NP-Hardness Results Related to PPAD*

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Abstract

Let $P = \{x \in R^n \mid Ax \leq b\}$ be a polytope satisfying that each row of A has at most one positive entry. The problem we consider is to determine whether there is an integer point in P, which is known to be an NP-complete problem. Applying an integer labeling rule and a triangulation of an augmented integral set in R^{n+1}, we show

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in this paper that determining whether there is an integer point in P can be reduced, in polynomial time, to certain decision problems related to PPAD (polynomial parity argument for directed graphs). Consequently, we prove that these decision problems are all NP-hard.

Key Words: Integer Point, Polytope, Integer Programming, Integer Labeling, Triangulation, Pivoting Procedure, Simplicial Method, PPAD, NP-hard.

1 Introduction

The problem we consider is as follows: Determine whether there is an integer point in a polytope given by $P = \{x \in R^n \mid Ax \leq b\}$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

satisfies that each row of A has at most one positive entry and $b = (b_1, b_2, \ldots, b_m)^\top$. It has been shown:

Theorem 1 (Lagarias, 1985)**.** *Determining whether there is an integer point in P is an NP-complete problem.*

Applying an integer labeling rule and a triangulation of an augmented integral set in R^{n+1}, we show in this paper that determining whether there is an integer point in P can be reduced, in polynomial time, to certain decision problems related to PPAD. Consequently, we prove these decision problems are all NP-hard. The idea of this paper is stimulated from the work in Dang (2009), Dang and Maaren (1998, 1999, 2001) and Dang and Ye (2009) and has its foundations in simplicial methods for computing fixed points of a continuous mapping that were originated in Scarf (1967) and substantially developed in the literature (e.g., Allgower and Georg, 2000; Dang, 1991, 1995; Eaves, 1972; Eaves and Saigal, 1972; Kojima and Yamamoto, 1984; Kuhn, 1968; van der Laan and Talman, 1979, 1981; Merrill, 1972; Scarf, 1973; Todd, 1976; Wright, 1981).

The rest of this paper is organized as follows. In Section 2, we introduce an integer labeling rule and a triangulation, and analyze their properties and structures. In Section 3, we show that the problem can be reduced in polynomial time to certain decision problems related to PPAD, and then draw our main conclusions.

2 Integer Labeling Rule and Triangulation

Let $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$, and $N_0 = \{1, 2, \ldots, n+1\}$. For $i \in M$, let $a_i^\top$ denote the ith row of A. Thus, $A = (a_1, a_2, \ldots, a_m)^\top$. Let $e = (1, 1, \ldots, 1)^\top \in R^n$. Without loss of generality, we assume throughout this paper that P is bounded and full dimensional. As a result of the property of A, one can easily obtain that, for any $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)^\top \in P$ and $x^2 = (x_1^2, x_2^2, \ldots, x_n^2)^\top \in P$,

$$\bar{x} = \max(x^1, x^2) = (\max\{x_1^1, x_1^2\}, \max\{x_2^1, x_2^2\}, \ldots, \max\{x_n^1, x_n^2\})^\top \in P.$$

This implies that $\max_{x \in P} e^\top x$ has a unique solution, which is denoted by $x^{\max} = (x_1^{\max}, x_2^{\max}, \dots, x_n^{\max})^\top$. Let $x^{\min} = (x_1^{\min}, x_2^{\min}, \dots, x_n^{\min})^\top$, where $x_j^{\min} = \min_{x \in P} x_j$, $j = 1, 2, \dots, n$. Clearly, $x^{\min} \le x \le x^{\max}$ for all $x \in P$.

For any real number α, let $\lfloor \alpha \rfloor$ denote the greatest integer less than or equal to α and $\lceil \alpha \rceil$ the smallest integer greater than or equal to α. Let

$$D(P) = \{x \in R^n \mid x^l \le x \le x^u\},$$

where $x^u = \lfloor x^{\max} \rfloor = (\lfloor x_1^{\max} \rfloor, \lfloor x_2^{\max} \rfloor, \dots, \lfloor x_n^{\max} \rfloor)^\top$ and $x^l = \lceil x^{\min} \rceil = (\lceil x_1^{\min} \rceil, \lceil x_2^{\min} \rceil, \dots, \lceil x_n^{\min} \rceil)^\top$. Thus, $x \in D(P)$ for all integer points $x \in P$ since $x^{\min} \le x \le x^{\max}$ for all $x \in P$. Without loss of generality, we assume that $x^l \le x^u$. The sizes of both x^l and x^u are bounded by polynomials of the size of A and b if they are rational, since x^l and x^u are obtained from the solutions of linear programs with rational data A and b.

For $x \in R^n$, let

$$f(x) = \begin{cases} 0 \in R^n & \text{if } x \in P, \\ \sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} a_i & \text{if } x \notin P, \end{cases}$$

where $I(x) = \{i \in M \mid a_i^\top x - b_i > 0\}$. Then, we have

Lemma 1. *For any $x \in R^n$, $f(x) = 0$ if and only if $x \in P$.*

Proof. We only need prove the "only" part. Suppose that there is some $x \in R^n$ with $f(x) = 0$ and $x \notin P$. Then, $I(x) \neq \emptyset$. For any given $y \in P$

(that is, $b_i - a_i^\top y \geq 0$ for all $i \in M$),

$$
\begin{aligned}
0 &= (x-y)^\top f(x) \\
&= (x-y)^\top \textstyle\sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} a_i \\
&= \textstyle\sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} a_i^\top (x-y) \\
&= \textstyle\sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} (a_i^\top x - b_i + b_i - a_i^\top y) \\
&\geq \textstyle\sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} (a_i^\top x - b_i) \\
&= \textstyle\sum_{i \in I(x)} \frac{(a_i^\top x - b_i)^2}{a_i^\top a_i} \\
&> 0.
\end{aligned}
$$

Thus, a contradiction occurs. This completes the proof. □

Let $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^\top$ be any given integer point of R^n and, for $y \in R^n$, $C \subseteq R^n$ and $d \in \{-1, 1\}$, let the augmented set

$$\Gamma(y, C, d) = \{t(x, 0) + (1-t)(y, d) \mid x \in C \text{ and } 0 \leq t \leq 1\} \subseteq R^{n+1}.$$

Then, we define the following integer labeling rule:

Definition 1 (An Integer Labeling Rule)**.** *For each integer point* $(x, \gamma) \in \Gamma(x^0, R^n, -1) \cup \Gamma(x^0, R^n, 1)$, *we assign to* (x, γ) *an integer label* $l(x, \gamma) \in \{0\} \cup N_0$ *as follows.*

1. $l(x^0, -1) = 1$ *and* $l(x^0, 1) = n + 1$.

