A Theory Approach to Local-to-Global Algorithms in Spatial Multi-Agent Systems

CS266, Fall 2007

Dan Yamins

Session 11: 12.06.2007

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Task (or "functionality")

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Last time ....

... the model.

Today ...

... some results.

Local Checkability

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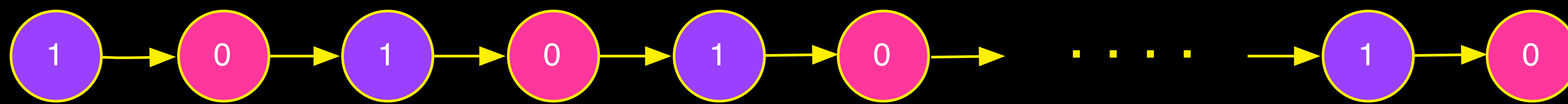
Let's take the simple 1-D repeat pattern T_{10}:



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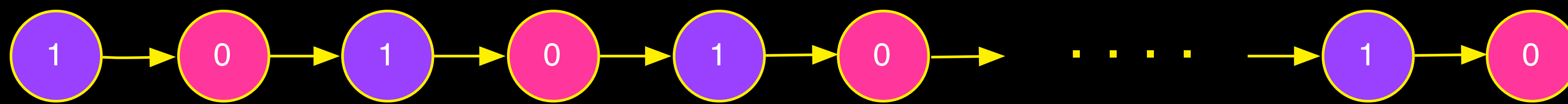

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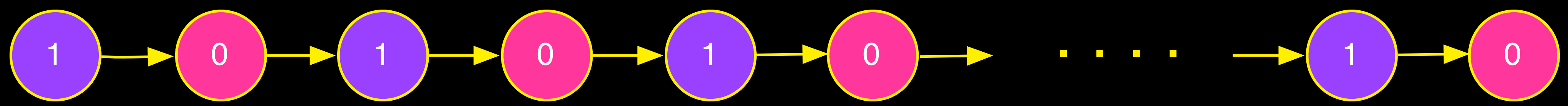

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Answer:

$$F(B) = \begin{cases} 1 - B(-1), & B \neq \text{left-end agent} \\ 1, & B = \text{left-end agent} \end{cases}$$

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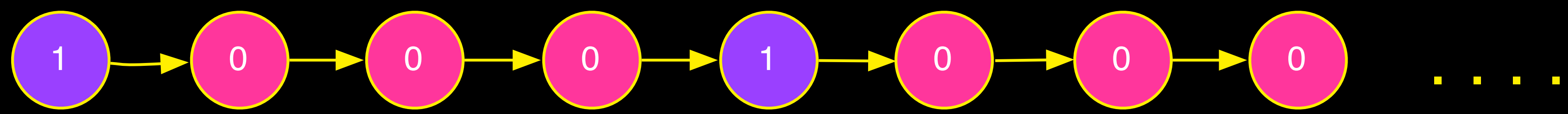

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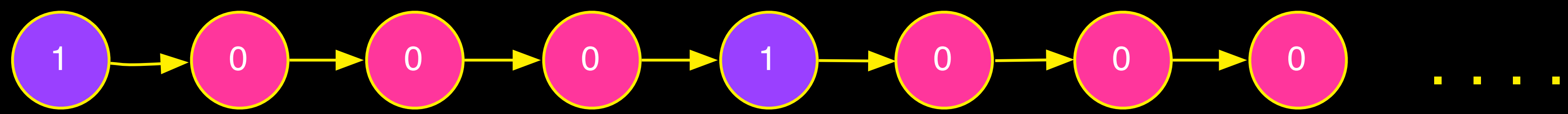

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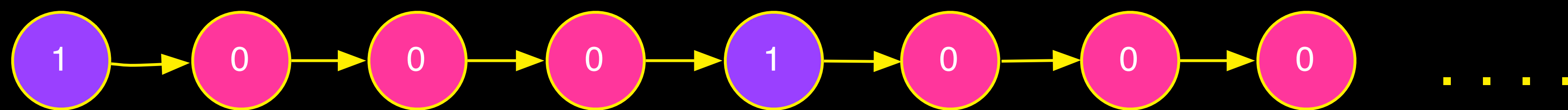
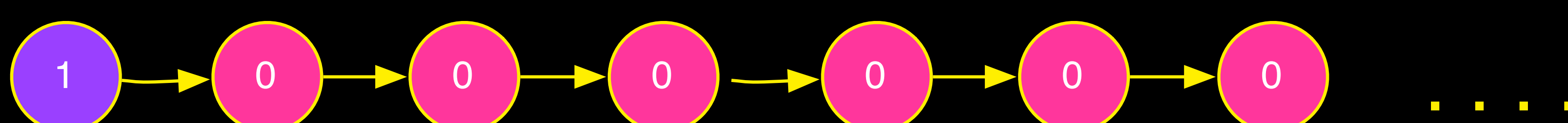


Can this pattern be solved robustly with a nearest-neighbor rule?

Answer: No. Because the with a radius 1 rule, 000 would have to be a fixed state.



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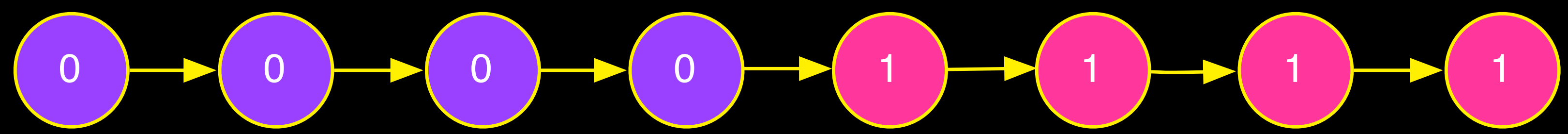

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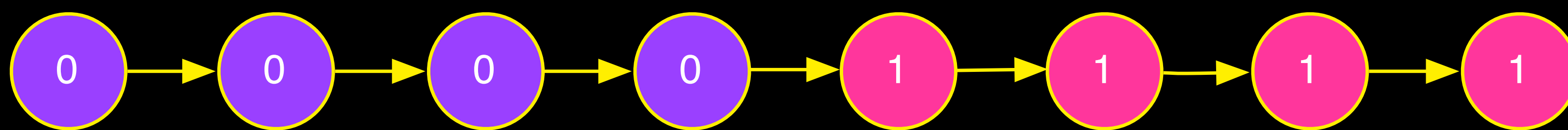

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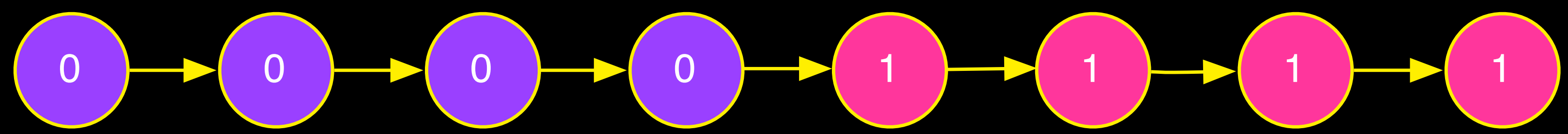

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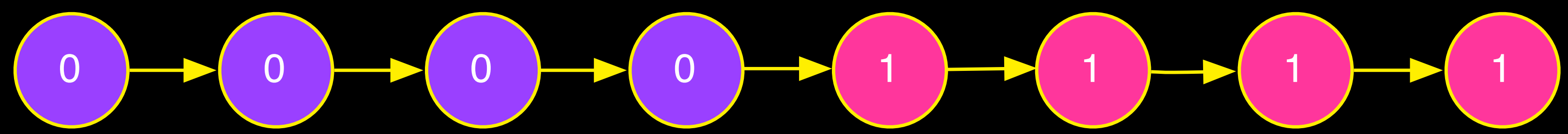

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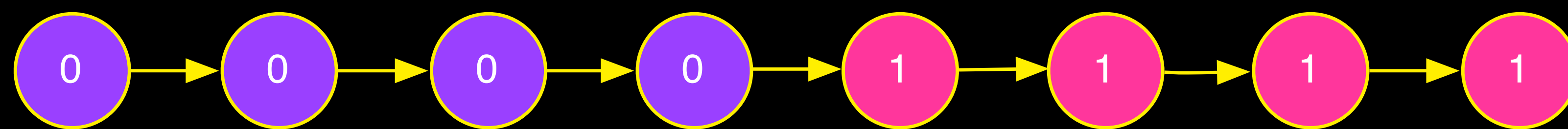

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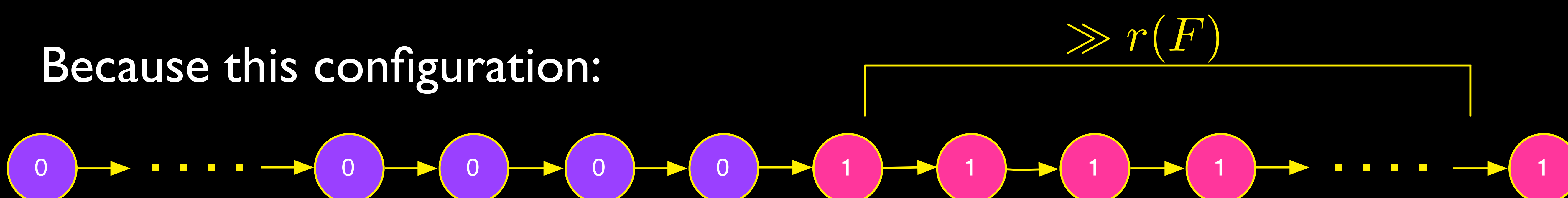

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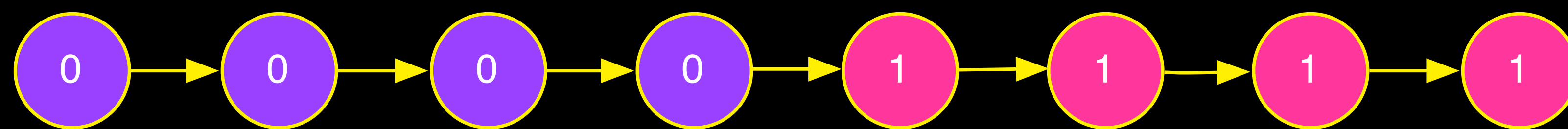

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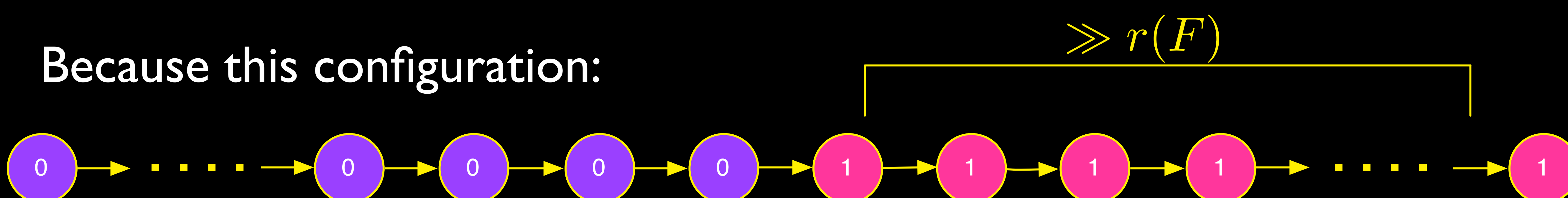

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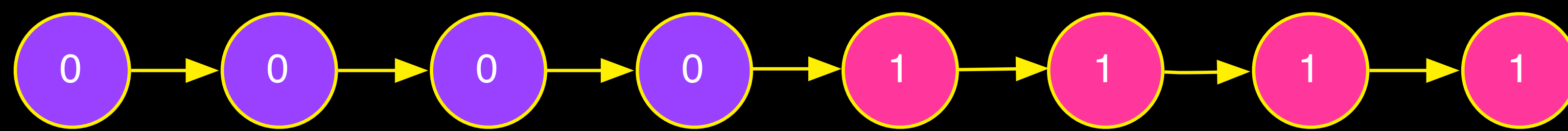

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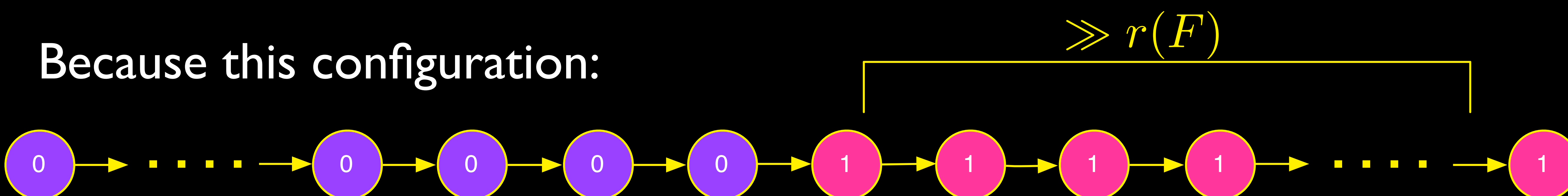

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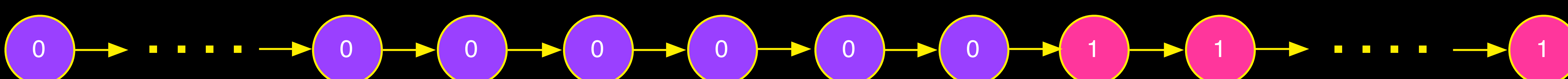

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- $\Theta[X] = \bigwedge_{i \in V(X)} (\Theta(B_r(i, X)) = 1) \quad \Rightarrow \quad X \in T$ *and*
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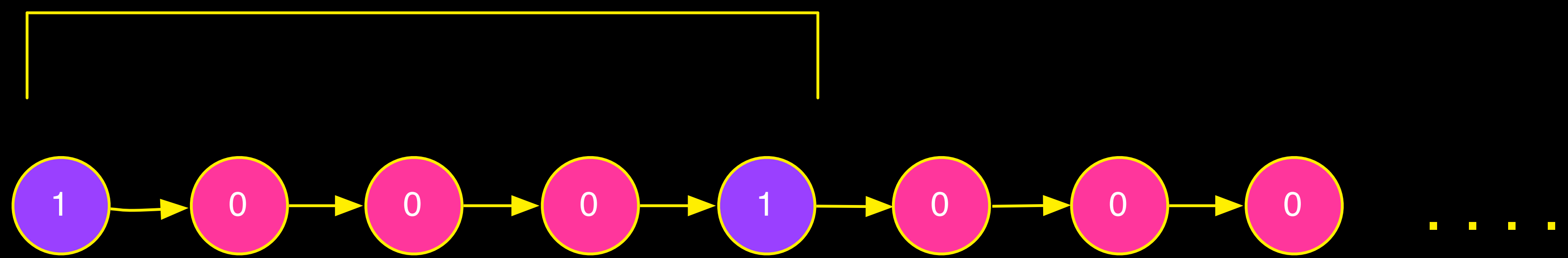

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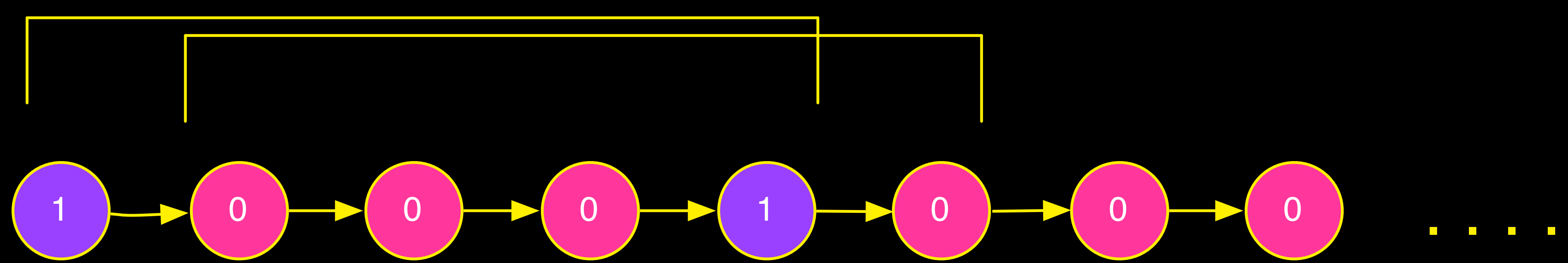

