ES 203 Fall 2004

Introduction to Linear Control Theory

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Up until now, we've concentrated on the analysis of dynamical equations:

$$dx = f(x)dt + \sum_i g_i(x)dw_i$$

We've analyzed these equations, and how to solve them. Of course, if we weren't doing stochastic control, we'd be looking at more regular differential equations, like this

$$dx = f(x)dt.$$

Infact, in the last handout I told you all about how to solve the special case of the linear inhomogeneous equation:

$$dx = A(t)x(t) + f(t).$$

We saw that the solution relied on the Peano-Baker series, and basically boiled down (in most cases) to matrix exponentiation. In a large class of cases, we saw that

$$x(t) = e^{\int_0^t A(s)ds}x(0)$$

for the homogenous equation.

Today, we're going to look at something different, something that will be useful later on in this course: control theory. Instead of just analyzing equations, we're going to be able to add control terms that can change the evolution of trajectories, and we're going to ask about what various control can do for us.

There are four fundamental problems in control theory:

- *Servomechanism* control of the path.
- *Ballistic* control of the system at specific time points.
- Optimal Control under some optimization criteria.
- Filtering and estimation in on-board control systems.

Today, we're going to consider the second problem, in the linear case. This problem is the typical standard problem. It is largely tractable, and a beautiful and clean theory has been developed to solve it. If you ever questioned the importance of learning linear algebra, this theory will end that questioning.

Consider the following equation:

$$\dot{x} = A(t)x(t) + B(t)u(t)$$

in which u is a real-valued vector function of time that you get to pick.

We want to study when we can, and how to, choose the function $u(t)$ such that if the system starts at $x(0) = x_0$, the system satisfies $x(t^*) = x^*$, a configuration of interest at a time of interest.

To begin, following Brockett, let's consider a special case of this so-called ballistic problem. Suppose

$$\dot{z}(t) = B(t)u(t) \quad z(0) = z_0$$

is an m-dimensional vector differential equation system and suppolse that $B(t)$ is an $n \times m$ continuous matrix function of time.

Given a value z^*, when can we find $u(t)$ so that the system is ensured to satisfy $z(t') = z^*$? This is the fundamental control problem at hand. Well, the obvious thing to do is to integrate:

$$z(t') = z_0 + \int_0^{t'} B(t)u(t)dt.$$

But now, let's play a linear algebra game. Consider the infinite dimensional vector space $C^m(0,t)$. But now, let L be the operator $L : C^m(t_0, t_1) \to \mathbb{R}^n$ defined by

$$L(u) = \int_{t_0}^{t_1} B(t)u(t)dt.$$

Because $B(t)$ is a matrix for each t, this is a linear operator as can easily be checked, between two well-behavior vector spaces. Now, it is clear that our basic control problem has a solution if and only if

$$z^* - z_0 \in Range(L).$$

What can we do with this? Well, consider the following (for the moment unsubstantiated) definition:

$$W(t_0, t_1) = \int_{t_0}^{t_1} B(t)B^\dagger(t)dt.$$

The point is that

$$Range(W(t_0, t_1)) = Range(L).$$

Why is this? Let's provide this be sandwiching inclusions. The easy direction is

$$Range(W) \subset Range(L).$$

Suppose $x \in Range(W)$. Then let $u = B^\dagger \eta$ where $W\eta = x$.

On the other hand, suppose $x \notin Range(W)$ but $x \in Range(L)$. Then there is an y such that $Wy = 0$ but $y^\dagger x \neq 0$ and a u such that $Lu = x$. Now consider $y^\dagger Lu$. We get

$$\int_{t_0}^{t_1} y^\dagger B(t)u(t)dt = y^\dagger x \neq 0.$$

On the other hand, now consider $y^\dagger W y$.

$$0 = y^\dagger W y = \int_{t_0}^{t_1} [y^\dagger B(t)][B^\dagger(t)y]dt = \int_{t_0}^{t_1} ||B^{da}(t)y||^2 dt.$$