2. For $(x,0)$ with $x \in D(P)$,

$$l(x,0) = \begin{cases} 0 & \textit{if } f(x) = 0 \textit{ or } x \in P, \\ \max\{k \mid f_k(x) = \max_{j \in N} f_j(x)\} & \textit{if } f_j(x) > 0 \textit{ for some } j \in N, \\ n+1 & \textit{if } f(x) \leq 0 \textit{ and } f(x) \neq 0. \end{cases}$$

3. For $(x,0)$ with $x_j > x_j^u$ for some $j \in N$,

$$l(x,0) = \max\{k \mid x_k - x_k^u = \max_{j \in N} x_j - x_j^u\}.$$

4. For $(x,0)$ with $x \leq x^u$ and $x_j < x_j^l$ for some $j \in N$,

$$l(x,0) = \begin{cases} n+1 & \textit{if } x < x^l, \\ \max\{k \mid x_k - x_k^l = \max_{j \in N} x_j - x_j^l\} & \textit{otherwise.} \end{cases}$$

Example 1. *Consider*

$$P = \left\{ x = (x_1, x_2)^\top \;\middle|\; \begin{array}{l} 2x_1 - x_2 \leq \frac{1}{2}, \\ -\frac{7}{6}x_1 + x_2 \leq \frac{1}{2}, \\ -x_1 - x_2 \leq \frac{9}{5} \end{array} \right\},$$

which has an integer point. Figure 1 illustrates the integer labeling rule of Definition 1 on $R^2 \times \{0\}$ for this polytope.

Example 2. *Consider*

$$P = \left\{ x = (x_1, x_2)^\top \;\middle|\; \begin{array}{l} 2x_1 - x_2 \leq \frac{1}{2}, \\ -x_1 + x_2 \leq -\frac{1}{5}, \\ -\frac{1}{5}x_1 - x_2 \leq \frac{8}{5} \end{array} \right\},$$

which has no integer point. Figure 2 illustrates the integer labeling rule of Definition 1 on $R^2 \times \{0\}$ for this polytope.

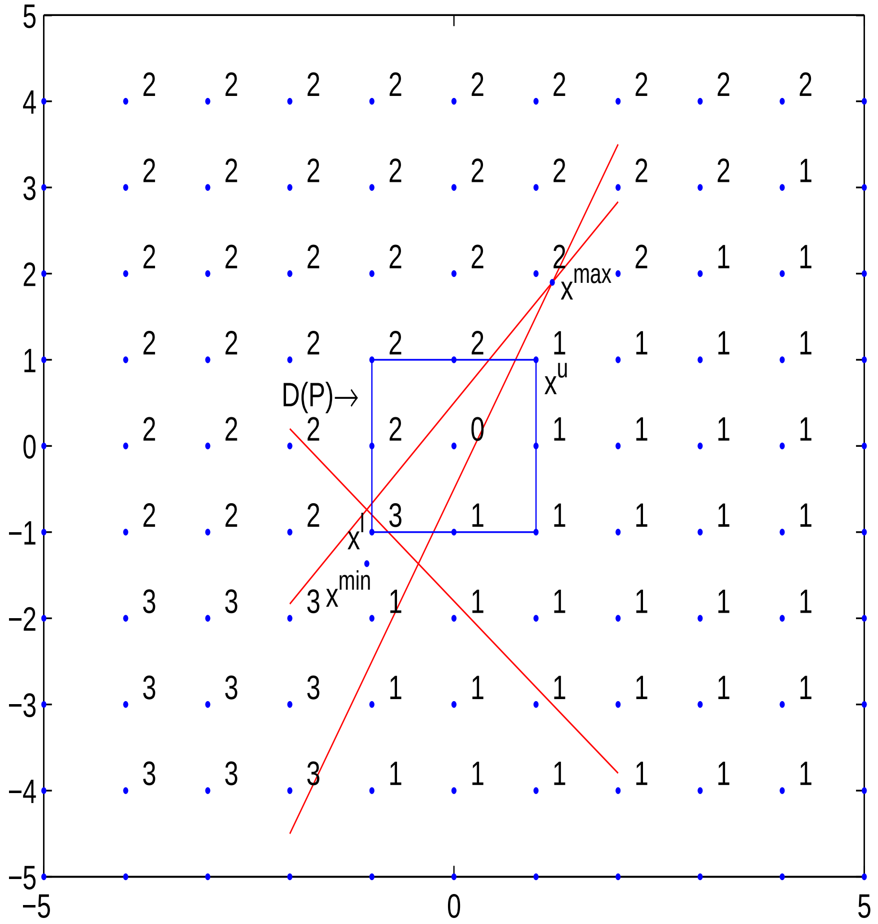

Figure 1: An Illustration of the Integer Labeling Rule on $R^2 \times \{0\}$ for Example 1