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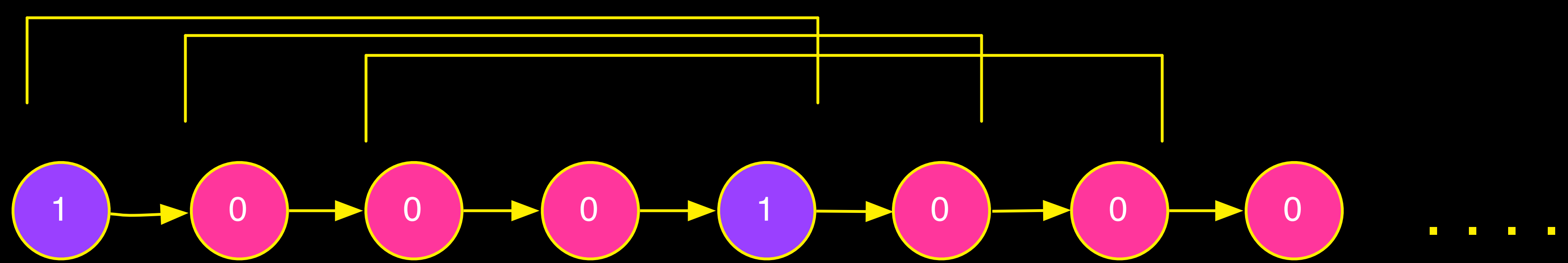

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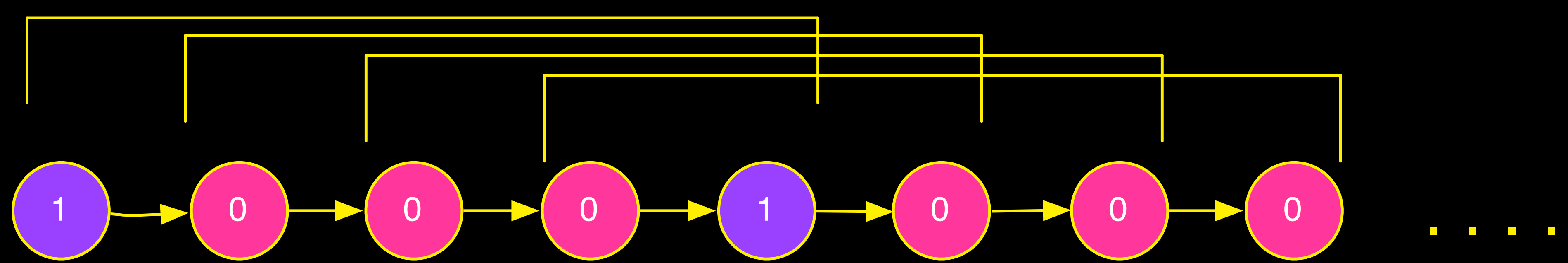

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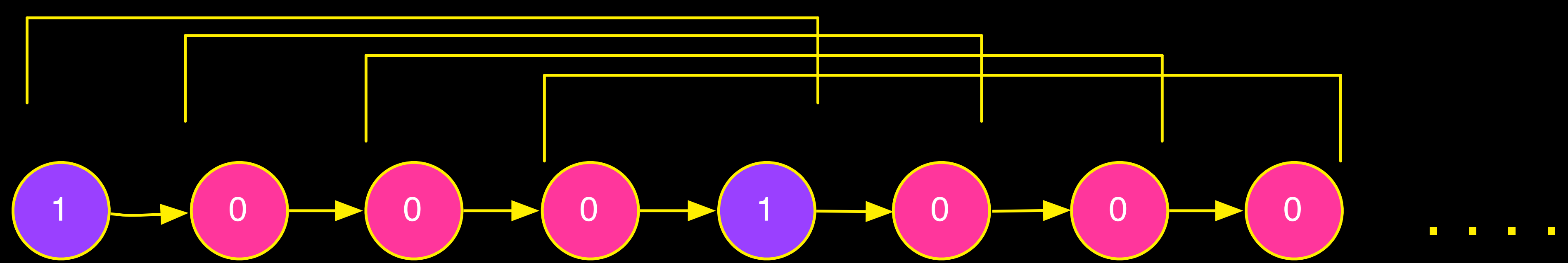

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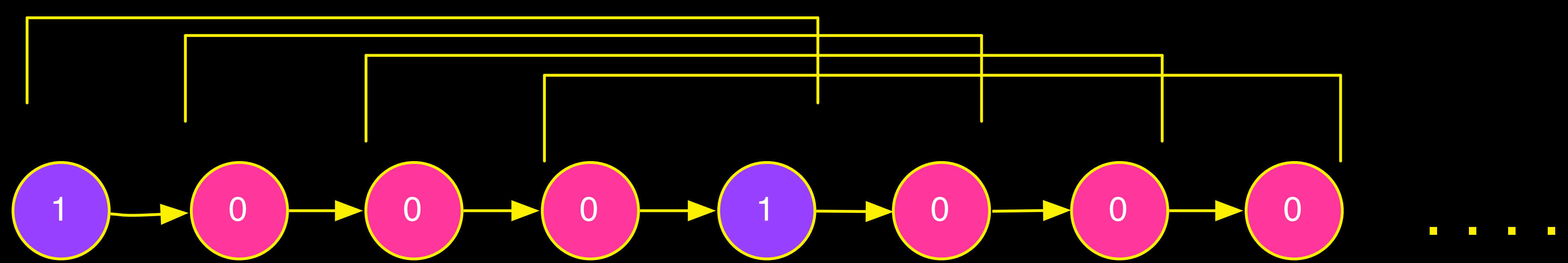

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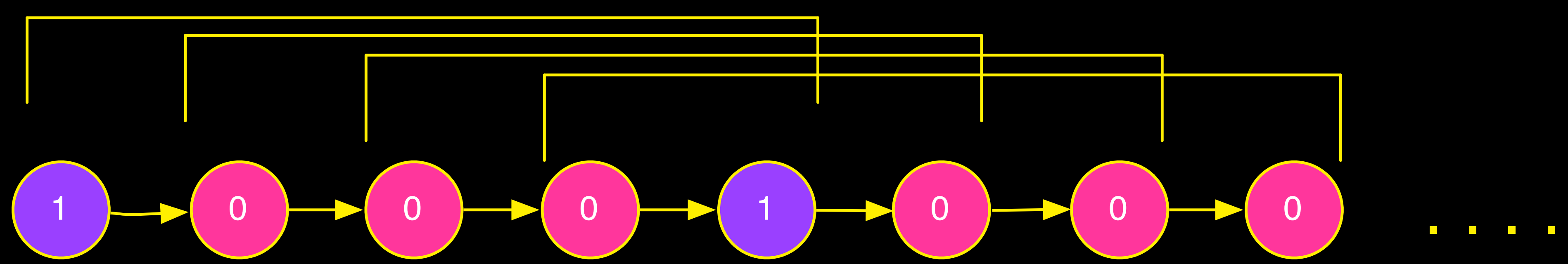

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Let LCR(T) denote the minimal radius of a check scheme for it -- this is T's "local check radius." T is "locally checkable" if LCR(T) is finite.

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Obvious next questions: 1) What kinds of patterns are locally checkable? And: 2) When is Local Checkability sufficient? Can we obtain sufficiency by making generic constructions?

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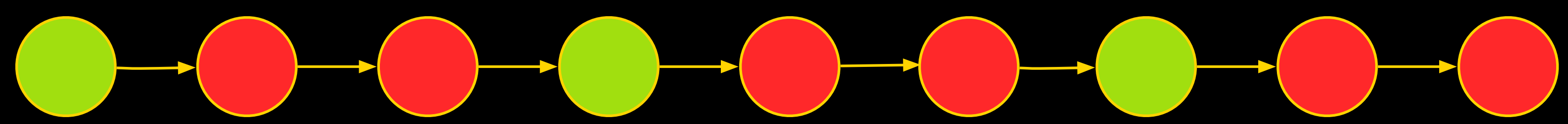

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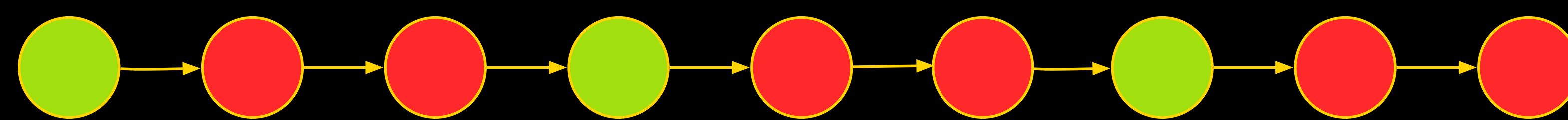

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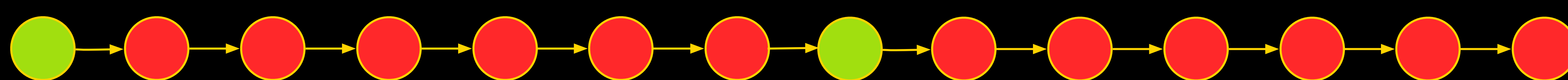

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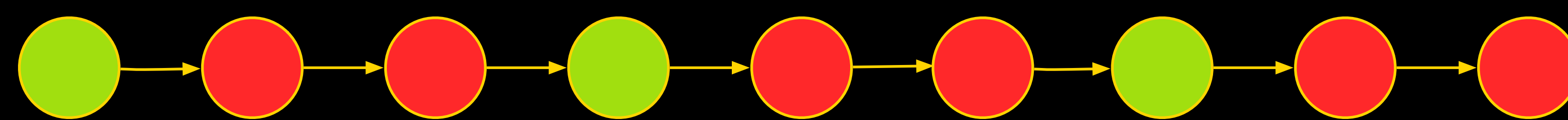

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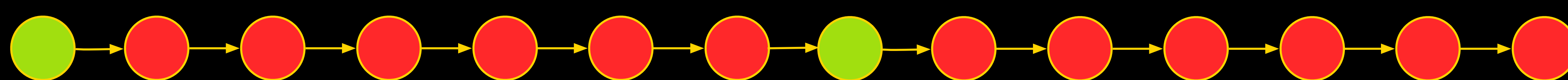

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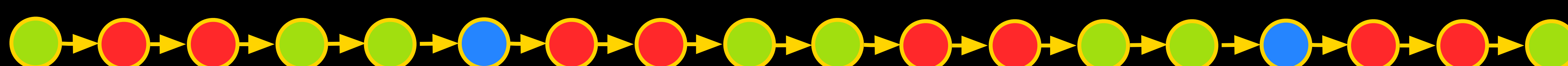

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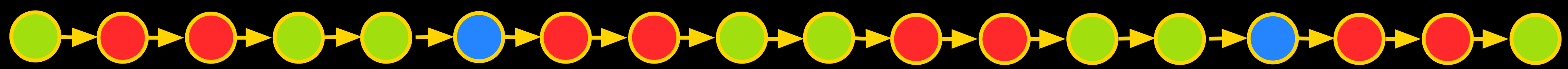

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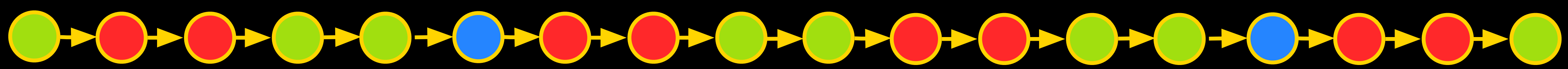

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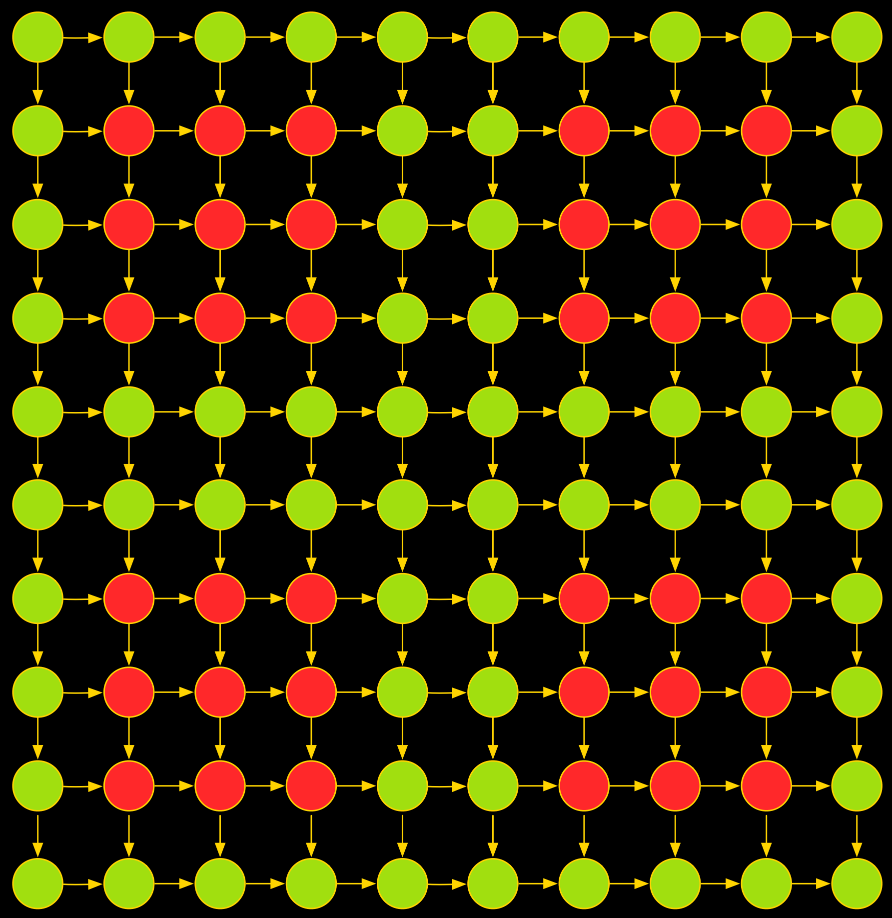

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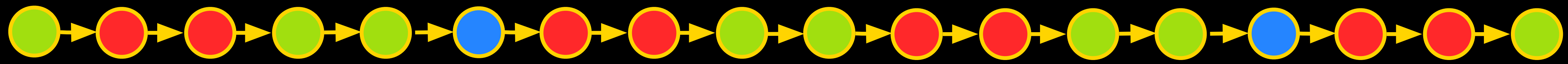