But since $B(t)$ is continuousm this means that $B(t)y = 0$ for all y. But this contradicts that

$$\int t_0{}^{t_1} y^\dagger B(t)u(t)dt \neq 0.$$

But this means that the control problem can be solved by evaluating a matrix integral and asking if the required difference vector is in the range of that matrix. That is:

$$z^* - z_0 \in Range(\int_{t_0}^{t_1} B(t)B^\dagger(t)dt).$$

And of course, if the transfer is possible than one transfering control is

$$u(t) = B^\dagger(t)\eta$$

where η is a solution of the linear equation

$$W(t_0, t_1)\eta = z^*.$$

Question: what happens if $B(t)$ is full-rank for all $t \in [t_0, t_1]$?

Now, we'd like to be able to deal with this problem in the case that there is drift, that is, a non-zero $A(t)$ term. Let's go back to the equation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t).$$

When can x_0 at t_0 be brought to x^* at t_1? We will now use a dirty trick!! We're going to change variables and treat this as if it were an inhomogeous equation.

Let $\Phi(t_0, t)$ be the transition function for the homogenous non-control system. (You all should either know about this or look in my handout.)

Then define $z(t)$ be the equation

$$\dot{z}(t) = \Phi(t_0, t)B(t)u(t).$$

The key fact is that

$$x(t) = \Phi(t, t_0)z(t).$$

Who can prove this?

Now, from the previous result, the z system can be brought from z_0 to z^* in the required time iff $z^* - z_0$ is in the range of

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t)B(t)B^\dagger(t)\Phi^\dagger(t_0, t)dt.$$

So the desired problem in x is doable iff $x_0 - \Phi(t_0, t_1)x^*$ is in the range space of $W(t_0, t_1)$. And of course, one such control accomplishes the transfer is

$$u(t) = -B^\dagger \Phi^\dagger(t_0, t)\eta$$

where η is a solution of the linear equation $x_0 - \Phi(t_0, t_1)x^* = W(t_0, t_1)\eta$.

So this is a BIG DEAL!

Another way to see this is the basic theorem from linear algebra:

$$Range(L) = Range(LL^*).$$

Because of this, $W(t_0, t_1)$ is called, as a function of t_0 and t_1, the *control grammian* of the system. Linear control boils down to studying the function W – and it will be important in optimization and non-linear control as well.

Here are some important formal properties of W.

- W is symmetric.
- W is non-negative definite.
- W satisfies the linear-matrix equation:

$$\dot{W}(t, t_1) = A(t)W(t, t_1) + W(t, t_1)A^\dagger(t) - B(t)B^\dagger(t).$$

- $W(t_1, t_1) = 0$.
- W satisfies the functional equation

$$W(t_0, t_1) = W(t_0, t) + \Phi(t_0, t)W(t, t_1)\Phi^\dagger(t_0, t).$$

Let's do the proof of some of these.

Now, what happens when A and B are constant matrices? In this case, the solutions simplify, as might be expected.

Theorem 1 *For A and B constant matrices and A being $n \times n$, then the system*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable from x_0 at t_0 to x_1 and t_1 IFF

$$x_1 - x_0 \in Range(C)$$

where

$$C = (B|AB|A^2B|\ldots|A^{n-1}B).$$

Let's look at the proof: it's based on proving that $W(t_0, t_1)$ has the same nullspace of $CC^\dagger$; then noting that since W is symmetric, they have the same range space; and then that $Range(CC^\dagger) = Range(C)$.

Of course this has the corollary that the system x is totally controllable iff $Rank(C) = n$.

If we have time, let's look at neat consequence of this way of thinking:

Proposition 1 *If the system*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable, then given any non-zero colunmn b_i of B, there is a matrix C_i such that

$$\dot{y}(t) = (A + BC_i)y(t) + b_i v(t)$$

is also controllable.

If we have time, we can go over the main results of linear observability.