorial task of LP is the partition of all variables into optimal basic (with positive values) and optimal nonbasic (with zero values) variables.

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In LP, a column generation techniques is popularly used when $n >> m$:

- Constructed a Restricted Master Problem (RMP) defined by a small subset of variables of the original problem
- Solve RMP and reselect initially unselected variables into RMP

Ideally, the initial RMP would already contain the set of $O(m)$ optimal basic variables and there is no need (or less) to do reselect!

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- Constructed a Restricted Master Problem (RMP) defined by a small subset of variables of the original problem
- Solve RMP and reselect initially unselected variables into RMP

Ideally, the initial RMP would already contain the set of $O(m)$ optimal basic variables and there is no need (or less) to do reselect!

This is precisely where the fast online LP algorithm does a good job - classify variables being positive or zero at an optimal solution in a short time.

Implementation in LP Solvers

More precisely, the fast online LP solution can be interpreted as a "score" of how likely a variable is to be optimal basic.

We run online algorithm to obtain $\hat{\mathbf{x}}$, set a threshold ε and select the columns in $\mathbb{I}_{\{\hat{\mathbf{x}}>\varepsilon\}}$. For benchmark LP problems that have more columns than rows (such as **rail4284**, **s82**, and **scpm1** from the Mittelmann's Simplex Benchmark), the online solution identifies more than 90% of the primal optimal basis on average.

This technique has been adopted in the emerging LP solver COPT - a new state of art LP solver.

Part (II): A "Fairer" Online LP Algorithm

Recall the online LP formulation (changing n to T as in the literature)

$$\max \sum_{t=1}^{T} r_t x_t \quad \text{s.t.} \quad \sum_{t=1}^{T} \mathbf{a}_t x_t \leq \mathbf{b}, \quad x_t \in [0,1]$$

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A finite-type assumption: $\mathbb{P}((r_t, \mathbf{a}_t) = (\mu_j, \mathbf{c}_j)) = p_j$ (unknown to the decision maker) for $j = 1, ..., J$. The offline problem with the knowledge:

$$\max \sum_{j=1}^{J} p_j \mu_j y_j \quad \text{s.t.} \quad \sum_{j=1}^{J} p_j \mathbf{c}_j y_j \leq \mathbf{b}/T, \quad y_j \in [0,1]$$

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	Benchmark	Regret Bound	Key Assumption(s)
Jasin and Kumar (2012)	Fluid	Bounded	Nondeg., distrib. known
Jasin (2015)	Fluid	$\tilde{O}(\log T)$	Nondeg.
Vera et al. (2019)	Hindsight	Bounded	Distrib. known
Bumpensanti and Wang (2020)	Hindsight	Bounded	Distrib. known
Asadpour et al. (2019)	Full flex.	Bounded	Long-chain, ξ-Hall condition
Chen, Li & Y (2021)	Fluid	Bounded	Partial Nondeg.

Behavior of the Simplex and Interior-Point

The key in Chen et al. (2021) paper is to use the interior-point algorithm for solving the sample LPs with sample proportion $\hat{p}_j$

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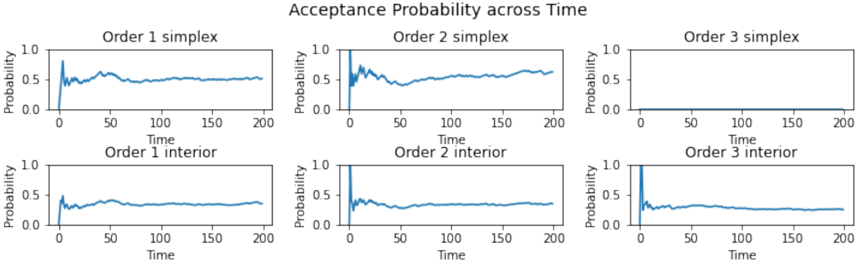

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But these individuals/groups could have different sensitive features, such as demographic, race, and gender, and areas in Hospital Admission and Hotel/Flight booking application.

Could we design an online algorithm/allocation-rule such as, while maintain the efficiency in objective value, all individual/groups get a fairer allocation shares?

Fairer Solution for the Offline Problem

We define $\mathbf{y}^*$, the fair offline optimal solution of the LP problem

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Let $\mathbf{y}_t$ be allocation rule at time t which encodes the accepting probabilities under algorithm π. Then we define the cumulative unfairness of the online algorithm π as

$$\mathrm{UF}_T(\pi) = \mathbb{E}\left[\sum_{t=1}^{T} \|\mathbf{y}_t - \mathbf{y}^*\|_2^2\right].$$

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This definition is consistent with the definition of fair classifiers/regressors in fair machine learning.

Our Result

We develop an algorithm [Chen, Li & Y (2021)] that achieves

$$\mathrm{UF}_T(\pi) = O(\log T)$$

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Key ideas in algorithm design:

- At each time t, we use interior-point method to obtain the sample analytic-center solution $\mathbf{y}_t$, and it is necessary to achieve the performance under weak non-degeneracy assumption and maintain fairness.
- We also adjust the right-hand-side properly to ensure (i) the depletion of binding resources and (ii) non-binding resources not affecting the fairness.

The use of interior-point method also relaxes a non-degeneracy assumption in previous analysis

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At each time $t \in [T]$, an arm i is selected to pull. The realized reward $\hat{r}_t$ and resources cost $\hat{\mathbf{c}}_t$ satisfying

$$\mathbb{E}[\hat{r}_t | i] = \mu_i, \quad \mathbb{E}[\hat{\mathbf{c}}_t | i] = \mathbf{c}_i.$$

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Goal: Select a subset of winning/optimal arms to maximize the total reward subject to the resource capacity constraints!