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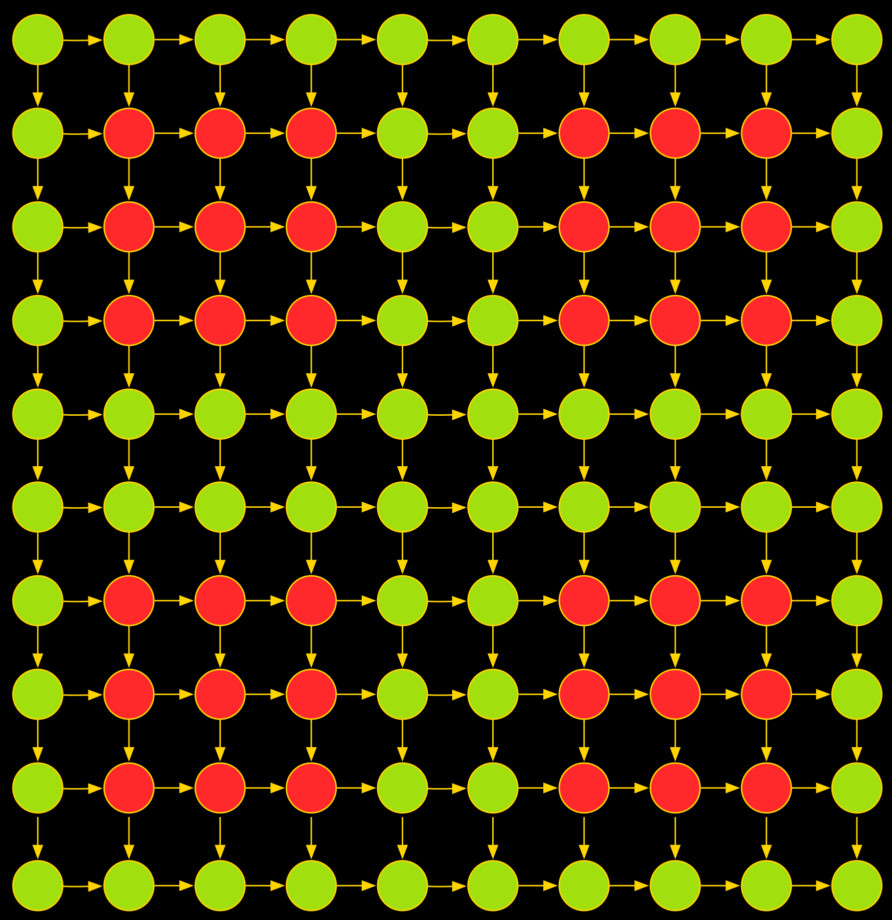

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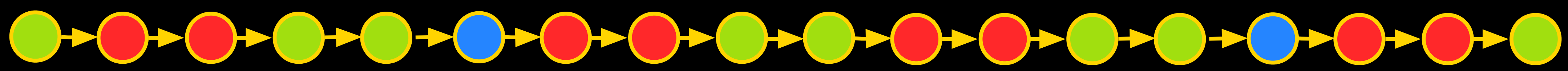

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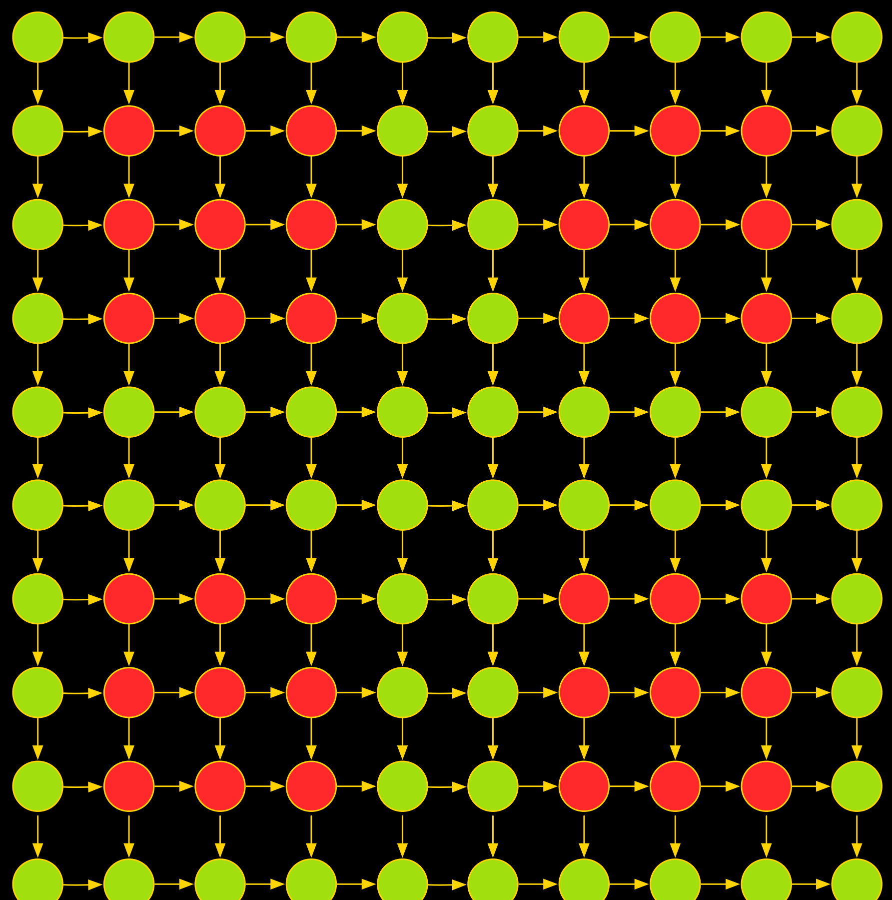

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In fact, whenever "repeat" is defined,

$$LCR(T_q) \leq |q|/2$$

where q is the unit being repeated.

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- Hence,

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... so all 1-D check schemes are combinations of things with periodicities

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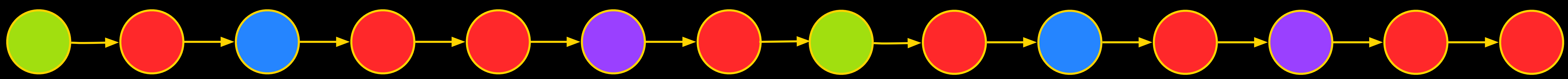

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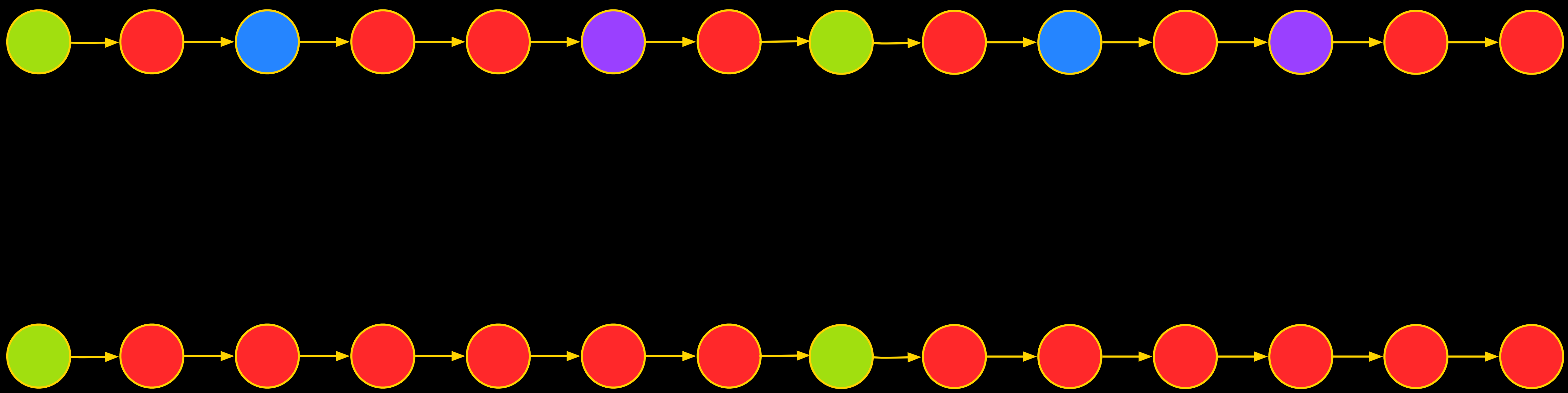

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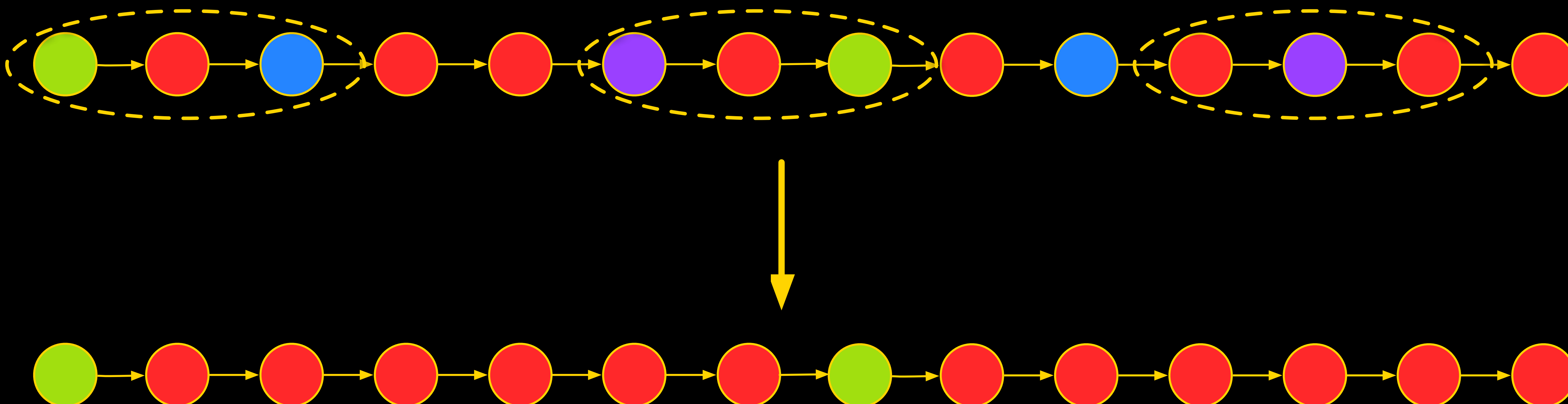

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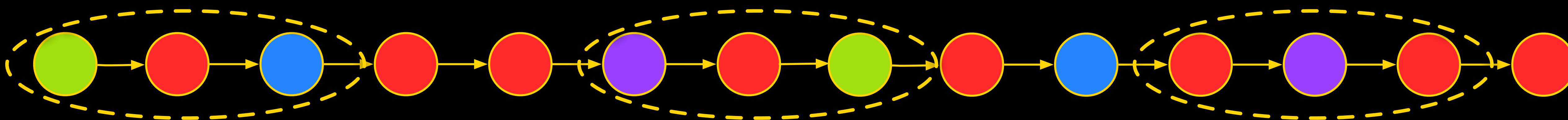
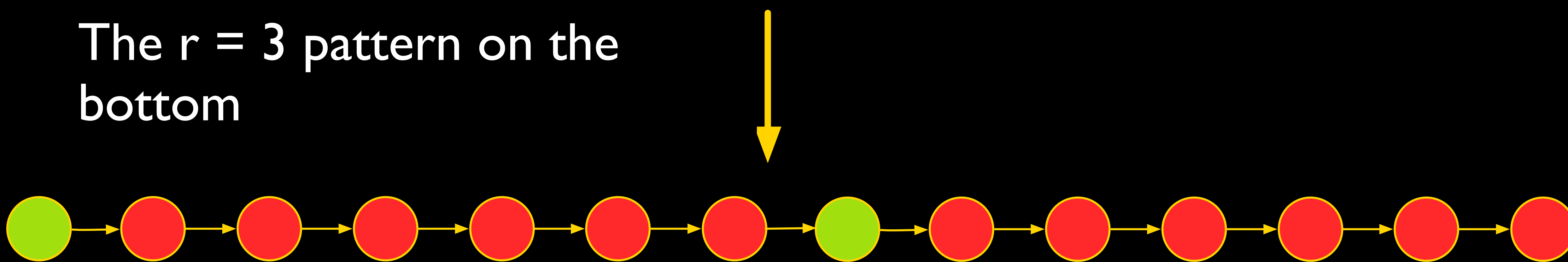


The r = 3 pattern on the bottom



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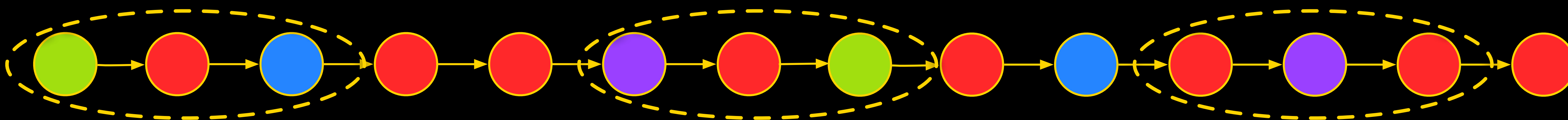

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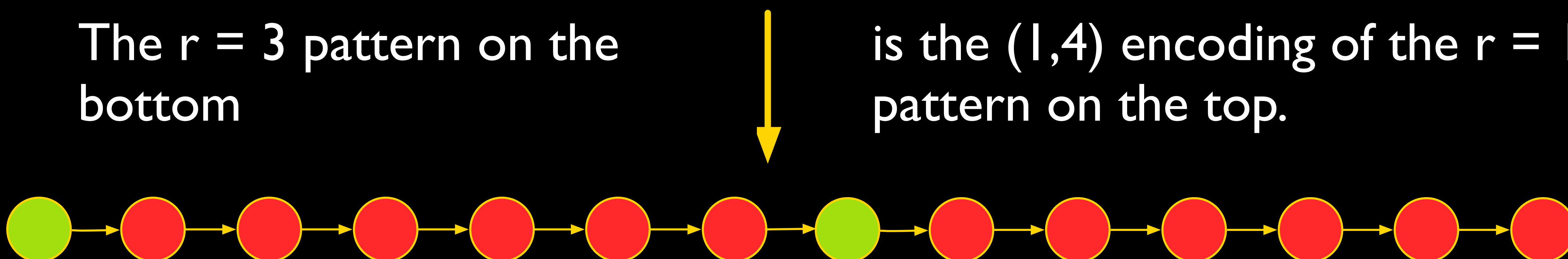

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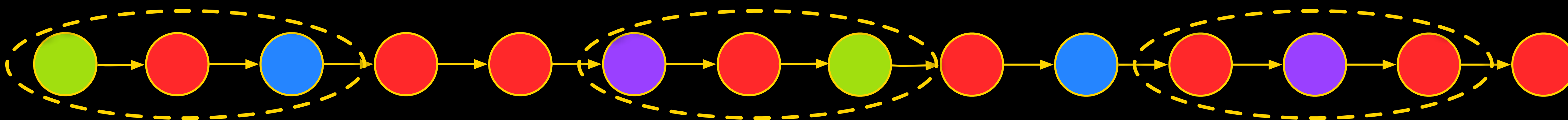

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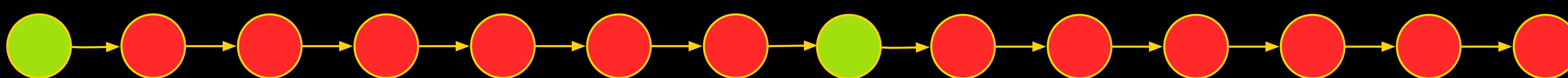

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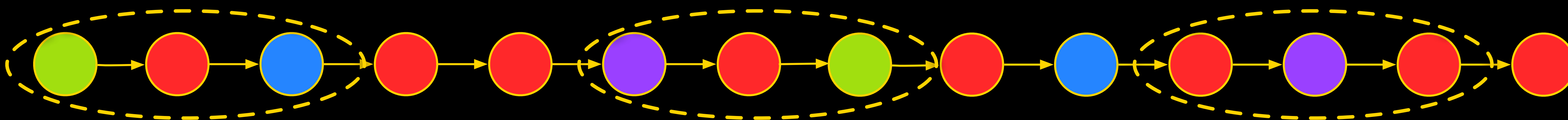

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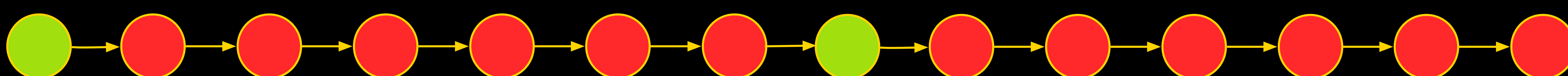

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Proposition. *In 1-D, all local encodings of locally checkable patterns are again locally checkable.*

