

International Series in Operations Research & Management Science

Volume 228

Series Editor

Camille C. Price
Stephen F. Austin State University, TX, USA

Associate Series Editor

Joe Zhu
Worcester Polytechnic Institute, MA, USA

Founding Series Editor

Frederick S. Hillier
Stanford University, CA, USA

More information about this series at <http://www.springer.com/series/6161>

David G. Luenberger • Yinyu Ye

Linear and Nonlinear Programming

Fifth Edition



Springer

David G. Luenberger
Department of Management Science
and Engineering
Stanford University
Stanford, CA, USA

Yinyu Ye
Department of Management Science
and Engineering
Stanford University
Stanford, CA, USA

ISSN 0884-8289 ISSN 2214-7934 (electronic)
International Series in Operations Research & Management Science
ISBN 978-3-319-18841-6 ISBN 978-3-319-18842-3 (eBook)
DOI 10.1007/978-3-319-18842-3

Library of Congress Control Number: 2015942692

Springer Cham Heidelberg New York Dordrecht London

© Springer International Publishing Switzerland 1973, 1984 (2003 reprint), 2008, 2016

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

*To Susan, Robert, Jill, and Jenna;
Daisun, Fei, Tim, Kaylee, and Rylee*

Preface

This book is intended as a text covering the central concepts of practical optimization techniques. It is designed for either self-study by professionals or classroom work at the undergraduate or graduate level for students who have a technical background in engineering, mathematics, or science. Like the field of optimization itself, which involves many classical disciplines, the book should be useful to system analysts, operations researchers, numerical analysts, management scientists, and other specialists from the host of disciplines from which practical optimization applications are drawn. The prerequisites for convenient use of the book are relatively modest; the prime requirement being some familiarity with introductory elements of linear algebra. Certain sections and developments do assume some knowledge of more advanced concepts of linear algebra, such as eigenvector analysis, or some background in sets of real numbers, but the text is structured so that the mainstream of the development can be faithfully pursued without reliance on this more advanced background material.

Although the book covers primarily material that is now fairly standard, this edition emphasizes methods that are both state-of-the-art and popular in emerging fields such as Data Sciences, Machine Learning and Decision Analytics. One major insight is the connection between the purely analytical character of an optimization problem, expressed perhaps by properties of the optimality conditions, and the behavior of algorithms used to solve a problem. This was a major theme of the first edition of this book and the fifth edition further expands and illustrates this relationship.

As in the earlier editions, the material in this fifth edition is organized into three separate parts. Part I is a self-contained introduction to classical and conic linear programming, a key component of optimization theory. The presentation in this part is fairly conventional, covering the main elements of the underlying theory of linear programming, many of the most effective numerical algorithms, and many of its important special and emerging applications. Part II, which is independent of Part I, covers the theory of unconstrained optimization, including both derivations of the appropriate optimality conditions and an introduction to basic algorithms. This part

of the book explores the general properties of algorithms and defines various notions of convergence. Part III extends the concepts developed in the second part to constrained optimization problems. Except for a few isolated sections, this part is also independent of Part I. It is possible to go directly into Parts II and III omitting Part I, and, in fact, the book has been used in this way in many universities. Each part of the book contains enough material to form the basis of a one-quarter course. In either classroom use or for self-study, it is important not to overlook the suggested exercises at the end of each chapter. The selections generally include exercises of a computational variety designed to test one's understanding of a particular algorithm, a theoretical variety designed to test one's understanding of a given theoretical development, or of the variety that extends the presentation of the chapter to new applications or theoretical areas. One should attempt at least four or five exercises from each chapter. In progressing through the book it would be unusual to read straight through from cover to cover. Generally, one will wish to skip around. In order to facilitate this mode, we have indicated sections of a specialized or digressive nature with an asterisk*.

New to this edition is, in Chap. 2, the introduction of quite a few problems in Machine Learning and Data Science that are closely related to linear programming. We added a section in Chap. 2 devoted to Farkas' Lemma and the Alternative-System theory. Consequently, we moved the Duality and Complementarity Chapter (Chap. 4) before the Simplex Method Chapter (Chap. 3). We restructured topics in Chap. 3 substantially, since linear programs are nowadays solved by computers rather than by hand. Therefore, we focus on introducing methods and algorithms most efficiently implementable by computer codes. Due to a recent breakthrough, we also add a section in (Chap. 3) on proving the efficiency of the Simplex method, which remains a dominate solver for linear programming.

As the field of optimization advances, researcher and practitioners face more challenges: addressing data-driven and dynamic programs, making decisions with uncertainty, developing online algorithms, and expanding the overall theory. We introduce modern optimization topics, such as Markov Decision Process, Reinforcement Learning, Distributionally Robust Stochastic Optimization and Online Optimization. In particular, we have added a section in Chap. 3 to illustrate online linear programming algorithms where the decisions need to be made "on the fly" in problem settings. One of the algorithms is related to the online Stochastic Gradient Decent method that is added in Chap. 8.