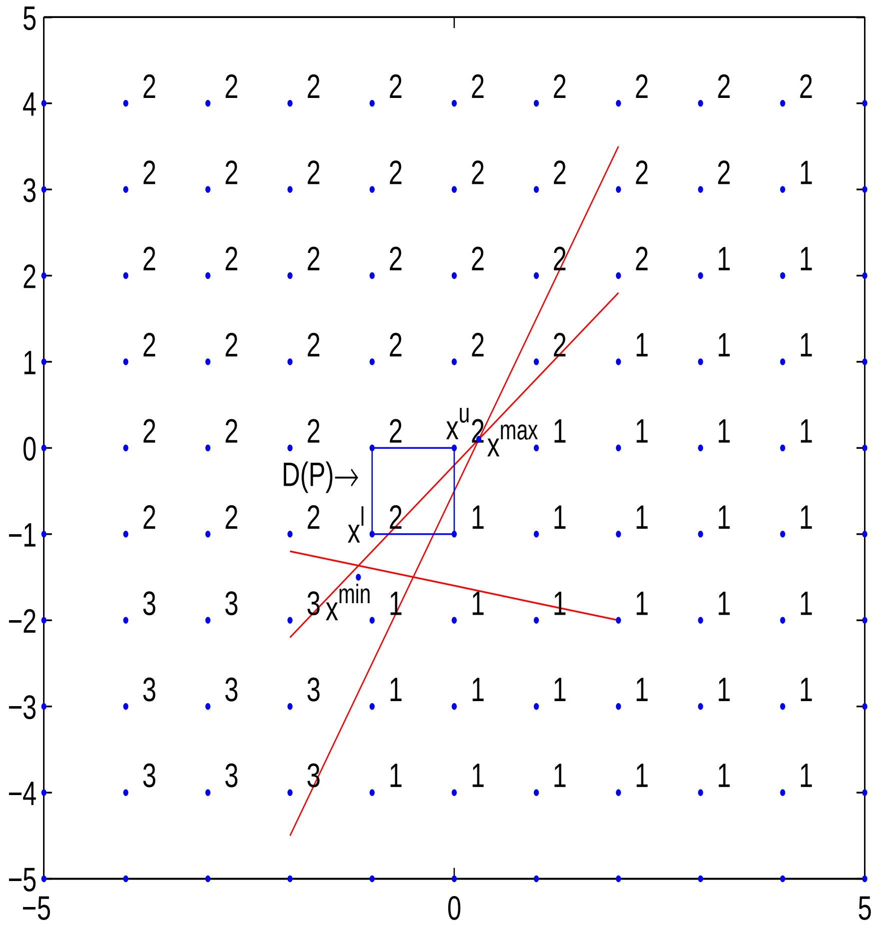

Figure 2: An Illustration of the Integer Labeling Rule on $R^2 \times \{0\}$ for Example 2

For our further development, we need a triangulation of $\Gamma(x^0, R^n, -1) \cup \Gamma(x^0, R^n, 1)$ that subdivides each of $\Gamma(x^0, R^n, d)$, $d \in \{-1, 1\}$, into simplices in such a way that every integer point of $\Gamma(x^0, R^n, -1) \cup \Gamma(x^0, R^n, 1)$ is a vertex of some simplex of the triangulation and every vertex of a simplex of the triangulation is an integer point of $\Gamma(x^0, R^n, -1) \cup \Gamma(x^0, R^n, 1)$. Any cubic triangulation of R^n is suitable for the purpose. For simplicity, we choose the K_1-triangulation in Freudenthal (1942), which is as follows.

A simplex of the K_1-triangulation of $\Gamma(x^0, R^n, -1) \cup \Gamma(x^0, R^n, 1)$ is the convex hull of $n+2$ vectors, y^0, y^1, $\ldots$, y^{n+1}, given by $y^0 = y$, $y^k = y^{k-1} + u^{\pi(k)}$, $k = 1, 2, \ldots, n$, and $y^{n+1} = (x^0, d)$, where $y = (y_1, y_2, \ldots, y_{n+1})^\top$ is an integer point in $R^n \times \{0\}$, $d \in \{-1, 1\}$, and $\pi = (\pi(1), \pi(2), \ldots, \pi(n), \pi(n+1))$ is a permutation of elements of N_0 with $\pi(n+1) = n+1$. Let K_1 be the set of all such simplices. Since a simplex of the K_1-triangulation is uniquely determined by y, d, and π, we use $K_1(y, d, \pi)$ to denote it. Two simplices of K_1 are adjacent if they share a common facet. For a given simplex $\sigma = K_1(y, d, \pi)$ with vertices y^0, y^1, $\ldots$, y^{n+1}, its adjacent simplex opposite to a vertex, say y^i, is given by $K_1(\bar{y}, \bar{d}, \bar{\pi})$, where $\bar{y}$, $\bar{d}$, and $\bar{\pi}$ are generated according to the pivot rules given in the following table.

Pivot Rules of the K_1-Triangulation of $\Gamma(x^0, R^n, -1) \cup \Gamma(x^0, R^n, 1)$

i	$\bar{y}$	$\bar{d}$	$\bar{\pi}$
0	$y + u^{\pi(1)}$	d	$(\pi(2), \ldots, \pi(n), \pi(1), \pi(n+1))$
$1 \leq i < n$	y	d	$(\pi(1), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n+1))$
n	$y - u^{\pi(n)}$	d	$(\pi(n), \pi(1), \ldots, \pi(n-1), \pi(n+1))$
$n+1$	y	$-d$	π

Let $\mathcal{K}_1$ be the set of faces of simplices of K_1. A q-dimensional simplex of $\mathcal{K}_1$ with vertices y^0, y^1, $\ldots$, y^q is denoted by $\langle y^0, y^1, \ldots, y^q\rangle$. For $\sigma \in \mathcal{K}_1$ with $\sigma \subset R^n \times \{0\}$, let $\text{grid}(\sigma) = \max\{\|x - y\| \mid (x, 0) \in \sigma \text{ and } (y, 0) \in \sigma\}$, where $\|\cdot\|$ denotes the infinity norm. We define $\text{mesh}(K_1) = \max\{\text{grid}(\sigma) \mid \sigma \in \mathcal{K}_1 \text{ and } \sigma \subset R^n \times \{0\}\}$. Then, $\text{mesh}(K_1) = 1$.