Offline Linear Program (LP) and Regret

With mean reward $\boldsymbol{\mu} = (\mu_1, ..., \mu_k)$ and mean cost $C = (\mathbf{c}_1, ..., \mathbf{c}_k)$ of all arms, consider the following deterministic offline LP,

$$\max_{\mathbf{x}} \sum_{i=1}^{k} \mu_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{k} \mathbf{c}_i x_i \leq \mathbf{b}, x_i \geq \mathbf{0}, i \in [k]$$

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Denote its optimal value as OPT (the benchmark) and let τ be the stopping time as soon as one of the resources is depleted. Then the problem-dependent regret

$$Regret(\mathcal{P}) = OPT - \mathbb{E}\left[\sum_{t=1}^{\tau} r_t\right],$$

where $\mathcal{P}$ encapsulates the parameters related to the underlying data distribution.

Literature and Our Result

	Paper	Result
$\mathcal{P}$-Independent	Badanidiyuru et. al. (13) Agrawal and Devanur (14)	$O(poly(m,k)\cdot\sqrt{T})$
$\mathcal{P}$-Dependent	Flajolet and Jaillet (15) Sankararaman and Slivkins (20) Li, Sun & Y (21)	$\tilde{O}(2^{m+k}\log T)$ $\tilde{O}(k\log T)$ for $m=1$ $\tilde{O}\left(m^4+k\log T\right)$

The problem-dependent bounds all involve parameters related to the non-degeneracy and the reduced cost of the underlying LP, while our work has the mildest assumption and requires no prior knowledge of these parameters.

Dual LP and Reduced Cost

$$\textit{Primal}: \max \ \boldsymbol{\mu}^\top \mathbf{x} \quad \text{s.t.} \ C\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0} \qquad \textit{Dual}: \min \ \mathbf{b}^\top \mathbf{y} \quad \text{s.t.} \ C^\top \mathbf{y} \ge \boldsymbol{\mu}, \mathbf{y} \ge \mathbf{0}$$

Denote $\mathbf{x}^* \in R^k$ and $\mathbf{y}^* \in R^m$ as optimal solutions

Define reduced cost (profit) for i-th arm $\Delta_i := \mathbf{c}_i^\top \mathbf{y}^* - \mu_i$ and the non-basic variable set $\mathcal{I}' = \{i : \Delta_i > 0\}$.

Proposition (Li, Sun & Y (2021))

The regret of a BwK algorithm has the following upper bound:

$$Regret(\mathcal{P}) \le \sum_{i \in \mathcal{I}'} \Delta_i \mathbb{E}[n_i(\tau)] + \mathbb{E}[\mathbf{b}^{(\tau)}]^\top \mathbf{y}^*$$

- $\mathbf{b}^{(t)}$: remaining resource at time t
- $n_i(t)$: the number of times that i-th (non-optimal) arm is played up to time t

Implications of the Regret Upper Bound

Two tasks to accomplish to reduce the regret:

Task I: Control the number of plays $n_i(\tau)$ for non-optimal arms $i \in \mathcal{I}'$ which corresponds to the first component in the regret bound

$$\sum_{i \in \mathcal{I}'} \Delta_i \mathbb{E}[n_i(\tau)]$$

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Task II is often overlooked in the existing BwK literature.

Our Approach: A Two-Phase Algorithm

- Phase I: Identify the optimal arms with as fewer number of plays as possible by designing an "importance score" for arm i:

$$\begin{aligned} OPT_i := \max \quad & \boldsymbol{\mu}^\top \mathbf{x} \\ \text{s.t.} \quad & C\mathbf{x} \leq \mathbf{b},\ x_i = 0, \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

  Implication: A larger value of $OPT - OPT_i \Rightarrow x_i$ important and likely to represent an optimal arm. Our algorithm then maintains upper confidence bound (UCB)/lower confidence bound (LCB) to estimate OPT and OPT_i based are samples.

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  After $t' = O(\frac{k \log T}{\sigma^2 \delta^2})$ times of Phase I, the non-optimal arm variables are identified as set $\mathcal{I}'$ and they would be removed from further consideration, and then we start

- Phase II: Use the remaining arms to exhaust the resource through an adaptive procedure such that no valuable resources are wasted.

Phase II: Exhausting the Binding Resources

Adaptive Algorithm for filling the knapsacks:

For $t = t' + 1, \ldots, T$

1. Solve the UCB-LP and denote its optimal solution as $\tilde{\mathbf{x}}$

$$\begin{aligned}
\max_{\mathbf{x}} \quad & \sum_{i=1}^{k} \left(\hat{\mu}_i(t) + \sqrt{\frac{2\log T}{n_i(t)}} \right) x_i \\
\text{s.t.} \quad & \sum_{i=1}^{k} \left(\hat{\mathbf{c}}_i(t) - \sqrt{\frac{2\log T}{n_i(t)}} \right) x_i \leq \mathbf{b}^{(t-1)} \\
& \mathbf{x} \geq \mathbf{0}, x_i = 0 \text{ for } i \in \mathcal{I}'
\end{aligned}$$

2. Normalize $\tilde{\mathbf{x}}$ into a probability and play an arm accordingly
3. Update the knapsack process $\mathbf{b}^{(t)}$ (remaining resource)

Combining the Two Phases

Proposition (Li, Sun & Ye 2021)

The regret of our two-phase algorithm is bounded by

$$O\left(\frac{m^4}{\sigma^2\delta^2} + \frac{k\log T}{\delta^2}\right).$$

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These condition numbers generalize the optimality gap for the original (unconstrained) multi-armed bandits (Lai and Robbins (1985), Auer et al. (2002)).

Summary

LP continues to play an important and significant role in today's online learning and decision-making!

Thank You