

Optimal Control Theory is Based on a Variational PDE

In this problem, we show how optimal in control theory can be viewed as solutions to a particular partial differential equation; and then show that the Euler-Lagrange equations of variational calculus can easily be derived from this point of view.

0.1 The HJB Equation

Consider the system given by the following non-linear vector differential equation:

$$\dot{x} = f(x(t), u(t), t); \quad x(t_0) \text{ given}$$

where $u(t)$ is some control parameter that the user gets to set.

The fundamental problem of optimal control theory is to find $u(t)$ under which the trajectories generated by the above system will minimize a following *performance index* of the form:

$$J = \phi(x(t_f), t_f) + \int_t^{t_f} L(x(s), u(s), t) ds$$

while subject to boundary conditions

$$\psi(x(t_f), t_f) = 0.$$

In the above, the functions ϕ , L , and ψ are just given functions (about which we know only that they satisfy some conditions on their smoothness). They can be thought of as the *fixed-cost*, *running cost* (or Lagrangian) and *infinite boundary costs*, respectively.

How are we to find the optimal $u(t)$ control function? Define the *optimal return function* by

$$J^*(x, t) = \min_{u(t)} \left\{ \phi(x(t_f), t_f) + \int_t^{t_f} L(x(s), u(s), t) ds \right\}.$$

This function of x and t represents the best possible return one could expect for a trajectory starting (x, t) . In other words, if one chose the optimal control $u(t)$ from some point in time t_0 , from a given point x_0 , then $J^*(x_0, t_0)$ is the value of J evaluated along the trajectory up to t_f . For the purposes of this problem, we will assume that J^* exists, is continuous, and has continuous second partial derivatives.

In this problem will see that J^* obeys a non-linear PDE: and in the course, lay the foundations for the most general formulation of the concept of Hamiltonian energy.

1. Suppose the system starts at a point (x, t) . Assume that the optimal control is taken from this point on. Given the assumption that J^* is continuous and has continuous partial derivatives, write its Taylor expansion about some given point (x, t) . In the Taylor expansion, expand both J^* and the trajectory of x to first order in t . In other words, get a formula of the kind

$$J^*(x(t + \delta t), t + \delta t) = J^*(x, t) + \delta t(\text{First Order Term}) + O(\delta t^2)$$

in which the First Order Term is the minimization over choices of u of expression involving derivatives of J^* at (x, t) .

2. Now, notice that there is a *very simple relationship* between $J^*(x(t + \delta t), t + \delta t)$ and $J^*(x, t)$, given the assumption that the optimal control has been taken from (x, t) onward. Using this simple relationship, show that in the limit that $\delta \rightarrow 0$, we have

$$-\frac{\partial J^*}{\partial t} = \min_u \left\{ L(x, u, t) + \frac{\partial J^*}{\partial x} f(x, u, t) \right\}.$$

3. Hold the thought from the previous part. In the meantime, let's adjoin something to the performance function:

$$\bar{J} = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(s), u(s), s) + \lambda^T(s)(f(x(s), u(s), s) - \dot{x}(s)) ds;$$

0.2 Deriving the Euler-Lagrange Equations

in other words, we've adjoined the system's differential equation $f - \dot{x}$ multiplied by a multiplier λ . Of course, in our system this \bar{J} is the same as J , since we've added something identically 0.

Now, define a new scalar function H (the *Hamiltonian associated with L*) by

$$H(x, u, \lambda, t) = L(x, u, t) + \lambda^T(t)f(x, u, t).$$

Using the variational techniques from class and manipulating \bar{J} , show that the choice of multiplier λ satisfying the ODE

$$\dot{\lambda}(t) = -\left(\frac{\partial f}{\partial x}\right)^T \lambda(t) - \left(\frac{\partial L}{\partial x}\right)^T$$

subject to the boundary condition

$$\lambda(t_f) = \left(\frac{\partial \phi}{\partial x}\right)_{t=t_f}^T$$

gives a stationary value of the original performance function. Notice that a stationary control $u(t)$ is given by solving

$$\frac{\partial H}{\partial u} = 0.$$

4. Show that it is possible to interpret the previous result as saying that

$$dJ^* = \lambda^T(t)dx - H(t)dt.$$

Then use this to show that

$$\lambda^T(t) = \frac{\partial J^*}{\partial x}$$

on the optimal trajectory and that

$$H = -\frac{\partial J^*}{\partial t}.$$

We can therefore write

$$-\frac{\partial J^*}{\partial t} = H^*\left(x, \frac{\partial J}{\partial x}, t\right)$$

where H^* is the minimizing value of the Hamiltonian H .

This is the famous *Hamilton-Jacobi-Bellman* equation; its solutions are the optimal values of the performance function.

5. What are the relevant boundary conditions with respect to which the HJB equation must be solved?

0.2 Deriving the Euler-Lagrange Equations

Use the following procedure to derive the Euler-Lagrange equations from the HJB equation.

1. Notice that along an optimal path, $\frac{\partial H}{\partial u} = 0$. Expand the LHS of this, and get one of the Euler-Lagrange equations.
2. Show that along an optimal path

$$\dot{\lambda}^T = \frac{\partial^2 J^*}{\partial x^2} \dot{x} + \frac{\partial^2 J^*}{\partial x \partial t}.$$

3. Take partial differentials of both sides of the HJB equation with respect to x . Use the fact that $\frac{\partial H}{\partial u} = 0$ along optimal trajectories to get that the coefficient of $\frac{\partial u}{\partial x} = 0$ in the result of above.
4. Plug the result of the previous part into the second part to get the other Euler-Lagrange equation.

0.3 An Application

Use the Hamilton-Jacobi-Bellman equation to solve the following optimal control problem.

Consider the system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -ku^2 & -uv \\ uv & -kv^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

in which k is a positive constant. From initial position $(0, 1)$, find $u(t), v(t)$ which maximize the value of $|x(t = \infty)|$, and for which $0 \leq |u(t)|, |v(t)| \leq 1$ for all t . And, what is the minimal value of J ?

Hint: To answer this question, formulate this problem in the form of the previous section, solve the HJB equation and find the explicit form of $J^*((0, 1), 0)$. Then, use this to find solve for the optimal controls by using that $\frac{\partial H}{\partial u} = 0$ for the optimal paths. (You should find that the controls spend a long time on the border of the square $[0, 1] \times [0, 1]$.)