Definition 2.

- *A q-dimensional simplex $\sigma = \langle y^0, y^1, \ldots, y^q\rangle$ of $\mathcal{K}_1$ is complete if $l(y^i) \neq l(y^j)$ for any $i \neq j$,*

- *A q-dimensional simplex $\sigma = \langle y^0, y^1, \ldots, y^q\rangle$ of $\mathcal{K}_1$ is almost complete if labels of $q + 1$ vertices of σ consist of q different integers.*

From Definition 2, it is easy to see that an almost complete simplex has exactly two complete facets that carry the same set of integer labels.

Lemma 2. *If $f(x) \leq 0$ and $f(x) \neq 0$, then, for any $y \in P$, there is some $k \in N$ satisfying that $x_k - y_k < 0$.*

Proof. Since $f(x) \neq 0$, $I(x) \neq \emptyset$. Let y be a point in P, that is

$b_i - a_i^\top y \geq 0$ for all $i \in M$. Suppose that $x - y \geq 0$. Then,

$$\begin{aligned} 0 \geq & \ (x-y)^\top f(x) \\ = & \ (x-y)^\top \textstyle\sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} a_i \\ = & \ \textstyle\sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} (a_i^\top x - b_i + b_i - a_i^\top y) \\ \geq & \ \textstyle\sum_{i \in I(x)} \frac{a_i^\top x - b_i}{a_i^\top a_i} (a_i^\top x - b_i) \\ = & \ \textstyle\sum_{i \in I(x)} \frac{(a_i^\top x - b_i)^2}{a_i^\top a_i} \\ > & \ 0. \end{aligned}$$

Thus, a contradiction occurs. The lemma follows immediately. $\square$

For $y \in R^n$ and $K \subseteq N$, let the "higher" level cone originated at y along certain directions given in K

$$H(y, K) = \{y + h \in R^n \mid 0 \leq h_j,\ j \in K, \text{ and } h_j = 0,\ j \notin K\}.$$

Lemma 3. *If z^0 is an integer point of P, then, for any $K \subseteq N$, each integer point of $H(z^0, K) \times \{0\}$ carries an integer label of either 0 or an integer in K.*

Proof. Let $(x, 0)$ be an integer point of $H(z^0, K) \times \{0\}$. Consider that $x \in D(P)$. From Lemma 2, we know that $l(x, 0) \neq n+1$, since $x \geq z^0$, which is equivalent to that either $f(x) = 0$ or $f_j(x) > 0$ for some $j \in N$.

Let $\lambda = x - z^0$. Then, $\lambda_j \geq 0$, $j \in K$, and $\lambda_j = 0$, $j \notin K$. Thus, for any constraint i with $a_{ij} \leq 0$ for all $j \in K$,

$$\begin{aligned} a_i^\top x &= a_i^\top z^0 + a_i^\top \lambda \\ &\leq b_i + a_i^\top \lambda \\ &= b_i + \textstyle\sum_{j \in K} a_{ij}\lambda_j \\ &\leq b_i. \end{aligned}$$

This implies that every violated constraint $i \in I(x)$, if it exists, must have $a_{ij} > 0$ for a $j \in K$. Since at most one coefficient $a_{ij} > 0$ for every i, we have $a_{ij} \leq 0$ for all $j \notin K$ and for every possible violated constraint. Therefore, if $f_j(x) > 0$, one must have $j \in K$, that is, either $l(x, 0) = 0$ or $l(x, 0) \in K$ from the labeling rule.

Consider $x \notin D(P)$. Since $x^l \leq z^0 \leq x$, hence, $x_j > x_j^u$ for some $j \in N$. From $x \in H(z^0, K)$, we know that $x_j = z_j^0 \leq x_j^u$ for all $j \notin K$. Therefore, according to the labeling rule, $l(x, 0) \in K$. This completes the proof. $\square$

This lemma plays an essential role in our development. As a corollary of Lemma 3, we obtain that

Corollary 1. *If P contains an integer point z^0, then there is no complete n-dimensional simplex in the higher level cone $H(z^0, N) \times \{0\}$ carrying all integer labels in N_0, and, for any $j \in N$ and $k \in N_0$, there is no complete $(n-1)$-dimensional simplex in $H(z^0, N \backslash \{j\}) \times \{0\}$ carrying all integer labels in $N_0 \backslash \{k\}$.*

Let

$$\Omega = \{x \in R^n \mid x^l - e \leq x \leq x^u + e\}$$

and $\partial\Omega$ denote the boundary of Ω. Clearly, Ω strictly contains $D(P)$. Figure 3 illustrates $\Gamma(x^0, \Omega, -1) \cup \Gamma(x^0, \Omega, 1)$ with integer labels for Example 1. Let the cube

$$C(x^u) = \{x \in R^n \mid x^u \leq x \leq x^u + e\}.$$

Then, $C(x^u) = H(x^u, N) \cap \Omega$.

Lemma 4. $\Gamma(x^0, C(x^u), 1)$ *contains all the complete* n*-dimensional simplices in* $\Gamma(x^0, \partial\Omega, 1)$ *carrying all integer labels in* N_0*.*

Proof. Let σ be a complete n-dimensional simplex in $\Gamma(x^0, \partial\Omega, 1)$ carrying all integer labels in N_0. Then, $(x^0, 1)$ must be a vertex of σ. Since $l(x^0, 1) = n + 1$, hence, the facet of σ opposite to $(x^0, 1)$ must be a complete $(n-1)$-dimensional simplex in $\partial\Omega \times \{0\}$ carrying all integer labels in N. Let $\tau = \langle (y^1, 0), (y^2, 0), \ldots, (y^n, 0) \rangle$ be a complete $(n-1)$-dimensional simplex in $\partial\Omega \times \{0\}$ carrying all integer labels in N. Without loss of generality, we assume that $l(y^i, 0) = i$, $i = 1, 2, \ldots, n$.