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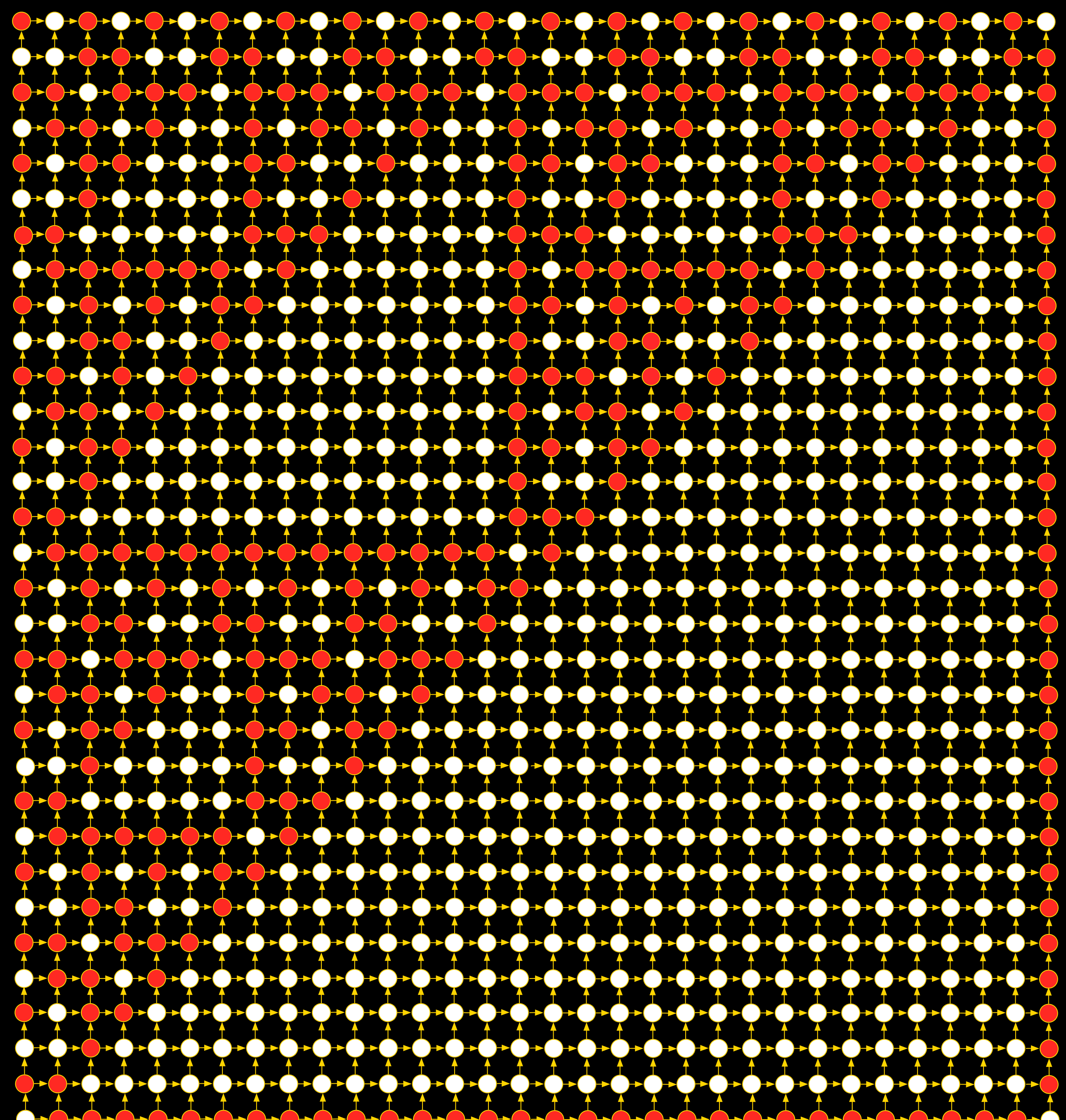

The Sierpinski Gasket has a radius-one check scheme.

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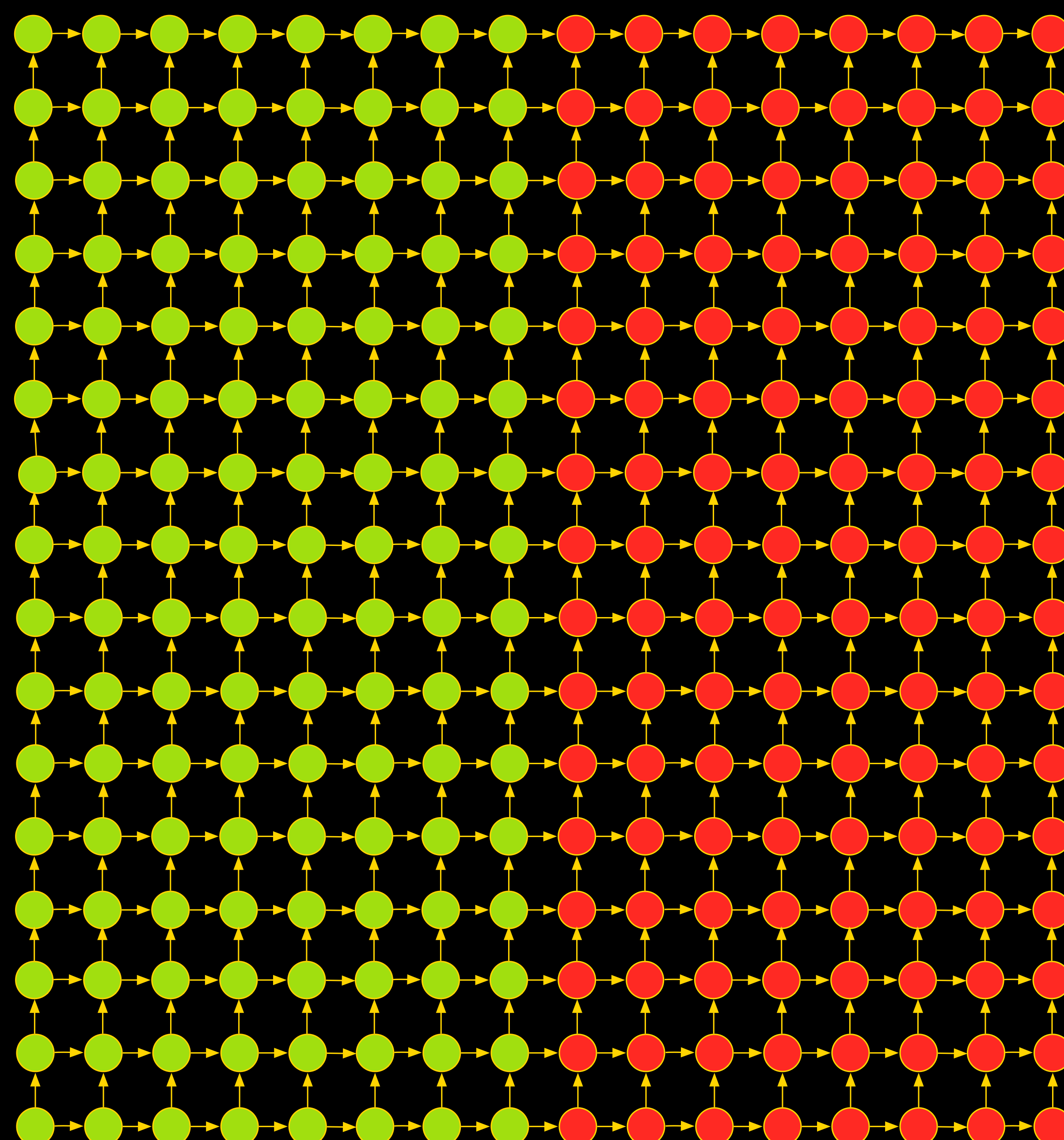

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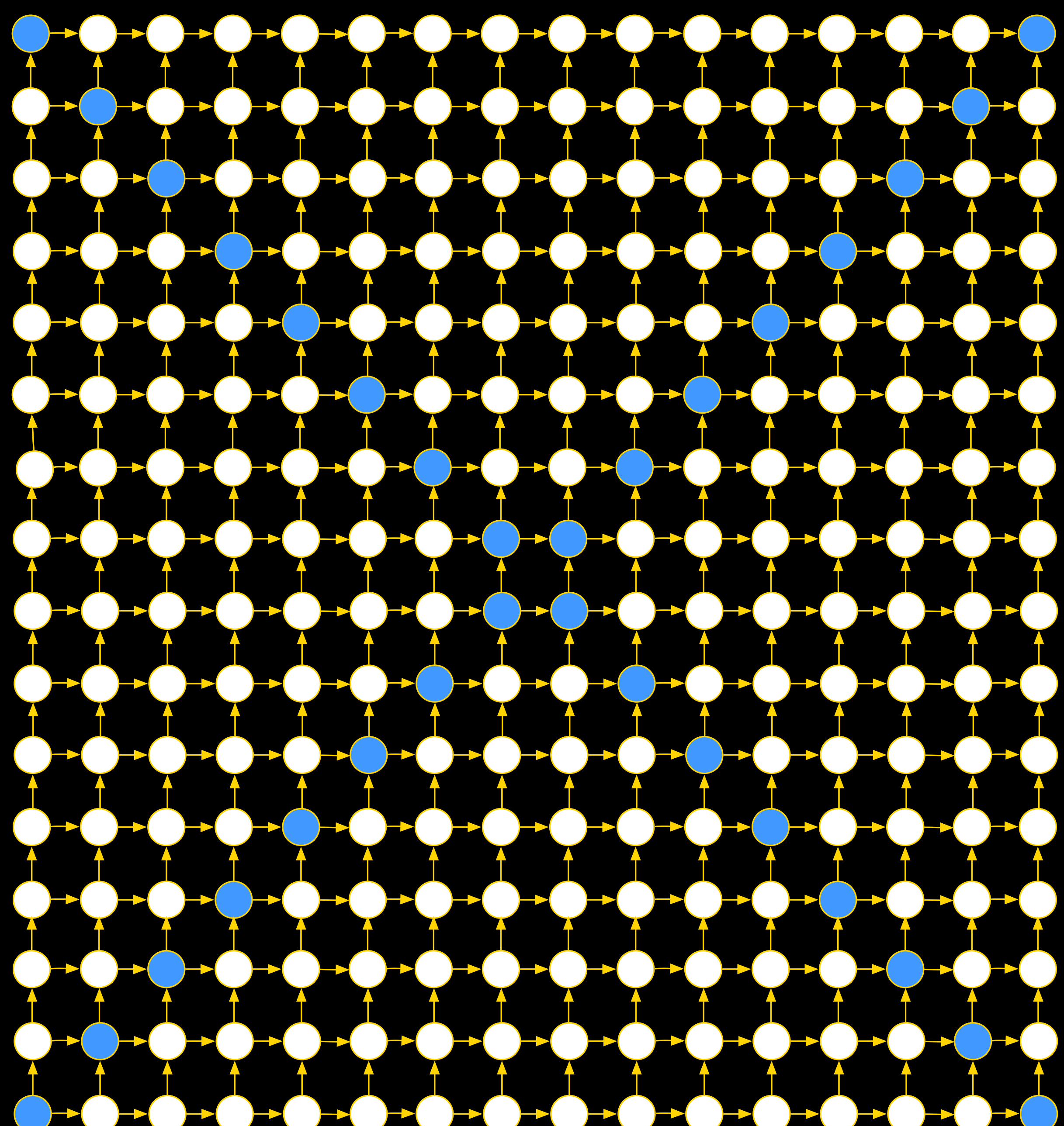
The Cross Pattern ($r = 1$, $m = 2$)



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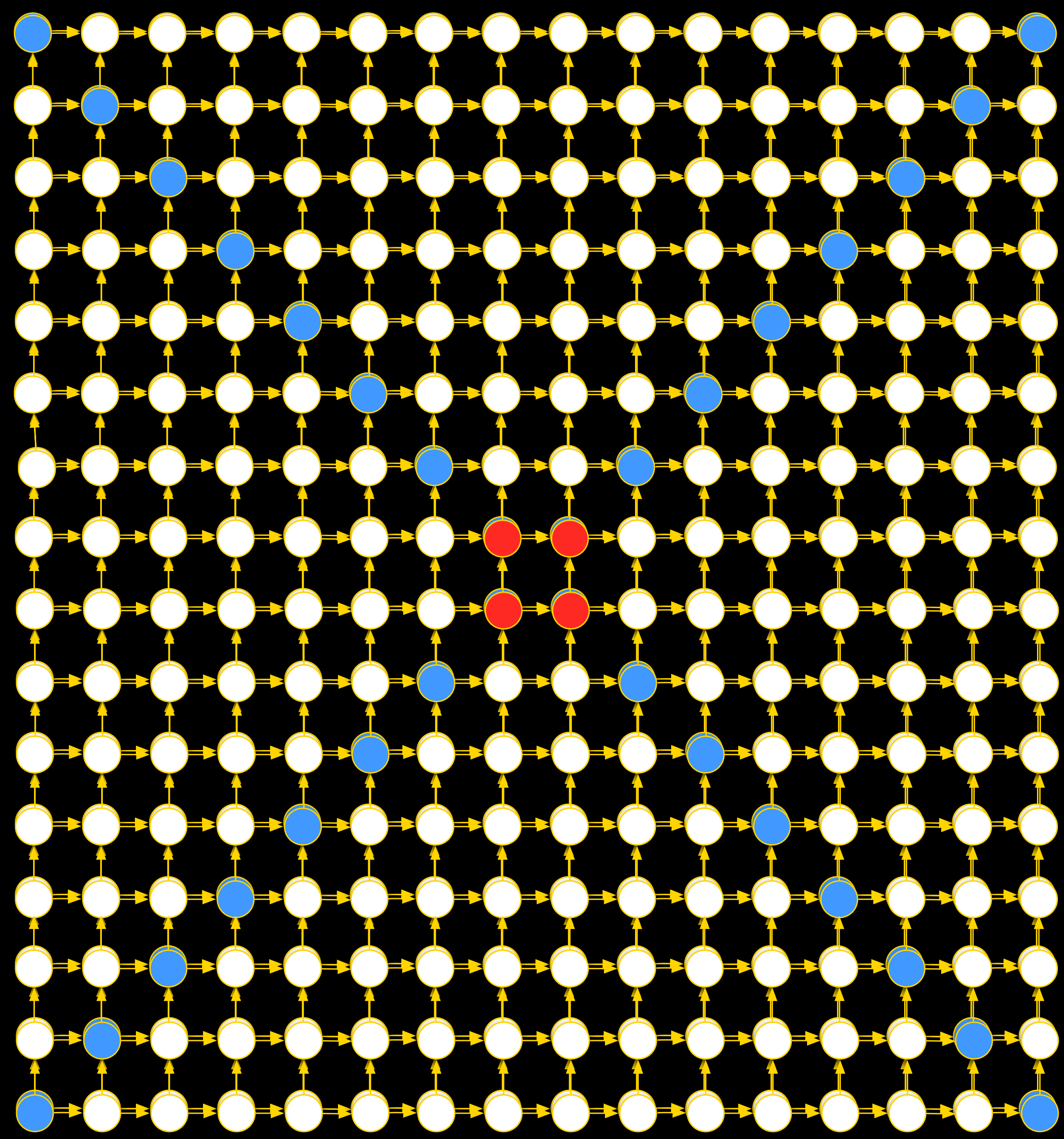
Center-Marked Pattern ($r = 1$, $m = 3$)



