Bootcamp: Interior Point Methods II

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The Various Notions of ‘Solving’ LP

...or how much error \( \varepsilon \) do we allow...

\[
\min \langle c, x \rangle : Ax = b, x \geq 0,
\]
\( m \) constraints, \( n \) variables

\( \varepsilon \)-approximate solution \( \bar{x} \):

\[
\langle c, \bar{x} \rangle \leq OPT + \varepsilon, \| A\bar{x} - b \| \leq \varepsilon, \| x^- \| \leq \varepsilon
\]

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<th>High</th>
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<td>( \text{poly}(1/\varepsilon) )</td>
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Improvement on weakly polynomial solvers for LP

...progress of recent years for high accuracy solvers...

\[ \min \langle c, x \rangle: Ax = b, x \geq 0, \]
m constraints, \( n \) variables

\[ \langle c, \bar{x} \rangle \leq OPT + \varepsilon, \| A\bar{x} - b \| \leq \varepsilon, \| x^- \| \leq \varepsilon \]

Running times (\( \log(1/\varepsilon) \) terms omitted):

\[ \sqrt{m}(\text{nnz}(A) + m^2) \quad \text{Lee-Sidford '13-'19} \]

\[ n^\omega \quad \text{Cohen, Lee, Song '19, van den Brand '20,}
\]
\[ \text{Jiang, Song, Weinstein, Zhang '21} \]

\[ nm + m^{2.5} \quad \text{van den Brand, Lee, Liu, Saranurak, Sidford, Song, Wang '21} \]

\[ \omega \ldots \text{matrix multiplication exponent: It takes } O(n^\omega) \text{ to multiply two } n \times n \text{ matrices.} \]
Scaling Invariance in LP Algorithms

- Let $\mathcal{D}$ be the set of diagonal matrices with positive nonzero diagonal entries, and let $D \in \mathcal{D}$.

\[
(P_{\text{scaled}}) \quad \min \ c^T Dx' \text{ s.t. } ADx' = b, \ x' \geq 0.
\]

\[
(D_{\text{scaled}}) \quad \max \ b^Ty \text{ s.t. } Dc - DA^Ty = s', \ s' \geq 0.
\]

$(P_{\text{scaled}})$ and $(D_{\text{scaled}})$ are equivalent to $(P)$ and $(D)$.

- An algorithm is called “scaling invariant” if it generates the (geometrically) the identical sequences when applied to $(P)$ and $(D)$, and $(P_{\text{scaled}})$ and $(D_{\text{scaled}})$.

- The simplex algorithm is not scaling invariant, whereas MTY-PC algorithm is scaling invariant.
Recall Predictor - Corrector Path Following
Mizuno-Todd-Y ‘93

- Given \( x^0 \) in ‘neighborhood’ around \( x_{\mu_0} \) for some \( \mu_0 > 0 \)
- Compute iterates \( x^1, \ldots, x^t \) by alternating between
  - **Predictor steps**: decrease \( \mu \) by moving ‘down’ the central path
  - **Corrector steps**: move back ‘closer’ to the central path for the same \( \mu \) (Newton step).

Each iteration requires a linear system solve.

Standard analysis as seen before: Decrease \( \mu \) by a factor of 2 in
\( O(\sqrt{n}) \) iterations.

\[
\min(c, x): A^T x \geq b, d \text{ variables, } n \text{ inequalities}
\]
Condition Numbers in LP Algorithms

- Let $\mathcal{B}$ be a set of indices of columns of $A$, and let $A_B$ be the submatrix associate with $B$.
- Let $\mathcal{B}$ be the set of index set $B$ such that $A_B$ is invertible.
- The condition number $\bar{\chi}_A$ of $A$ is defined as follows:
  \[ \bar{\chi}_A = \max_{B \in \mathcal{B}} \| A_B^{-1} A \|. \]
- If the input bit length of $A$ is $L_A$, then, $\bar{\chi}_A = 2^{O(L_A)}$.
- $\bar{\chi}_A$ is not scaling invariant; namely, $\bar{\chi}_{AD} \neq \bar{\chi}_A$.
  \[ \bar{\chi}_A^* = \min_{D \in \mathcal{D}} \bar{\chi}_{AD} \]
  ($\mathcal{D}$ is the set of diagonal matrices with positive diagonal entries.)

If an algorithm is scaling invariant, then we can use $\bar{\chi}_A^*$ for evaluating complexity.

Introduced by Dikin’67. Used in Stewart ’89, Todd ’90, Vavasis-Y ’96, Monteiro and Tsuchiya...
Condition Number Based Complexity Analyses

Theorem (Vavasis and Y ‘96):
LP of the form $\min(c, x), Ax = b, x \geq 0$ can be solved exactly within $O(n^{3.5}\log(\chi_A))$ many iterations, each of which requires $O(n)$ linear system solves.