Since τ is an $(n-1)$-dimensional simplex on the boundary of $\Omega \times \{0\}$, there must be an index $h \in N$ such that $y^1_h = y^2_h = \ldots = y^n_h$. Suppose

$$\tau \subset \{x \in \Omega \mid x_h = x^l_h - 1\} \times \{0\},$$

that is, this common entry hits the lower bound side of Ω. But for any integer point $(x, 0) \in R^n \times \{0\}$, from (4) of the labeling rule, we have $l(x, 0) = n + 1$ if $x < x^l$. Hence, $y^i \not< x^l$ for every vertex $(y^i, 0)$ of τ. In particular,

$$y^h_h - x^l_h = \max_{j \in N} y^h_j - x^l_j \geq 0.$$

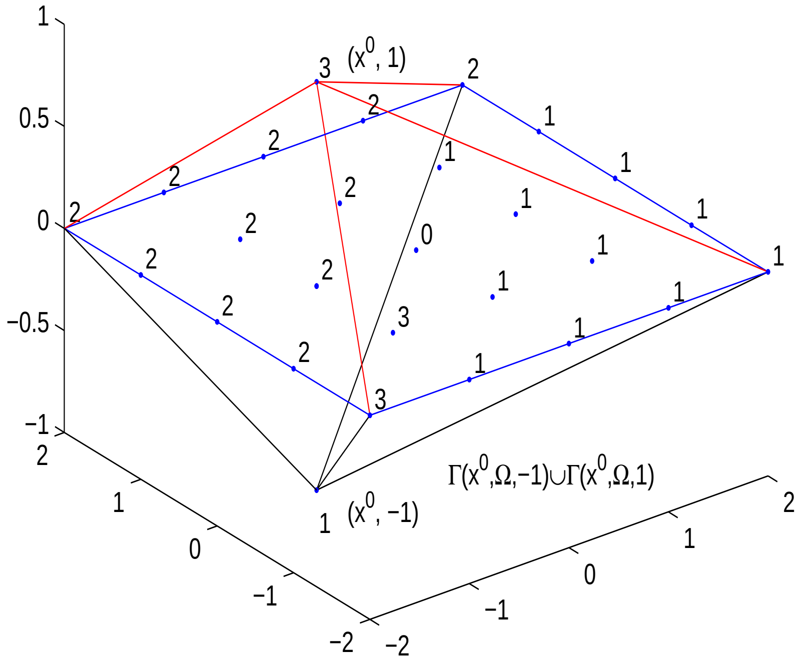

Figure 3: An Illustration of $\Gamma(x^0, \Omega, -1) \cup \Gamma(x^0, \Omega, 1)$ with Integer Labels for Example 1

This contradicts with $y^h_h = x^l_h - 1 < x^l_h$. Thus, the common entry of τ must hits the upper bound of Ω, or

$$\tau \subset \{x \in \Omega \mid x_h = x^u_h + 1\} \times \{0\}.$$

For all $i = 1, 2, \ldots, n$, from $l(y^i, 0) = i = \max\{k \mid y^i_k - x^u_k = \max_{j \in N} y^i_j - x^u_j\}$ and $y^i_h = x^u_h + 1$, we derive that

$$y^i_i - x^u_i = \max_{j \in N} y^i_j - x^u_j = 1.$$

Therefore, as a result of $\text{mesh}(K_1) = 1$, we obtain that $\tau \subset C(x^u) \times \{0\}$. This completes the proof. □

Lemma 4 says that each of possible complete n-dimensional simplices in $\Gamma(x^0, \partial\Omega, 1)$ carrying all integer labels in N_0 must be contained by $\Gamma(x^0, C(x^u), 1)$ formed from $(x^0, 1)$ and the cube $C(x^u)$. Next, we will prove that such a complete n-dimensional simplex exists and it is unique.

Let $\widetilde{\tau}_1 = \langle y^1, y^2, \ldots, y^{n+1}\rangle$ and $\widetilde{\sigma}_1 = \langle y^0, y^1, \ldots, y^{n+1}\rangle$ with

$$y^0 = (x^u, 0),$$

$$y^k = y^{k-1} + u^k,\ k = 1, 2, \ldots, n, \text{ and}$$

$$y^{n+1} = (x^0, 1).$$

Then, $l(y^i) = i$, $i = 1, 2, \ldots, n+1$. Thus, $\widetilde{\tau}_1$ is a complete n-dimensional simplex in $\Gamma(x^0, \partial\Omega, 1)$ carrying all integer labels in N_0.

Lemma 5. *$\widetilde{\tau}_1$ is a unique complete n-dimensional simplex in $\Gamma(x^0, \partial\Omega, 1)$ carrying all integer labels in N_0.*

Proof. Let $\tau = \langle v^1, v^2, \ldots, v^n\rangle$ be a complete $(n-1)$-dimensional simplex in $\partial\Omega \times \{0\}$ with $l(v^i) = i$, $i = 1, 2, \ldots, n$. From Lemma 4, we obtain that

$\tau \subset C(x^u) \times \{0\}$. Thus, from (3) of the labeling rule and the definition of the K_1-triangulation, we drive that $v^1 = (x^u, 0) + u^1$ and $v^i = v^{i-1} + u^i$, $i = 2, 3, \ldots, n$. Therefore, $\langle v^1, v^2, \ldots, v^n, (x^0, 1)\rangle = \widetilde{\tau}_1$. This completes the proof. $\square$

Let $\widetilde{\tau}_{-1} = \langle y^0, y^1, \ldots, y^{n-1}, y^{n+1}\rangle$ and $\widetilde{\sigma}_{-1} = \langle y^0, y^1, \ldots, y^{n+1}\rangle$ with

$$y^0 = (x^l - e, 0),$$

$$y^k = y^{k-1} + u^{k+1},\ k = 1, 2, \ldots, n-1,$$

$$y^n = y^{n-1} + u^1,\ \text{and}$$

$$y^{n+1} = (x^0, -1).$$

Then, $l(y^0) = n+1$, $l(y^k) = k+1$, $k = 1, 2, \ldots, n-1$, and $l(y^{n+1}) = 1$. Thus, $\widetilde{\tau}_{-1}$ is a complete n-dimensional simplex in $\Gamma(x^0, \partial\Omega, -1)$ carrying all integer labels in N_0.