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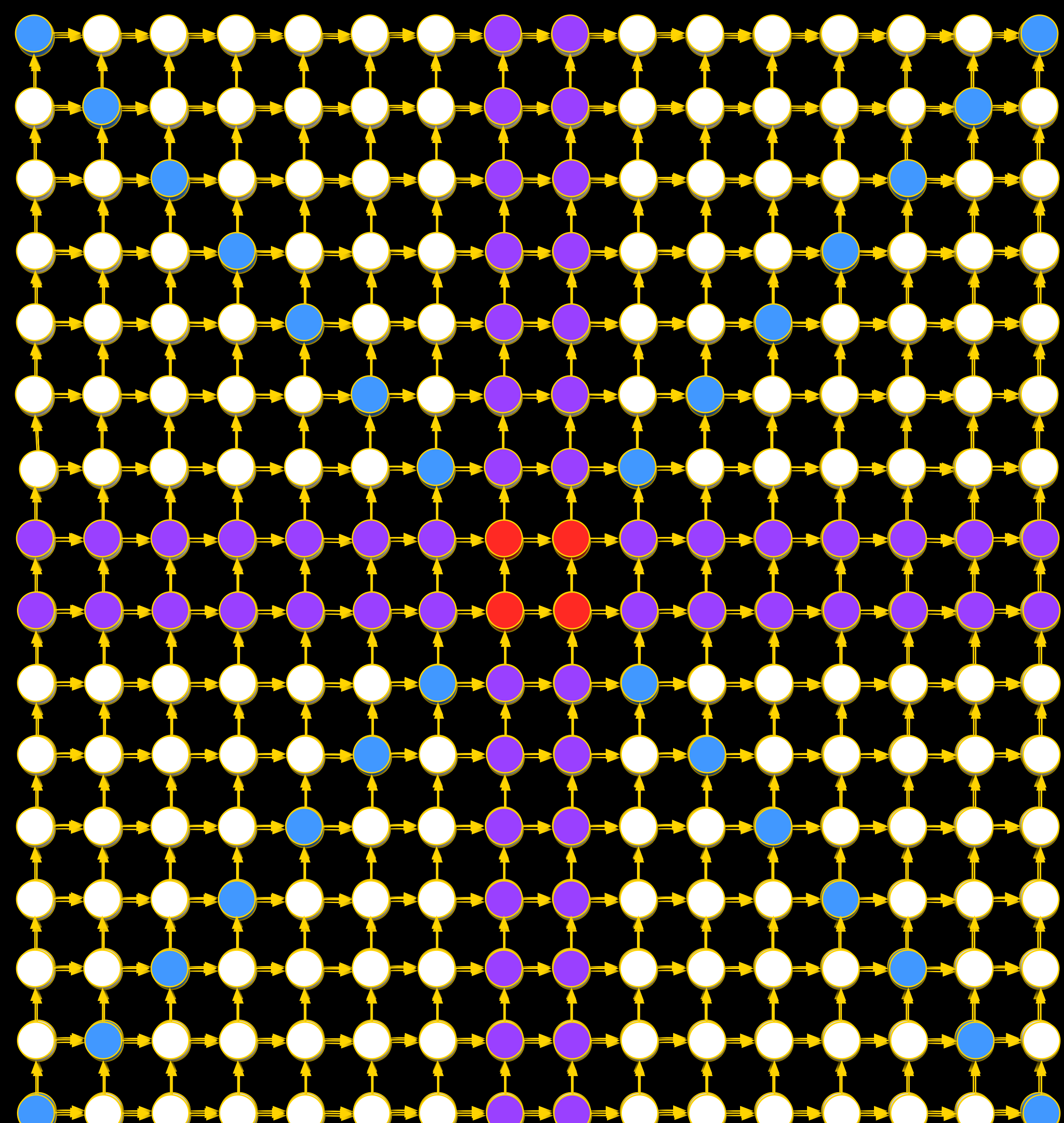
Axis pattern ($r = 2$, $m = 3$)



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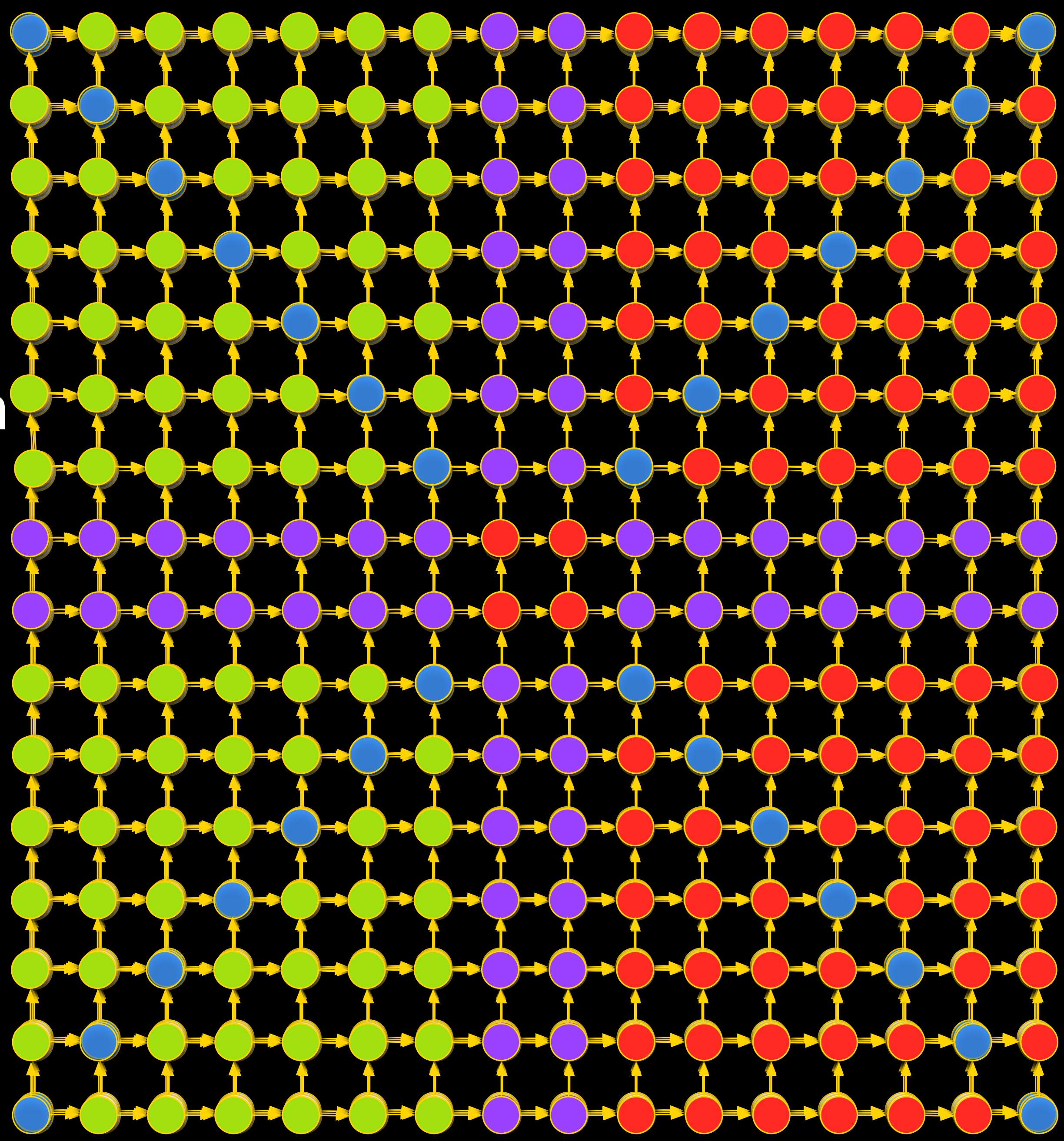
Half-proportion with skeleton pattern (r = 2, m = 3)



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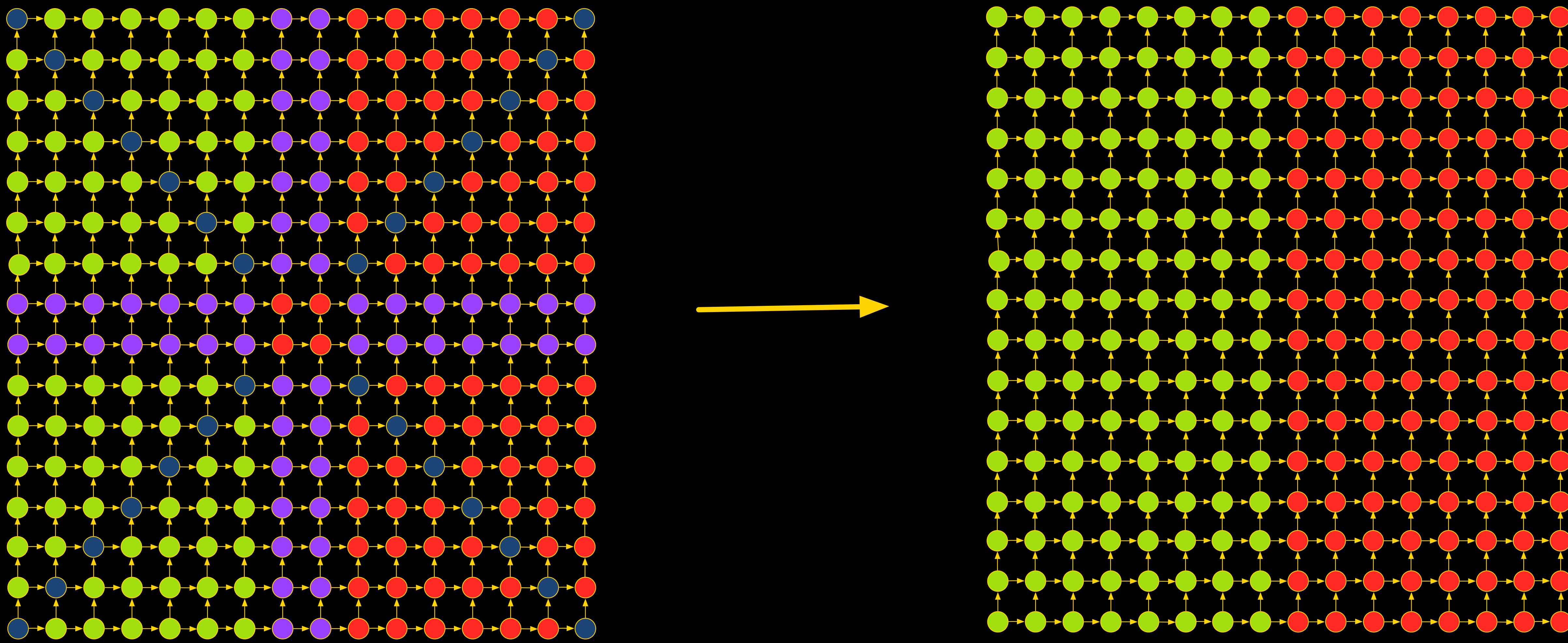

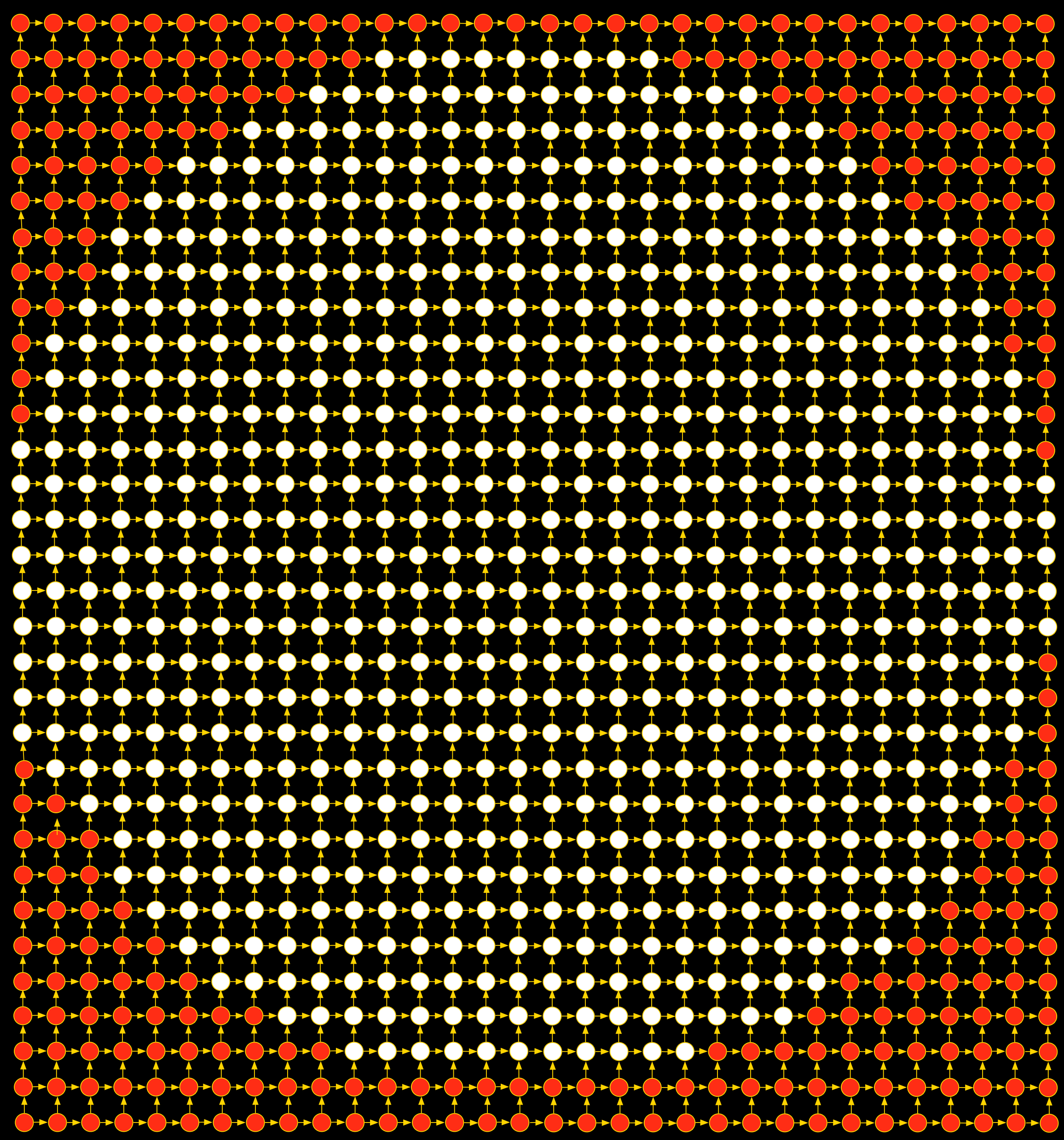

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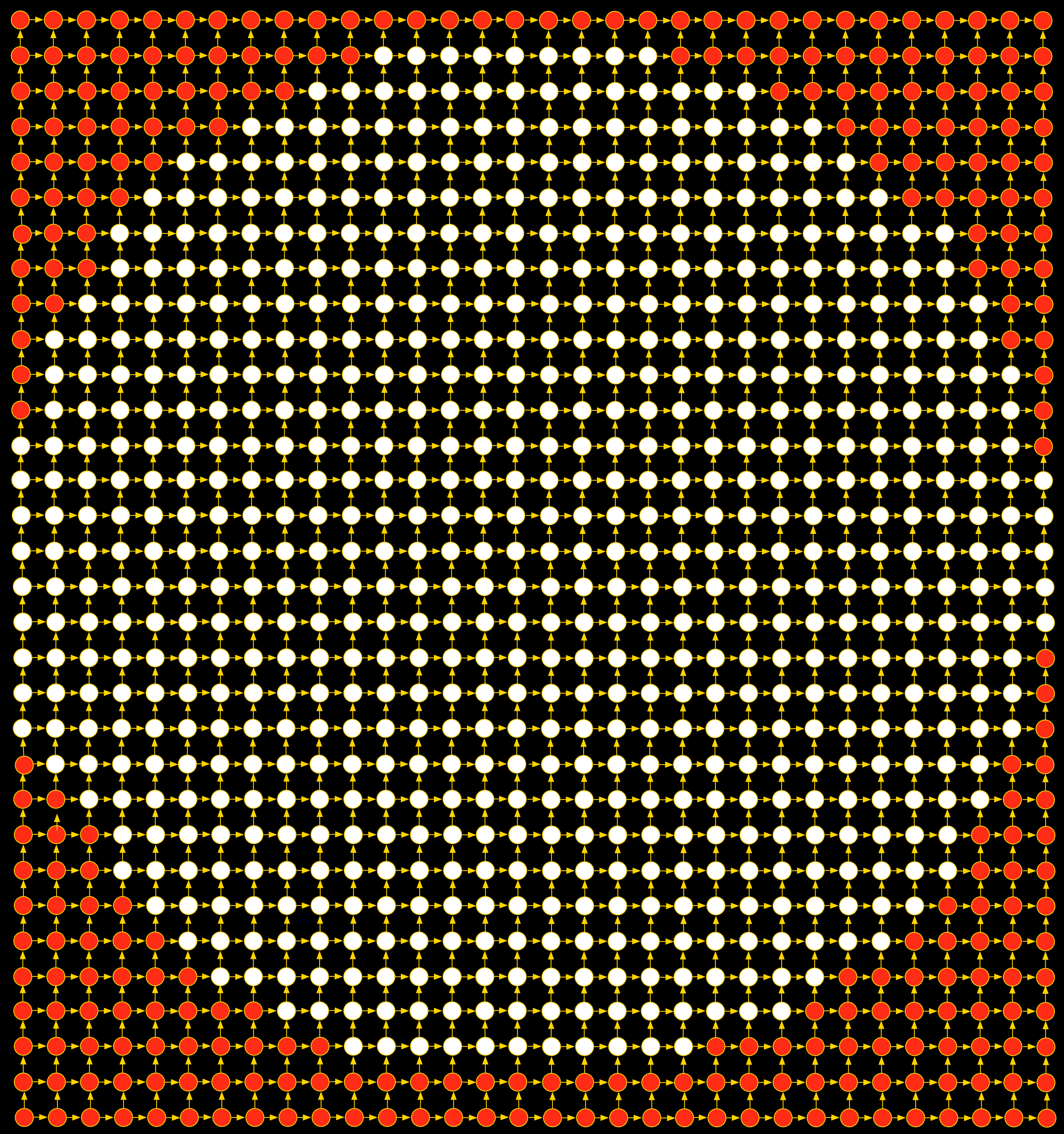