Note: Number of iterations independent of bit-encoding $b$ and $c$. This captures many combinatorial problems, with nice constraint matrix $A$ but arbitrary $b$ and $c$.

The key technique is the departure from affine-scaling to the layered-least-squares linear system solver.
Layered-Least-Squares Linear System

\[ \min \langle c, x \rangle : Ax = b, x \geq 0, m \text{ constrains, } n \text{ variables} \]

Standard affine scaling:

\[ \Delta x : = \arg \min \sum_{i=1}^{n} \left( \frac{x_i + \Delta x_i}{x_i} \right)^2 \text{ s.t. } A\Delta x = 0 \]

Vavasis-Y Layered-Least-Squares step:

Step 1: \( z : = \arg \min_{\Delta x} \sum_{i \in L_2} \left( \frac{x_i + \Delta x_i}{x_i} \right)^2 \text{ s.t. } A\Delta x = 0 \)

Step 2: \( \Delta x : = \arg \min_{\Delta x} \sum_{i \in [n]} \left( \frac{x_i + \Delta x_i}{x_i} \right)^2 \text{ s.t. } A\Delta x = 0, \Delta x_{L_2} = z_{L_2} \)
Key Idea of the V-Y Analysis

Central path consists of
• $O(n^2)$ short curved segments of length $O(\sqrt{\chi_A})$ in $\mu$
  $\Rightarrow \sqrt{n} \cdot n \log(\chi_A)$ iterations required to traverse each.

• $O(n^2)$ long straight segments
  $\Rightarrow$ single iteration of LLS step sufficient to traverse each, even if unbounded length in $\mu$.

The algorithm is not scale-invariant.
**Scale-Invariant Improvements**

There is an exact algorithm (Vavasis and Ye 1996) that

- Finds an exact optimal solution in $O(n^{3.5} \log (\bar{\chi}_A + n))$ iterations. Thus, the computational complexity does not depend on $b$ nor $c$.
- The algorithm is not scaling invariant.
- The number of iterations to reduce the duality gap by a factor of $\varepsilon$ by MTY-PC algorithm [Monteiro and Tsuchiya 2004]:

  \[ O(n^2 \log \log(1/\varepsilon) + n^{3.5} \log(\bar{\chi}_A^* + n)) \]

- Analysis is based on Vavasis-Ye framework. Compared with Vavasis-Ye algorithm, $\bar{\chi}_A$ is replaced by $\bar{\chi}_A^*$ (scaling invariance), but $n^2 \log \log(1/\varepsilon)$ is there (no finite termination).

**Theorem (Dadush, Huiberts, Natura, Végh ’20):**

LP of the form $\min(c, x), Ax = b, x \geq 0$ can be solved exactly within $O(n^{2.5} \log(\bar{\chi}_A^*))$ many iterations, each of which requires $O(n)$ linear system solves,
Key Question: How ‘Curved’ is the Central Path?

\[ \min \langle c, x \rangle : A^T x \geq b, \ d \text{ variables, } n \text{ inequalities} \]

\[ x_\mu := \arg\min \langle c, x \rangle - \mu \sum_{i=1}^{n} \log(\langle a_i, x \rangle - b_i) \]

- Parameter \( \mu \) - optimality gap
- Multiplicative decrease in \( \mu \) in each iteration
- \( x_\mu \) is ‘as far away as possible’ from constraints subject to having optimality gap \( \mu \)

Question: How many iterations to solve an LP exactly?

Can the condition numbers be bounded Polynomially in dimensions?

As \( c \to e_2 \) convergence to \( x^* \) is arbitrarily slow

Path can be arbitrarily curved or “long” in parameter space
The Central Path Curvature and Iteration # I

Let \((\dot{x}, \dot{s}, \dot{y})\) be the derivative of \((x(\nu), s(\nu), y(\nu))\), which satisfies

\[
\dot{x} \circ s + x \circ \dot{s} = -e, \quad A\dot{x} = 0, \quad A^T \dot{y} + \dot{s} = 0.
\]

Sonnevend Curvature [Sonnevend, Stoer, Zhao 1991]:

\[
\kappa(\nu) = \sqrt{\nu |\dot{x} \circ \dot{s}|}
\]

Sonnevend Curvature integral:

\[
I_{PD}(\nu_{ini}, \nu_{fin}) = \int_{\nu_{ini}}^{\nu_{fin}} \frac{\kappa(\nu)}{\nu} d\nu.
\]
The Central Path Curvature and Iteration # II for CP-Based IPMs