Lemma 6. $\widetilde{\tau}_{-1}$ *is a unique complete n-dimensional simplex in* $\Gamma(x^0, \partial\Omega, -1)$ *carrying all integer labels in N_0.*

Proof. Let $\tau = \langle v^0, v^1, v^2, \ldots, v^{n-1}\rangle$ be a complete $(n-1)$-dimensional simplex in $\partial\Omega \times \{0\}$ with $l(v^0) = n+1$ and $l(v^i) = i+1$, $i = 1, 2, \ldots, n-1$. From the labeling rule, we know that $(x^l - e, 0)$ is a unique integer point in $\partial\Omega \times \{0\}$ carrying integer label $n+1$. Thus, $v^0 = (x^l - e, 0)$. Then, from (4) of the labeling rule and the definition of the K_1-triangulation, we drive that $v^i = v^{i-1} + u^{i+1}$, $i = 1, 2, \ldots, n-1$. Therefore, $\langle v^0, v^1, \ldots, v^{n-1}, (x^0, -1)\rangle = \widetilde{\tau}_{-1}$. This completes the proof. $\square$

3 Polynomial-Time Reduction to Certain Decision Problems Related to PPAD

Recently, as a subclass of total search problems, a PPAD class was proposed by Papadimitriou (1994) (see also Daskalakis et al. (2009)), which leads us to the following work. We show in this section that determining whether there is an integer point in P can be reduced in polynomial time to certain decision problems related to PPAD. In the following discussions,

1. a C(n)S stands for a complete n-dimensional simplex in $\Gamma(x^0, \partial\Omega, -1) \cup \Gamma(x^0, \partial\Omega, 1)$ carrying all integer labels in N_0;

2. an AC($n+1$)S stands for an almost complete $(n+1)$-dimensional simplex in $\Gamma(x^0, \Omega, -1) \cup \Gamma(x^0, \Omega, 1)$ carrying only integer labels in N_0; and

3. a C($n+1$)S stands for a complete $(n+1)$-dimensional simplex in $\Gamma(x^0, \Omega, -1) \cup \Gamma(x^0, \Omega, 1)$ carrying all integer labels in $\{0\} \cup N_0$.

Let Φ be a graph defined as follows. Nodes of Φ consist of

1. all C(n)Ss,

2. all AC($n+1$)Ss, and

3. all C($n+1$)Ss.

There is an edge between two nodes of graph Φ if one is a complete facet of the other or they have a common complete facet carrying all integer labels in N_0.

Let us first determine the degree of each node in graph Φ.

1. Consider node σ of a C(n)S. From the definition of Φ, one can obtain that node σ is only adjacent to a node given by one of the following:

   (a) an AC($n+1$)S or

   (b) a C($n+1$)S.

   Thus, node σ has degree one (which is called an unbalanced node).

2. Consider node σ of an AC($n+1$)S. From the definition of Φ, one can obtain that node σ is only adjacent to a pair of nodes given by one of the following pairs:

   (a) two C(n)Ss,

   (b) a C(n)S and an AC($n+1$)S,

   (c) a C(n)S and a C($n+1$)S,

   (d) two AC($n+1$)Ss,

   (e) an AC($n+1$)S and a C($n+1$)S, or

   (f) two C($n+1$)Ss.

   Thus, node σ has degree two (which is called a balanced node).

3. Consider node σ of a C(n+1)S. From the definition of Φ, one can obtain that node σ is only adjacent to a node given by one of the following:

   (a) a C(n)S,

(b) an AC($n+1$)S, or

(c) a C($n+1$)S.

Thus, node σ has degree one (an unbalanced node).

From the construction of graph Φ, one can see that each node of Φ belongs uniquely to one of these three categories. The above results show that the degree of each node of Φ is at most two and that C($n+1$)Ss and C(n)Ss are only nodes that have degree one. Therefore, we come to

Lemma 7. *Each connected component of graph Φ has one of the following forms:*

- *A simple circuit, in which each of nodes has degree two.*
- *A simple path, in which each of end nodes has degree one and is given by one of*

  1. *a C(n)S, or*
  2. *a C($n+1$)S.*

We show next how to determine the direction of an edge in a path of Φ using the orientation of simplices given in Eaves and Scarf (1976) and Todd (1976a). Let $\tau = \langle y^0, \ldots, y^{i-1}, y^{i+1}, \ldots, y^{n+1} \rangle$ be a complete facet of $\sigma = \langle y^0, y^1, \ldots, y^{n+1} \rangle \in K_1$ carrying all integer labels in N_0. Let $p = (p_1, p_2, \ldots, p_{n+1})$ be the permutation of elements of $\{0, \ldots, i-1, i+1, \ldots, n+1\}$ such that $l(y^{p_i}) = i$, $i = 1, 2, \ldots, n+1$.