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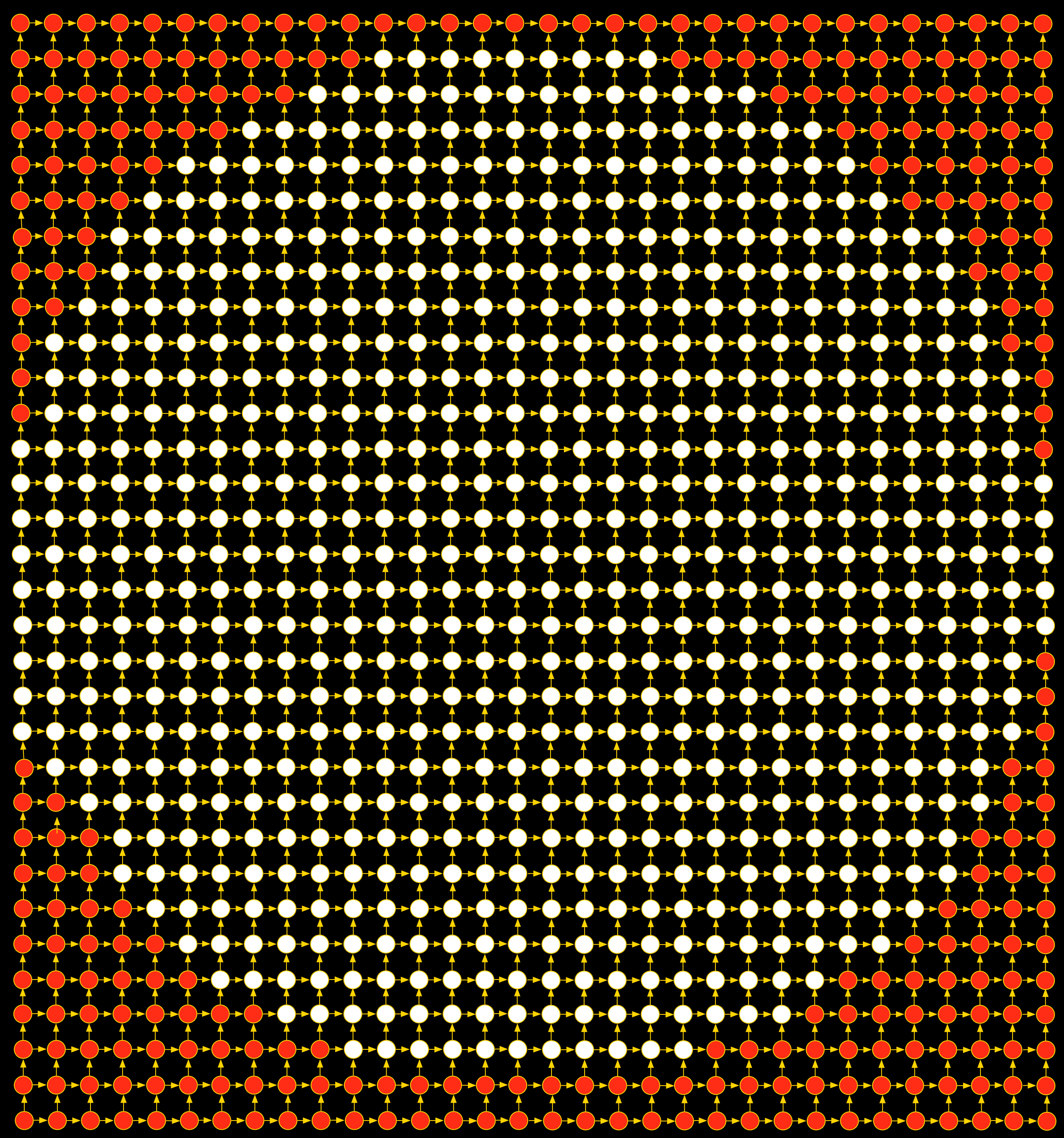

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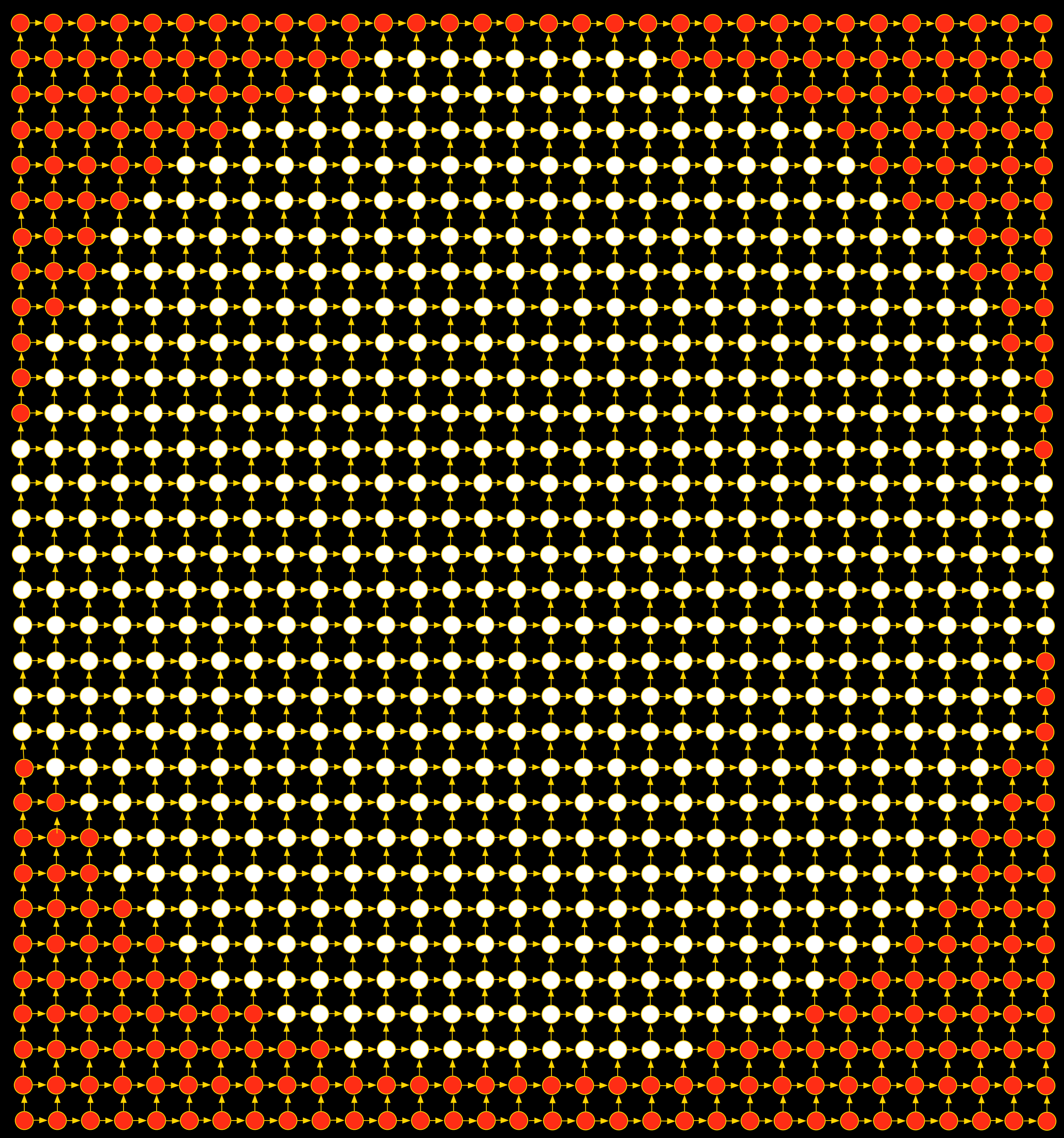

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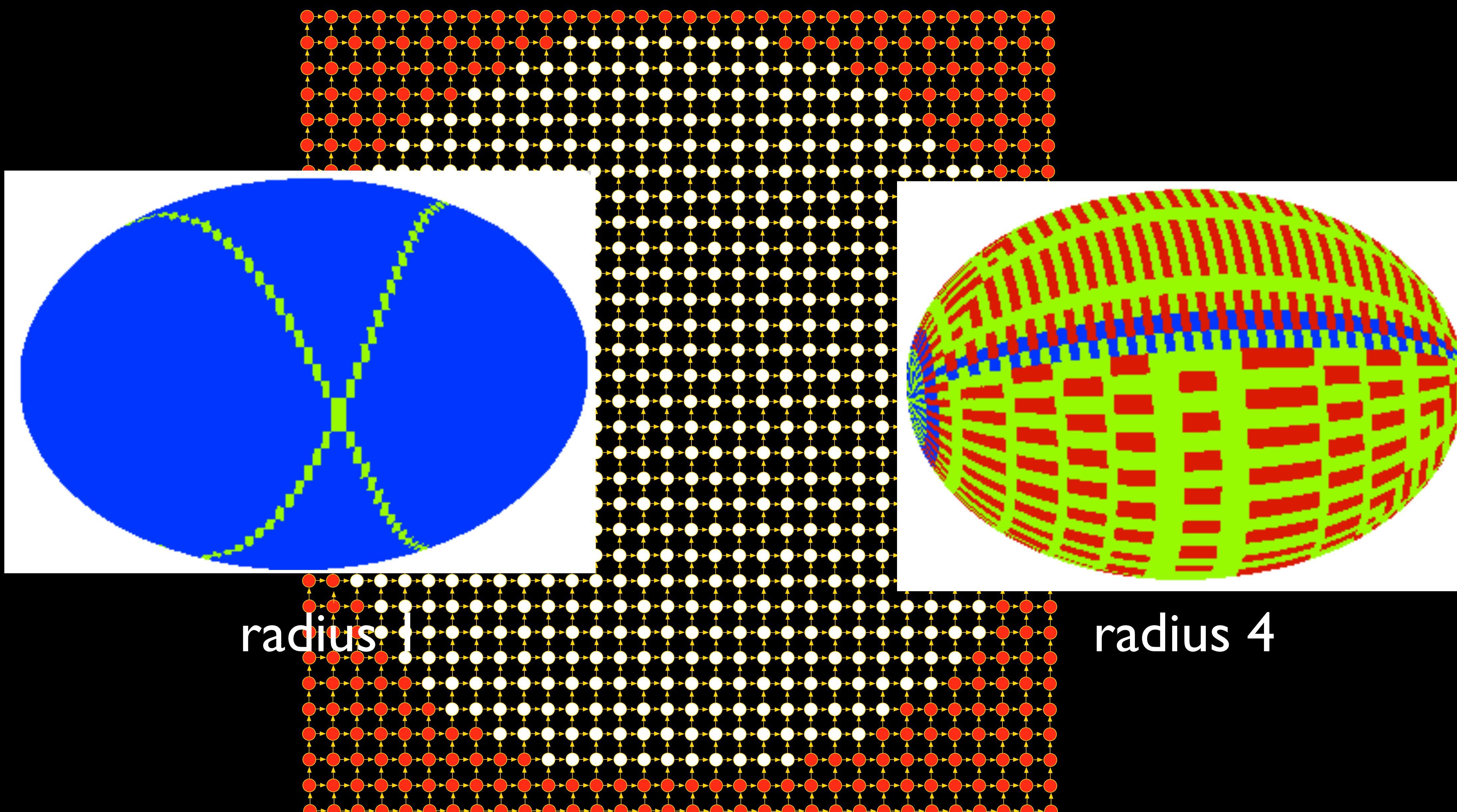