Another new topic is multiplicative descent-direction methods that exhibit good convergence properties in Chap. 8. We have included the affine-scaling and mirror-descent methods that are especially effective for optimization where decision variables are subject to nonnegativity constraints. We have also added a couple of globally convergent Newton's methods there.

We have added a section on Lagrangian duality for constrained nonlinear optimization in Chap. 11. The Lagrangian duality plays a fundamental role, as the duality does for linear optimization, in both theory and algorithm design. We introduce detailed rules on how to construct the dual explicitly for certain type of problems, such as the support vector machine problem.

Then we have added two sections into Chap. 12. The first is a “descent-first and feasible-second” steepest descent projection method for linear and nonlinear constrained optimization, which is simple and effective in practice. The second is an interior trust-region sequential quadratic optimization method which is suitable for computing a solution that meets the second-order optimality condition. The convergence analyses of the two methods are presented.

We have added a new section in Chap. 14 to introduce the randomized multi-block alternative direction method with multipliers, which are effective for optimization problems arising of both private and distributed data.

Finally, we have added two sections in Chap. 15 introducing the nonlinear monotone complementarity problem that includes the optimality condition problem as a special case. We also present the homogeneous model/algorithm that is a one-phase algorithm with capability to detect possible primal or dual infeasibility, which becomes an important task in nonlinear optimization.

In this revision, we have also removed a few sections where the methods and/or materials are not suitable for large-scale optimization and computer-coding in our modern computation age.

We wish to thank the many students and researchers who over the years have given us comments concerning the book and those who encouraged us to carry out this revision. We are especially thankful to Xiaocheng Li and Robert Luenberger for their careful readings and comments for this new revision.

Stanford, CA, USA
Stanford, CA, USA
August 2021

D.G. Luenberger
Y. Ye

Contents

1	Introduction	1
1.1	Optimization	1
1.2	Types of Problems	2
1.3	Complexity of Problems	5
1.4	Iterative Algorithms and Convergence	7
Part I Linear Programming		
2	Basic Properties of Linear Programs	11
2.1	Introduction	11
2.2	Examples of Linear Programming Problems	14
2.3	Basic Feasible Solutions	21
2.4	The Fundamental Theorem of Linear Programming	22
2.5	Relations to Convex Geometry	25
2.6	Farkas' Lemma and Alternative Systems	29
2.7	Summary	31
2.8	Exercises	31
3	Duality and Complementarity	37
3.1	Dual Linear Programs and Interpretations	37
3.2	The Duality Theorem	43
3.3	Geometric and Economic Interpretations	45
3.4	Sensitivity and Complementary Slackness	48
3.5	Selected Applications of the Duality	51
3.6	Max Flow–Min Cut Theorem	56
3.7	Summary	61
3.8	Exercises	61
4	The Simplex Method	69
4.1	Adjacent Basic Feasible Solutions (Extreme Points)	70
4.2	The Primal Simplex Method	73

4.3 The Dual Simplex Method 80

4.4 The Simplex Tableau Method 84

4.5 The Simplex Method for Transportation Problems 92

4.6 Efficiency Analysis of the Simplex Method 104

4.7 Summary 106

4.8 Exercises 107

5 Interior-Point Methods 117

5.1 Elements of Complexity Theory 119

5.2 *The Simplex Method Is Not Polynomial-Time 120

5.3 *The Ellipsoid Method 121

5.4 The Analytic Center 125

5.5 The Central Path 128

5.6 Solution Strategies 133

5.7 Termination and Initialization 140

5.8 Summary 146

5.9 Exercises 146

6 Conic Linear Programming 151

6.1 Convex Cones 151

6.2 Conic Linear Programming Problem 152

6.3 Farkas’ Lemma for Conic Linear Programming 157

6.4 Conic Linear Programming Duality 160

6.5 Complementarity and Solution Rank of SDP 169

6.6 Interior-Point Algorithms for Conic Linear Programming 174

6.7 Summary 177

6.8 Exercises 177

Part II Unconstrained Problems

7 Basic Properties of Solutions and Algorithms 183

7.1 First-Order Necessary Conditions 184

7.2 Examples of Unconstrained Problems 187

7.3 Second-Order Conditions 191

7.4 Convex and Concave Functions 193

7.5 Minimization and Maximization of Convex Functions 196

7.6 Global Convergence of Descent Algorithms 198

7.7 Speed of Convergence 206

7.8 Summary 211

7.9 Exercises 211

8 Basic Descent Methods 215

8.1 Line Search Algorithms 216

8.2 The Method of Steepest Descent: First-Order 230

8.3 Applications of the Convergence Theory and Preconditioning 241

8.4 Accelerated Steepest Descent 245

- 8.5 [Multiplicative Steepest Descent](#) 248
- 8.6 Newton’s Method: Second-Order 251
- 8.7 [Sequential Quadratic Optimization Methods](#) 257
- 8.8 Coordinate and [Stochastic Gradient](#) Descent Methods 262
- 8.9 Summary 268
- 8.10 Exercises 269