Definition 3. *The orientation of τ with respect to σ is given by*

$$or_\sigma(\tau) = sign(\det\left(\begin{pmatrix} 1 \\ y^{p_1} \end{pmatrix}, \begin{pmatrix} 1 \\ y^{p_2} \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ y^{p_{n+1}} \end{pmatrix}, \begin{pmatrix} 1 \\ y^{i} \end{pmatrix}\right)).$$

It is easy to see that if σ is an almost complete $(n+1)$-dimensional simplex of K_1 carrying only integer labels in N_0 and τ_1 and τ_2 are two complete facets of σ, then $\mathrm{or}_\sigma(\tau_1) = -\mathrm{or}_\sigma(\tau_2)$. It is well known that adding a multiple of a column to another column does not change the determinant of a matrix and that exchanging two columns just changes the sign of the determinant of a matrix. Thus, applying these two column operations and the pivoting rules of the K_1-triangulation, one can derive that

Theorem 2. *If τ is a common complete n-dimensional facet of σ_1 and σ_2 in K_1 carrying all integer labels in N_0, then $or_{\sigma_1}(\tau) = -or_{\sigma_2}(\tau)$.*

From Lemma 5 and Lemma 6, we know that there are only two C(n)Ss, which are given by $\widetilde{\tau}_{-1}$ and $\widetilde{\tau}_1$. We use $\widetilde{\tau}_1$ to determine the direction of an edge of graph Φ.

Definition 4. *Let σ be a node of Φ given by either an $A(n+1)S$ or a $C(n+1)S$ and τ a complete facet of σ carrying all integer labels in N_0. If $or_\sigma(\tau) = -or_{\widetilde{\sigma}_1}(\widetilde{\tau}_1)$, the edge leaves σ from τ. If $or_\sigma(\tau) = or_{\widetilde{\sigma}_1}(\widetilde{\tau}_1)$, the edge enters σ from τ.*

A direct calculation yields that $\mathrm{or}_{\widetilde{\sigma}_{-1}}(\widetilde{\tau}_{-1}) = -\mathrm{or}_{\widetilde{\sigma}_1}(\widetilde{\tau}_1)$. Given this definition, Corollary 1, Lemma 5, Lemma 6 and Lemma 7 together imply that

Theorem 3. *There are two given unbalanced nodes $\widetilde{\tau}_1$ and $\widetilde{\tau}_{-1}$ for the directed graph Φ, and one inward and one outward. If P contains no integer point,* *then the graph has no other unbalanced nodes, which implies that it has a unique path from $\widetilde{\tau}_1$ to $\widetilde{\tau}_{-1}$. On the other hand, if P contains an integer point, then other unbalanced nodes exist (should be at least two) and each of them should be a $C(n+1)S$ that has an integer point of P as its vertex. Furthermore, the path from τ_1 will end at a $C(n+1)S$.*

Proof. An unbalanced node is either a C(n)S or a C($n+1$)S. Lemma 5 and Lemma 6 together show that there are only two C(n)Ss, which are $\widetilde{\tau}_1$ and $\widetilde{\tau}_{-1}$. Lemma 7 implies that the graph has one path starting from $\widetilde{\tau}_1$, denoted by $P(\widetilde{\tau}_1)$, and one path ending at $\widetilde{\tau}_{-1}$, denoted by $P(\widetilde{\tau}_{-1})$. Consider the case that P contains no integer point. Then, there is no C($n+1$)S. Thus, $P(\widetilde{\tau}_1)$ must end at the other C(n)S ($\widetilde{\tau}_{-1}$) since it is contained in the bounded set $\Gamma(x^0, \Omega, -1) \cup \Gamma(x^0, \Omega, 1)$. Therefore, the graph has a unique path.

Consider the case that P contains an integer point. Let z^0 be an integer point of P. From Corollary 1, we derive that there is no complete n-dimensional simplex in the boundary of $\Gamma(x^0, H(z^0, N), 1)$ carrying all integer labels in N_0. This together with Lemma 5 imply that $P(\widetilde{\tau}_1)$ is contained in the bounded set $\Gamma(x^0, H(z^0, N) \cap \Omega, 1)$. Thus, $P(\widetilde{\tau}_1)$ must end at a C($n+1$)S.

From Corollary 1, we derive that there is no complete n-dimensional simplex in the boundary of $\Gamma(x^0, H(z^0, N), -1) \cup \Gamma(x^0, H(z^0, N), 1)$ carrying all integer labels in N_0. This together with Lemma 5 and Lemma 6 imply that $P(\widetilde{\tau}_{-1})$ is contained in the bounded set $\Gamma(x^0, L(z^0, \Omega), -1) \cup \Gamma(x^0, L(z^0, \Omega), 1)$, where $L(z^0, \Omega) = \{x \in \Omega \mid x_j \leq z_j^0 \text{ for some } j \in N\}$. Therefore, $P(\widetilde{\tau}_{-1})$ must start from a C($n+1$)S. This completes the proof. $\square$

This theorem reduces the problem of determining whether there is an integer point in P to certain decision problems related to the PPAD graph, since graph Φ has only two alternatives: either the unbalanced node on the other end of the path $P(\widetilde{\tau}_1)$ is a C$(n+1)$S that implies P containing an integer point, or it is the only other known unbalanced node C(n)S.

Figure 4 and Figure 5 illustrate the paths projected on $R^2 \times \{0\}$ for Example 1 and the path projected on $R^2 \times \{0\}$ for Example 2, respectively. In Example 1, the paths, either starting from unbalanced node $\widetilde{\tau}_1$ or ending at unbalanced node $\widetilde{\tau}_{-1}$, either ends at or starts from a C$(n+1)$S, respectively. In Example 2, the path starting from the unbalanced node $\widetilde{\tau}_1$ ends at the only other unbalanced node $\widetilde{\tau}_{-1}$, where $(x^0, 1)$ and $(x^0, -1)$ are switched in the middle of the path.

Finally, the following method shows how the unique successor node of a predecessor node in a simple path can be computed in *polynomial time* in graph Φ.

Initialization: Choose $\tau_0 \in \{\widetilde{\tau}_{-1}, \widetilde{\tau}_1\}$. Let σ_0 be the unique simplex in $\Gamma(x^0, \Omega, -1) \cup \Gamma(x^0, \Omega, 1)$ with τ_0 being one of its facets, y^+ the vertex of σ_0 opposite to τ_0, and $k = 0$, and go to **Step 1**.