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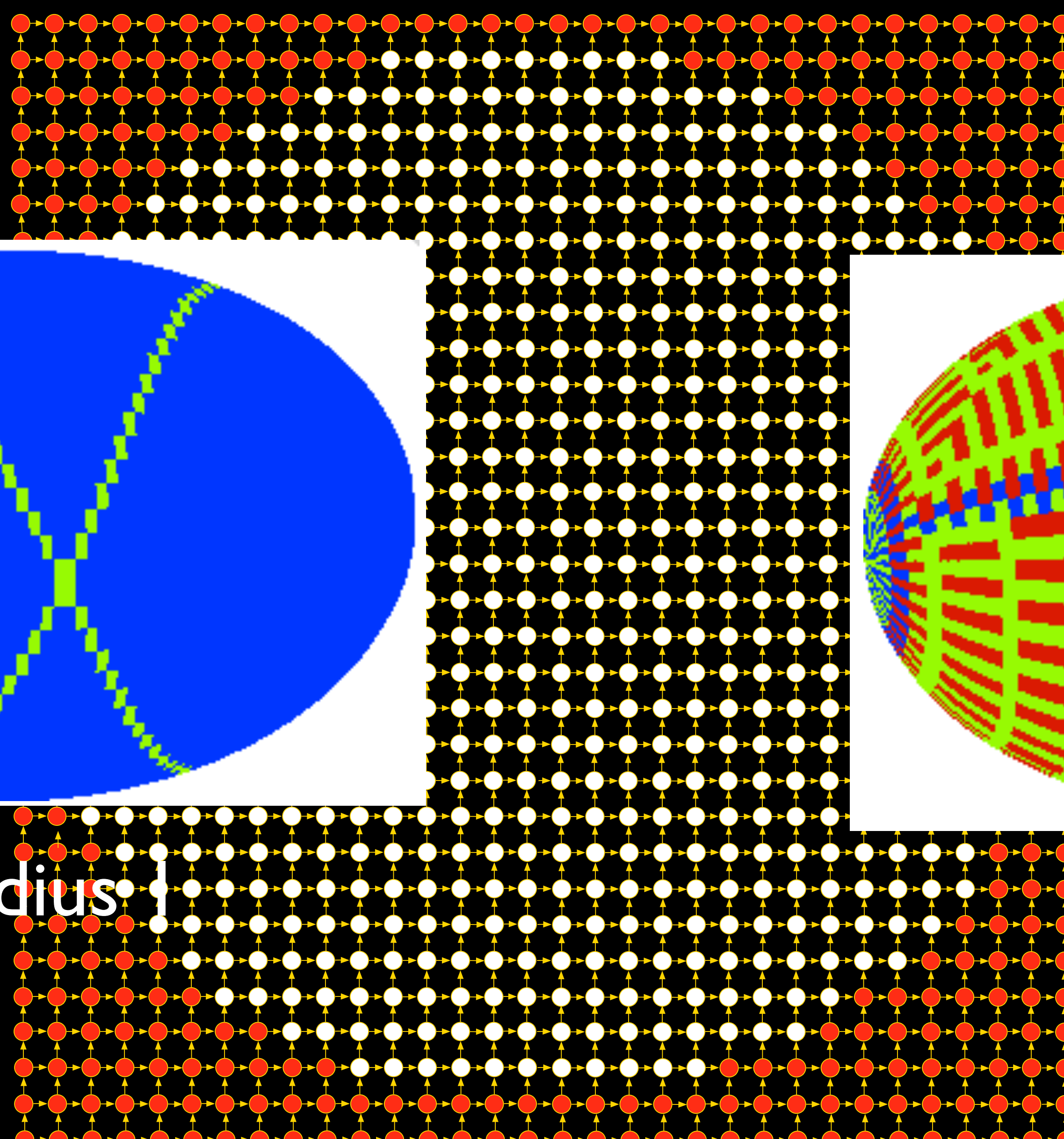
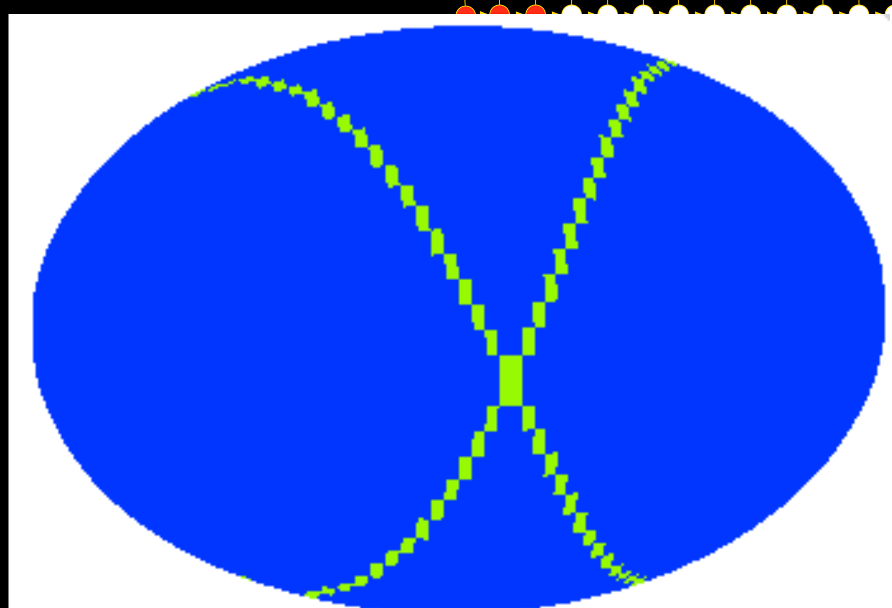
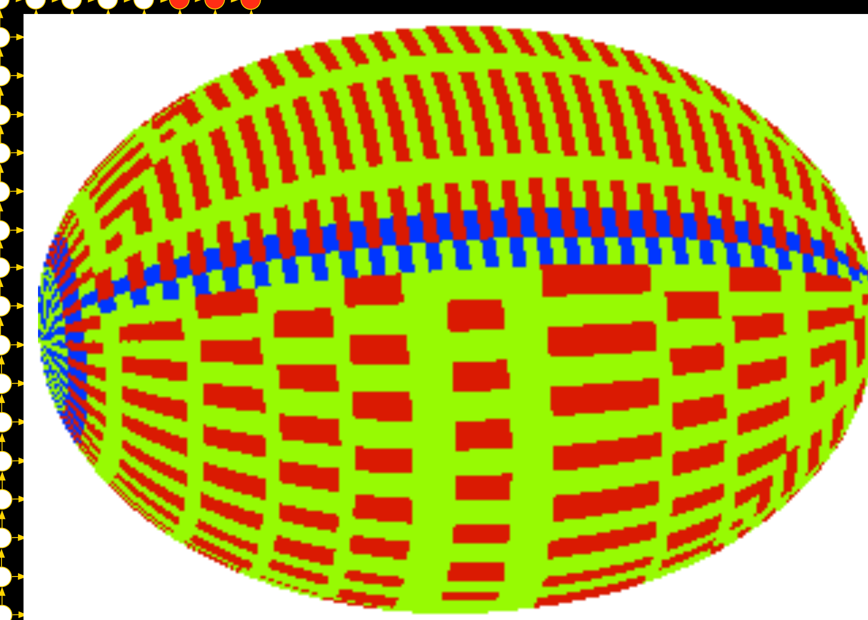




radius 1



radius 4

So, in effect, a vector pattern language is available in regular structures above 1 dimension.

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- <u>Graph-theoretically.</u>

Local Check Schemes as Graphs

Definition. *Given an underlying geometry $\mathcal{G}$ and label set S, the* length-n shift graph over $\mathcal{G}, S$ *is the derived graph*

$$\mathcal{D}_n(\mathcal{G}, S) = (V, E)$$

where

$$V = \{\textit{diameter-}n \textit{ induced subgraphs in } S\textit{-configurations over } \mathcal{G}\}$$

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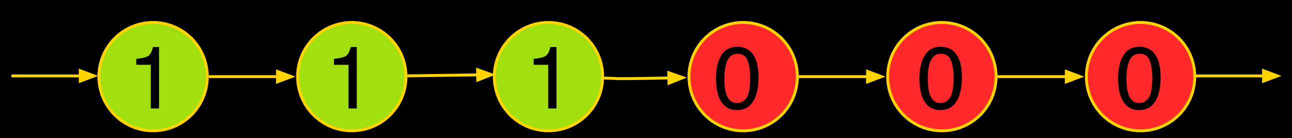

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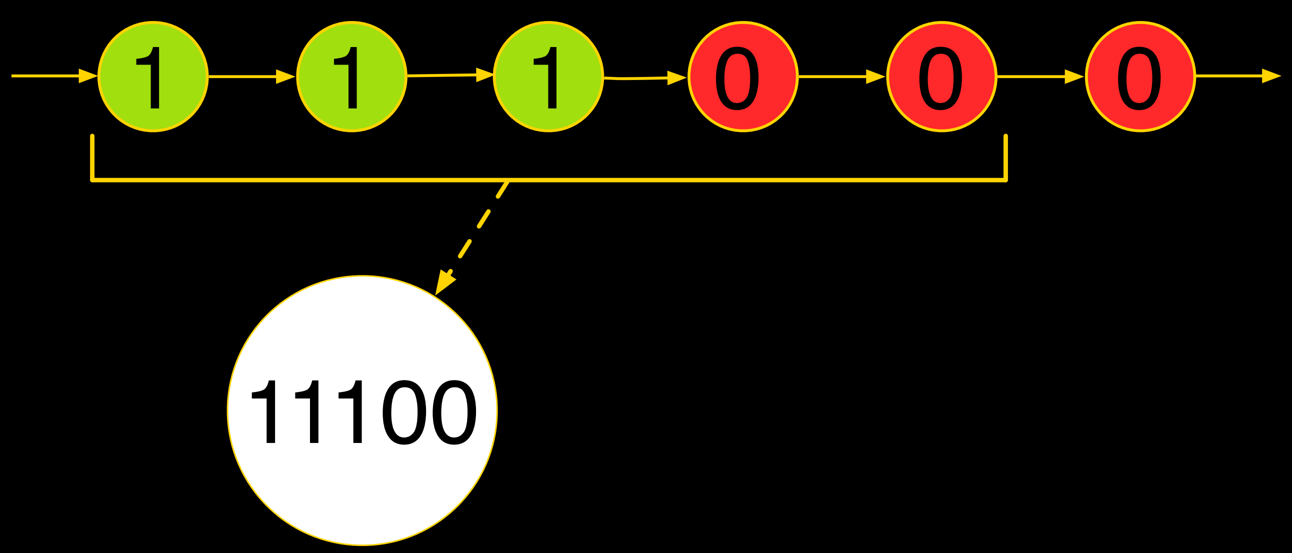

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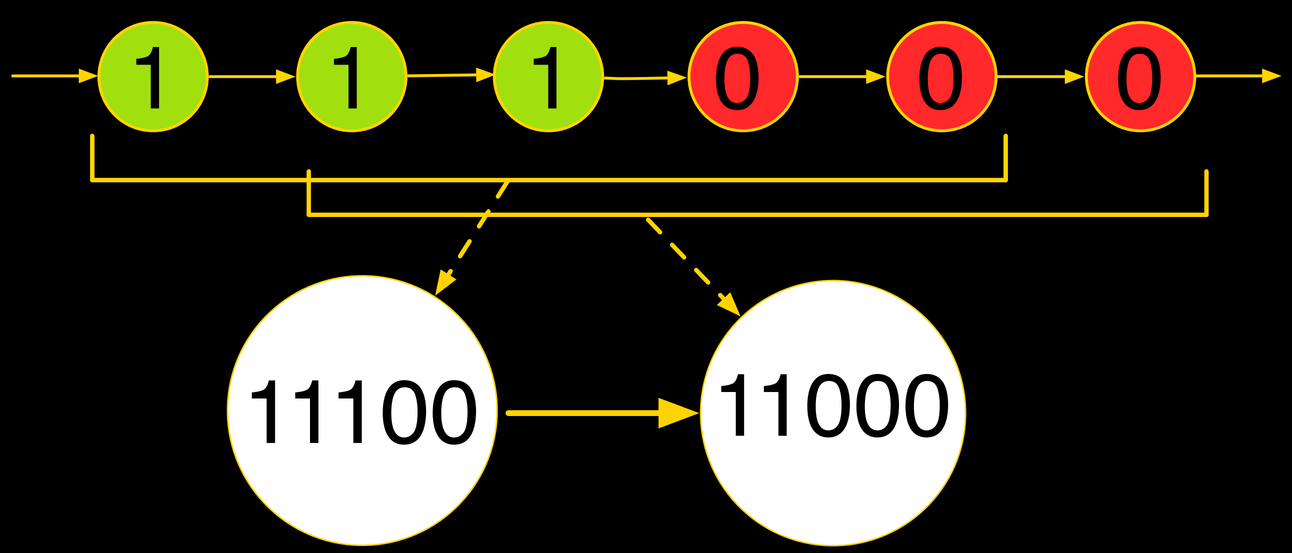

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$\mathcal{D}_n(\mathbb{Z}, 2)$ is known (from other contexts) as the DeBruijn graph, so the generalized DeBruijn graphs are the ``ambient spaces" of locally checkable patterns.

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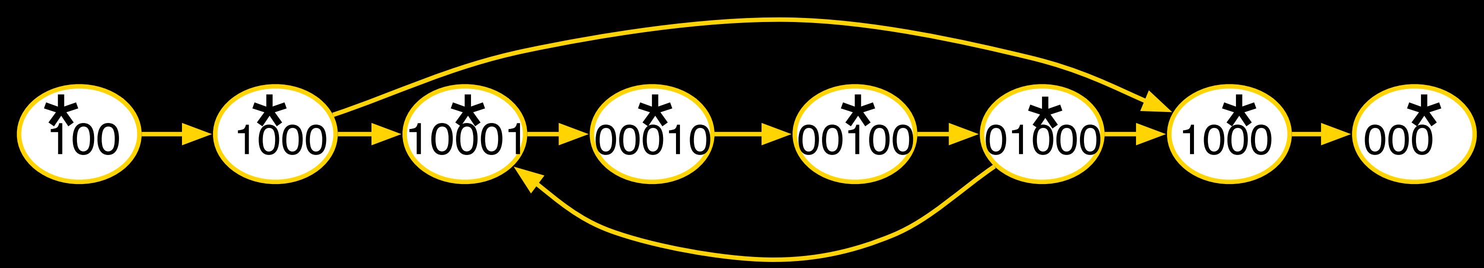

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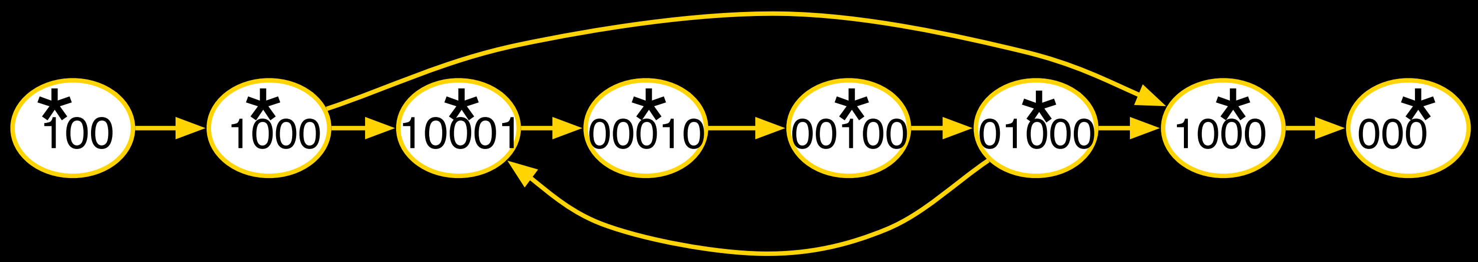

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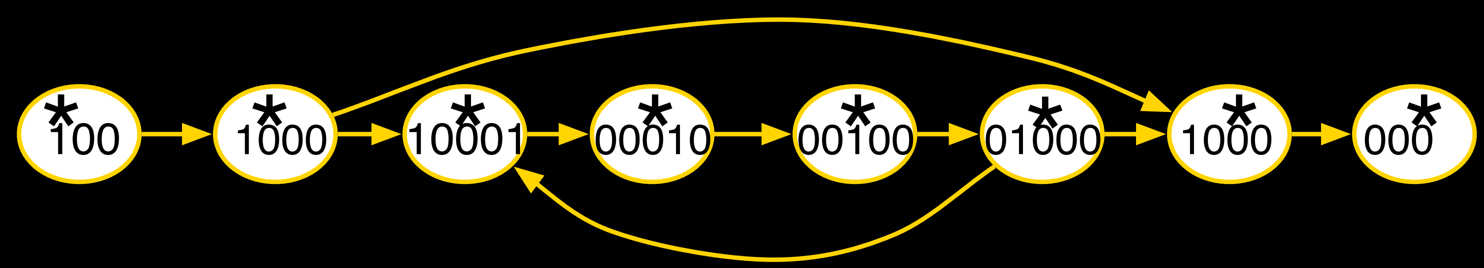

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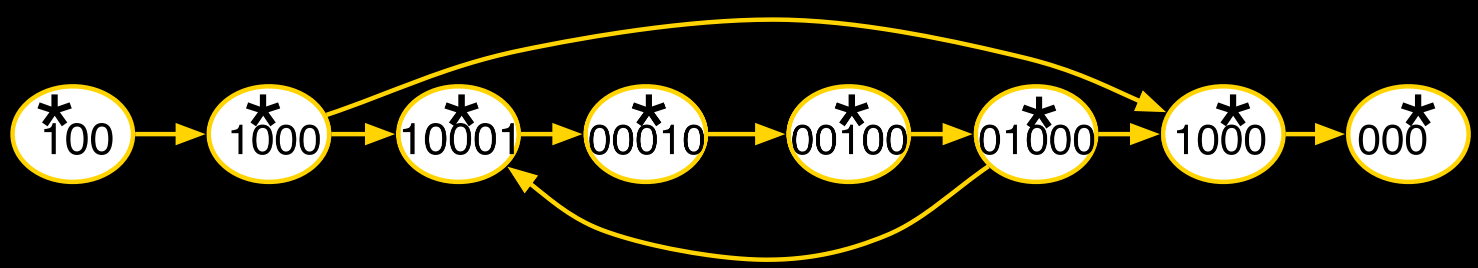