- The iteration number of IPM following $C$ from $\nu_{\text{ini}}$ to $\nu_{\text{fin}}$ is approximated as follows:

$$\# \text{ of iterations} \sim \frac{l_{PD}(\nu_{\text{ini}}, \nu_{\text{fin}})}{\sqrt{\beta}}$$

((The value of integral) $\sim$ (# of iterations when $\beta = 1$))

- By using Vavasis-Ye analysis, it can be shown that

$$l_{PD}(0, \infty) = O\left(n^{3.5} \log(\bar{x}_A + n)\right).$$

[Monteiro and Tsuchiya 2008]

- $l_{PD}(\nu_{\text{ini}}, \nu_{\text{fin}})$ is rigorously represented as differential geometric quantity [Kakihara, Ohara and Tsuchiya 2013] by using information geometry.
Lower Bounds I
Condition-Number vs Iteration Count

How many iterations needed to go from parameter $\mu_0$ to parameter $\mu_1$?

Lower bound:
$\min \text{ #pieces of any piecewise linear curve from } x_{\mu_0} \text{ to } x_{\mu_1} \text{ that stays inside some neighborhood.}$

...lower bound depends on which neighborhood we use.

$\min (c, x): A^T x \geq b,$
$d$ variables, $n$ inequalities
Central path can visit all (small) neighborhood of a variant of the Klee-Minty cube [Deza, Nematollahi and Terlaky 2008] \( n \sim m^3 2^{3m} \).

- Sonnevend curvature integral of a similar Klee-Minty type instance is exponential order [Mut and Terlaky 2014].

- It was an open problem to construct an instance with exponentially many sharp turns (in \( m \)) of central path with \( n = O(poly(m)) \). This problem is solved by using Tropical geometry [Allamigeon, Benchimol, Gaubert, Joswig 2018].

\[
\min (c, x) : A^T x \geq b, \\
d \text{variables}, n \text{ inequalities}
\]
New Prospect: Straight Line IPM Complexity (SLC)

- Initial \((x_0, \mu_0) \in N\) (a neighborhood of the path).

**Straight Line Complexity:**

\[
SLC(N, \mu_0) := \text{minimum \# of pieces of any piecewise-linear traversing } N \text{ from } (x_0, \mu_0) \text{ to } (x^*, 0)
\]

Multiplicative neighborhood \(N_\infty(\theta)\) for \(\theta \in (0,1)\):

\[
N_\infty(\theta): = \{(x, \mu): \frac{a_i^\top x_\mu - b_i}{a_i^\top x - b_i} \in [1 - \theta, (1 - \theta)^{-1}], i \\
\in [n]\}
\]

\[
\min(c, x): A^\top x \geq b,
\]

d variables, \(n\) inequalities
Straight Line Complexity to IPM complexity

**THEOREM**: (Allamigeon, Dadush, Loho, Natura, Végh ‘22):
Given \((x_0, \mu_0) \in N_2(0.1)\) there exists an IPM that stays in \(N_2(0.1)\) and solves the LP *exactly* using a number of iterations that is bounded by

\[ O(n^{1.5} \log \left( \frac{n}{1 - \theta} \right) \text{SLC}(N_\infty(\theta), \mu_0)), \forall \theta \in (0,1) \]

- Every ‘reasonable’ IPM traverses \(N_\infty(1 - 1/\text{poly}(n))\)
- How large can the straight line complexity \(\text{SLC}(N_\infty(1 - 1/\text{poly}(n)), \mu_0)\) be?
  - Vavasis-Y ‘96: \(O(n^{3.5} \log(n \bar{X}_A))\)
  - Dadush, Huiberts, N., Végh ‘20: \(O(n^{2.5} \log(n \bar{X}_A))\)
- vertices of the polytope, i.e. \(\binom{n}{d}\)
Which LPs are strongly polynomially solvable?

Strongly polynomial (known before 2022) LP in small dimension \( d = O(\log^2(n) / \log \log n) \)

### Strongly Polynomial Straight Line Complexity (SLC)

**Combinatorial LP:** \( A \) integral, \( \| A \|_\infty = 2^{O(\text{poly}(n))} \)

- Shortest Path
- Bipartite Matching
- Maximum flow
- Minimum-cost flow
- Multi-commodity flow
- Lattice polytopes
- Maximum generalized flow
- 2-variable-per-inequality feasibility systems
- Discounted Markov Decision Processes (MDP)
- Deterministic MDP

Klee-Minty cubes
Markov Decision Processes
Minimum-cost generalized flow

LP remains an open research field…