Step 1: Compute $l(y^+)$. If $l(y^+) = 0$, the method terminates and an integer point of P has been found. Otherwise, let y^- be the vertex of σ_k other than y^+ and carrying integer label $l(y^+)$, and τ_{k+1} the facet of σ_k opposite to y^-, and go to **Step 2**.

Step 2: If $\tau_{k+1} \subset \Gamma(x^0, \partial\Omega, -1) \cup \Gamma(x^0, \partial\Omega, 1)$, the method terminates and

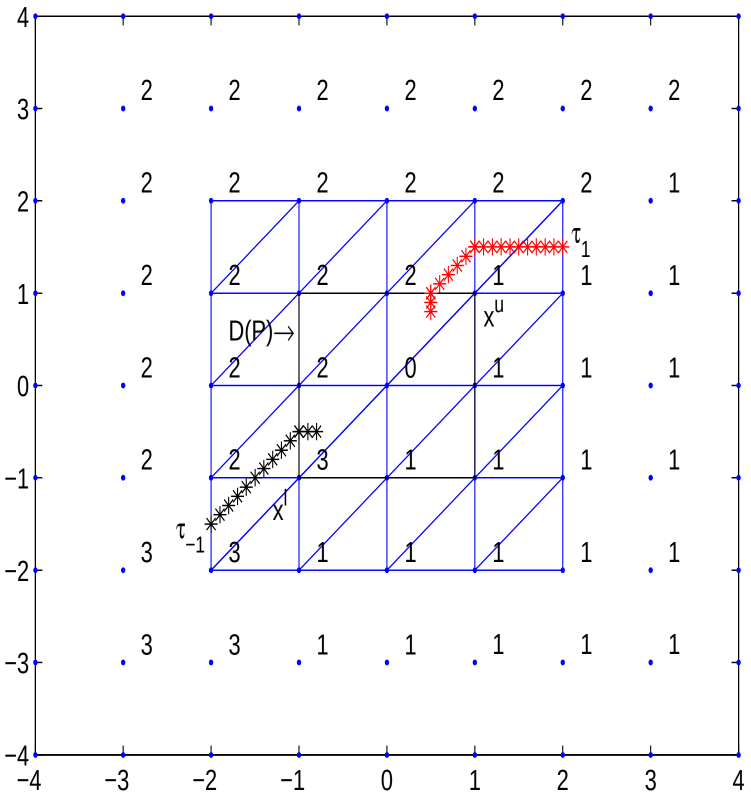

Figure 4: An Illustration of the Paths Projected on $R^2 \times \{0\}$ for Example 1

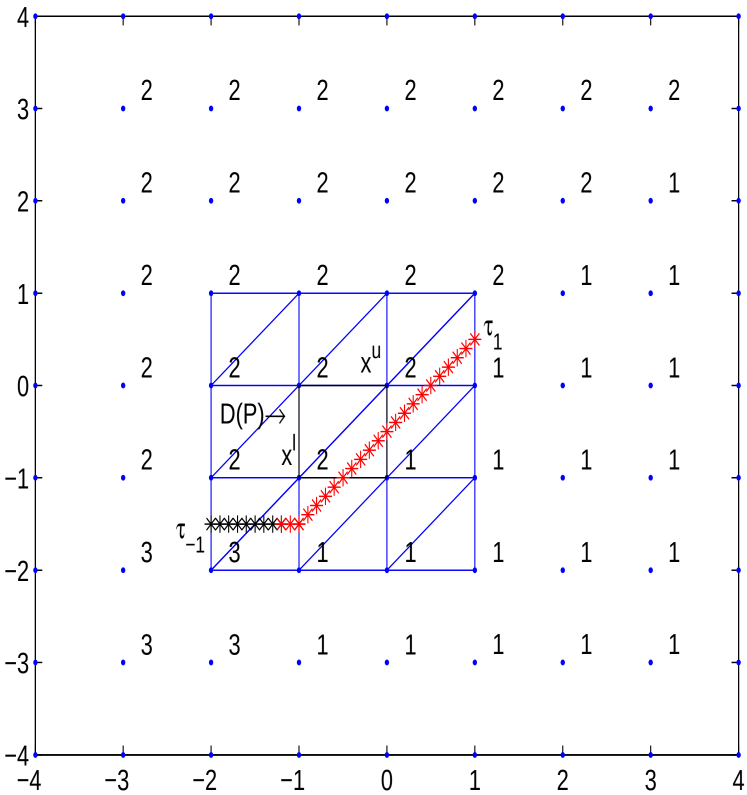

Figure 5: An Illustration of the Path Projected on $R^2 \times \{0\}$ for Example 2

P has no integer point. Otherwise, let σ_{k+1} be the unique $(n+1)$-dimensional simplex that is adjacent to σ_k and has τ_{k+1} as a facet, y^+ the vertex of σ_{k+1} opposite to τ_{k+1}, and $k = k+1$, and go to **Step 1**.

It is easy to see that the computational complexity of an iteration of the method is bounded by $O(n^2)$ arithmetic operations over integers. Thus, the unique successor of a predecessor in a simple path can be generated in polynomial time. Therefore, we come to our major conclusions:

Theorem 4. *Given a directed graph G and* TWO *unbalanced nodes (with degree one), one inward and one outward, it is NP-hard to decide*

- *Are the two given unbalanced nodes connected?*
- *Is the path from the given inward unbalanced node unique in G?*
- *Are the given two unbalanced nodes the only unbalanced nodes in G? Or its complement: are there other unbalanced nodes in G besides the given two?*

It is also NP-hard to find

- *any pair of connected unbalanced nodes in G;*
- *the unbalanced node connected to a known unbalanced node.*

We need to mention that a similar result for decising Nash Matrix Game equilibria has been developed in Gilboa and Zemel [14]: given a matrix game, does there exists a unique Nash equilibrium in the game? Their result does not imply our results presented in Theorem 4.

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