- 9 Conjugate Direction Methods** 275
 - 9.1 Conjugate Directions 275
 - 9.2 Descent Properties of the Conjugate Direction Method 278
 - 9.3 The Conjugate Gradient Method 280
 - 9.4 The C–G Method as an Optimal Process 282
 - 9.5 The Partial Conjugate Gradient Method 285
 - 9.6 Extension to Nonquadratic Problems 288
 - 9.7 *Parallel Tangents 292
 - 9.8 Exercises 294

- 10 Quasi-Newton Methods** 297
 - 10.1 Modified Newton Method 298
 - 10.2 Construction of the Inverse 300
 - 10.3 Davidon-Fletcher-Powell Method 302
 - 10.4 The Broyden Family 305
 - 10.5 Convergence Properties 308
 - 10.6 Scaling 312
 - 10.7 Memoryless Quasi-Newton Methods 316
 - 10.8 *Combination of Steepest Descent and Newton’s Method 318
 - 10.9 Summary 321
 - 10.10 Exercises 322

Part III Constrained Optimization

- 11 Constrained Optimization Conditions** 329
 - 11.1 Constraints [and Tangent Plane](#) 329
 - 11.2 First-Order Necessary Conditions (Equality Constraints) 334
 - 11.3 Equality-Constrained Optimization Examples 337
 - 11.4 Second-Order Conditions (Equality Constraints) 343
 - 11.5 Inequality Constraints 348
 - 11.6 [Mix-Constrained Optimization Examples](#) 353
 - 11.7 Lagrangian Duality and Zero-Order Conditions 356
 - 11.8 [Rules of Constructing the Lagrangian Dual Explicitly](#) 360
 - 11.9 Summary 363
 - 11.10 Exercises 363

12	Primal Methods	369
12.1	Infeasible Direction and the Steepest Descent Projection Method	370
12.2	Feasible Direction Methods: Sequential Linear Programming	375
12.3	The Gradient Projection Method	377
12.4	Convergence Rate of the Gradient Projection Method	383
12.5	The Reduced Gradient Method	391
12.6	Convergence Rate of the Reduced Gradient Method	395
12.7	Sequential Quadratic Optimization Methods	402
12.8	Active Set Methods	405
12.9	Summary	409
12.10	Exercises	410
13	Penalty and Barrier Methods	415
13.1	Penalty Methods	416
13.2	Barrier Methods	420
13.3	Lagrange Multipliers in Penalty and Barrier Methods	422
13.4	Newton's Method for the Logarithmic Barrier Optimization	428
13.5	Newton's Method for Equality Constrained Optimization	431
13.6	Conjugate Gradients and Penalty Methods	434
13.7	Penalty Functions and Gradient Projection	435
13.8	Summary	439
13.9	Exercises	440
14	Local Duality and Dual Methods	445
14.1	Local Duality and the Lagrangian Method	446
14.2	Separable Problems and Their Duals	452
14.3	The Augmented Lagrangian and Interpretation	456
14.4	The Augmented Lagrangian Method of Multipliers	460
14.5	The Alternating Direction Method of Multipliers	465
14.6	The Multi-Block Extension of the Alternating Direction Method of Multipliers	468
14.7	*Cutting Plane Methods	471
14.8	Exercises	476
15	Primal-Dual Methods	479
15.1	The Standard Problem and Monotone Function	479
15.2	A Simple Merit Function	483
15.3	Basic Primal-Dual Methods	484
15.4	Relation to Sequential Quadratic Optimization	490
15.5	Primal-Dual Interior Point (Barrier) Methods	495
15.6	The Monotone Complementarity Problem	498
15.7	Detect Infeasibility in Nonlinear Optimization	501
15.8	Summary	504
15.9	Exercises	505

- A Mathematical Review** 509
 - A.1 Sets 509
 - A.2 Matrix Notation 510
 - A.3 Spaces 511
 - A.4 Eigenvalues and Quadratic Forms 512
 - A.5 Topological Concepts 513
 - A.6 Functions 514

- B Convex Sets** 519
 - B.1 Basic Definitions 519
 - B.2 Hyperplanes and Polytopes 521
 - B.3 Separating and Supporting Hyperplanes 523
 - B.4 Extreme Points 525

- C Gaussian Elimination and Pivot Operation** 527
 - C.1 The LU Decomposition 527
 - C.2 Pivots 529

- D Basic Network Concepts** 533
 - D.1 Flows in Networks 535
 - D.2 Tree Procedure 535
 - D.3 Capacitated Networks 537

- Bibliography** 539

- Index** 557