Let S and Ω be anti-symmetric real matrices of dimensions $n \times n$ and $k \times k$ respectively. Furthermore, let G and B be real matrices such that

$$rank([G|SG|S^2G|\dots|S^{n-1}G]) = n$$

and

$$rank([B|\Omega B|\Omega^2 B|\dots|\Omega^{k-1}B] = n + k.$$

Of course, the dimensions of G are $n \times r$ for some r, and the dimensions of B are $n \times k$.

Consider the following stochastic differential equation system:

$$\begin{bmatrix} dx \\ dz \end{bmatrix} = \begin{bmatrix} S - GG^T & B \\ -B^T & \Omega \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} Gdw \\ 0 \end{bmatrix}.$$

Find the steady-state variance for this system. Extra credit will be given for interpreting the physical meaning of the result that you get. You may use the following theorem:

Theorem 1 *Let*

$$\dot{x}(t) = Ax(t) + Bu(t)$$

be a controllable time-invariant system in which $A^T = -A$. Then all solutions to the system

$$\dot{z}(t) = (A - BB^T)z(t)$$

converge to zero as $t \to \infty$.

I suggest following these steps:

1. Use the Ito equations to set up a diffential equation for the second moment $E[(x\ z)(x\ z)^T]$ of the system. This equation will be of the form

   $$L(X) = AX + XA^T + C$$

   where A and C are matrices and $X = [x\ z]^T$. To do this part of the problem you just need to find A and C.

2. Find, by inspection, a solution to setting $L(X) = 0$. HINT: The solution is extremely simple. Try the simplest invertible matrix you can think of, and then tweak it by a constant factor.

3. All you have to do now is show that the solution you found in the previous part is unique. Toward this end, notice that $L(X)$ is a linear operator and solutions to

   $$L(X) = AX + X^T + C = 0$$

   are unique iff A has no zero equigenvalues. So now you must simply show that A has no zero eigenvalues. You should show in fact that the real parts of the eigenvalues of A are strictly negative. To do this, suppose otherwise. That is, that there is a nonzero eigenvector $X^* = [x^*\ z^*]^T$ such that

   $$AX = \lambda X$$

where $Re(\lambda) \geq 0$. Work out, using the expression for A that you derived in the first part, what this actually means in terms two linear equations relating x^* and z^*.

4. Now, consider the theorem noted above. It has a consequence for the sign of the real part of the eigenvalues of the matrix $A - BB^T$. Use this to draw conclusions about the real parts of the eigenvalues of the matrices $S - GG^T$ and $\Omega - B^T R^T RB$ where R is *any* invertible matrix.

5. Use this to be able to invert certain matrices in the two linear equations you derived, and get a single linear eigenvalue equation in one of the variables, say z^*. Finally, use the same information about the real parts of the eigenvalues of those matrices to derive a contradiction from assuming that $Re(\lambda) \geq 0$ in this eigenvalue equation for z^*.

As for interpretation: think in terms of statistical thermodynamics and the famous "equipartition" theorem. Ω is a heat bath and B is a term coupling the stochastic system

$$\dot{x} = (S - GG^T)x + Gdw$$

in to the heat bath.