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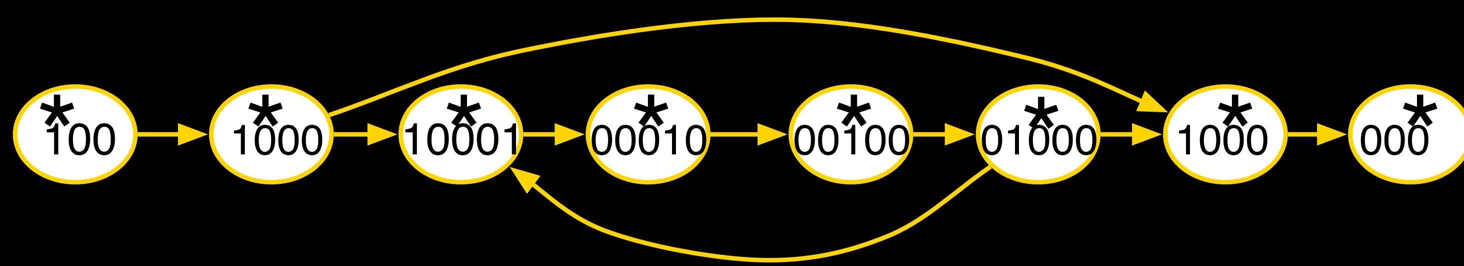

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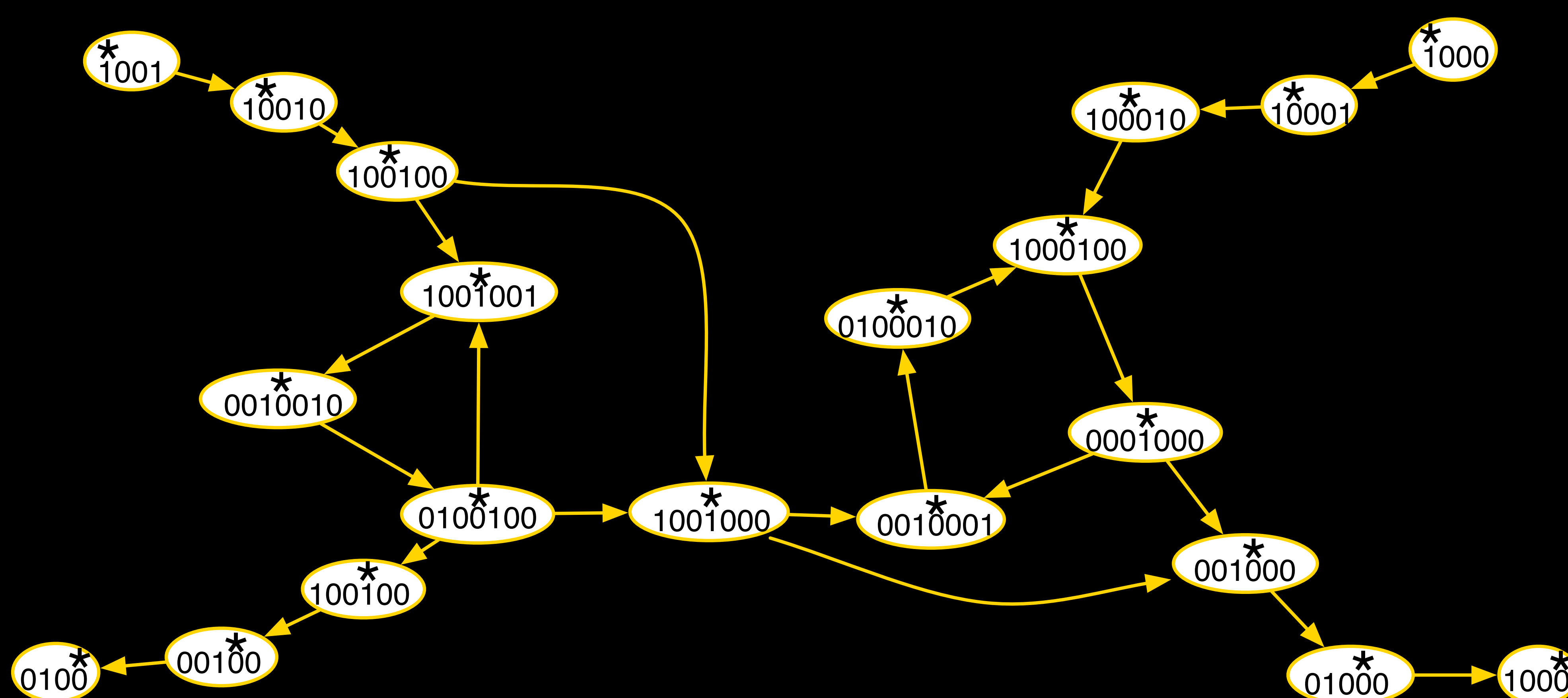

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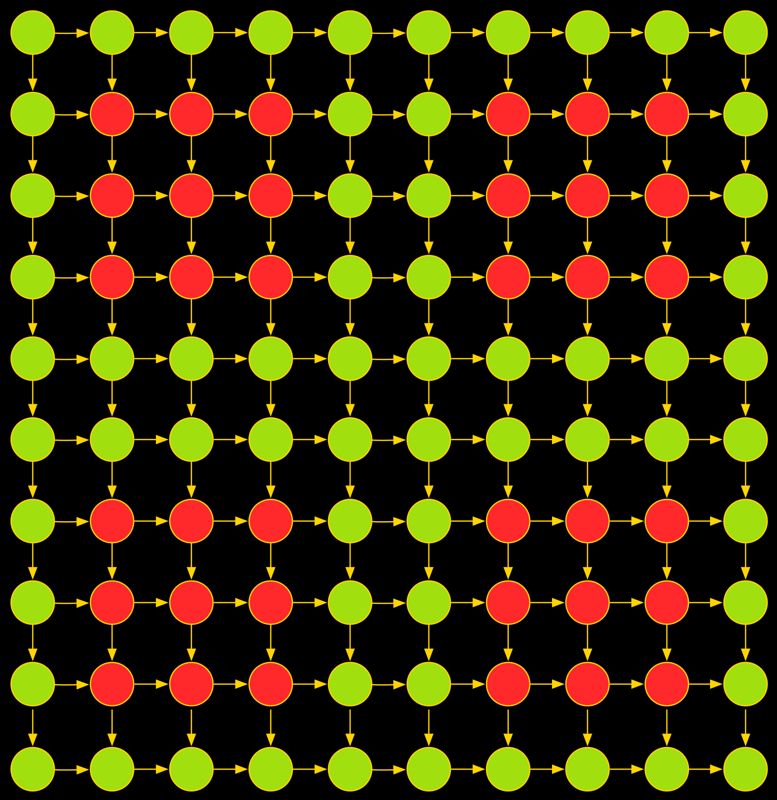

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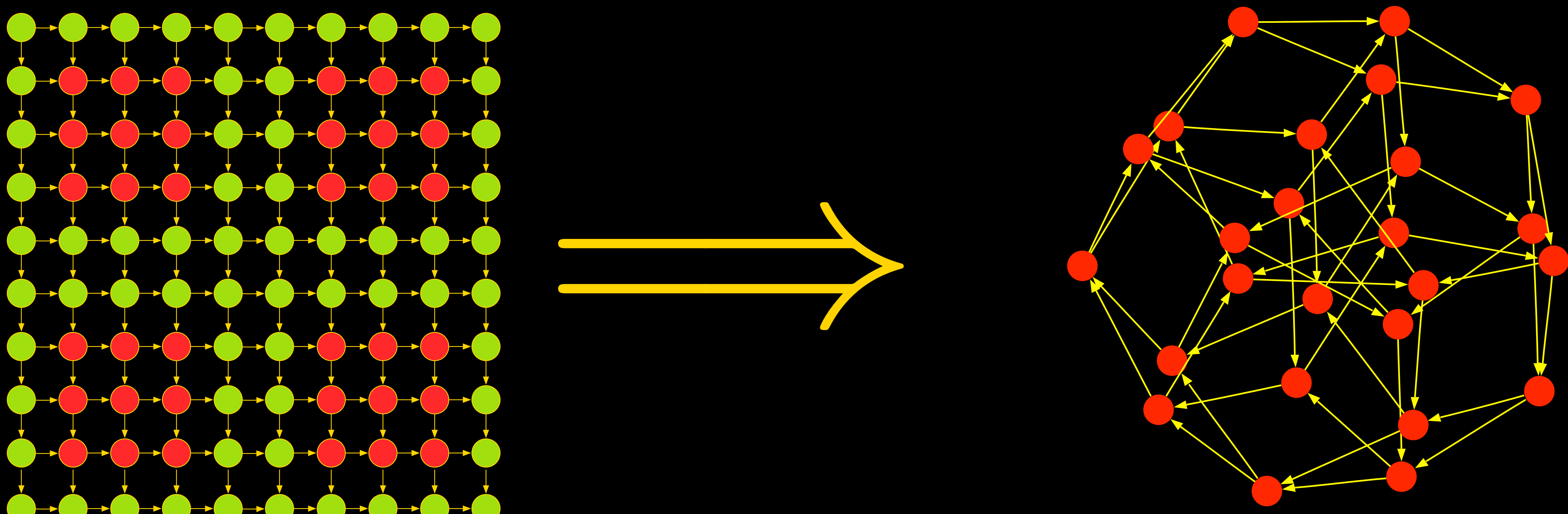

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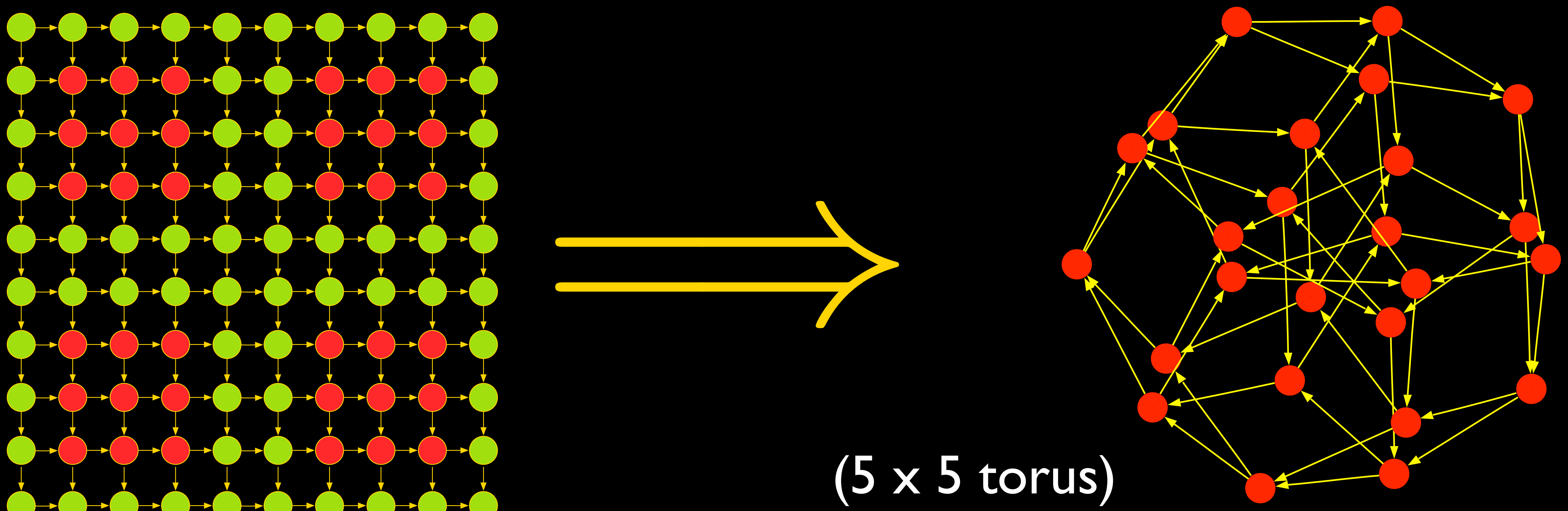

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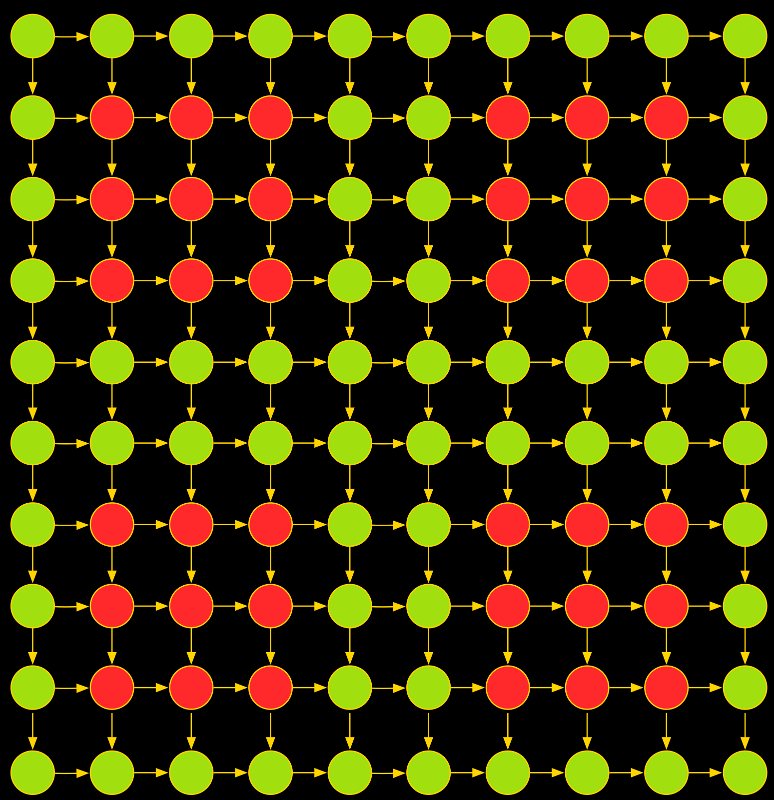

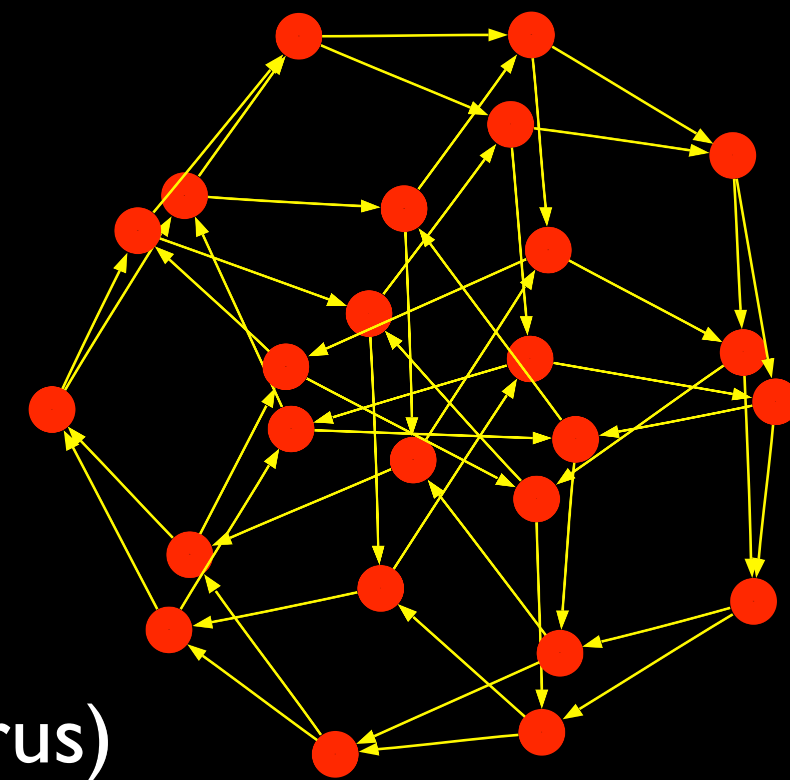

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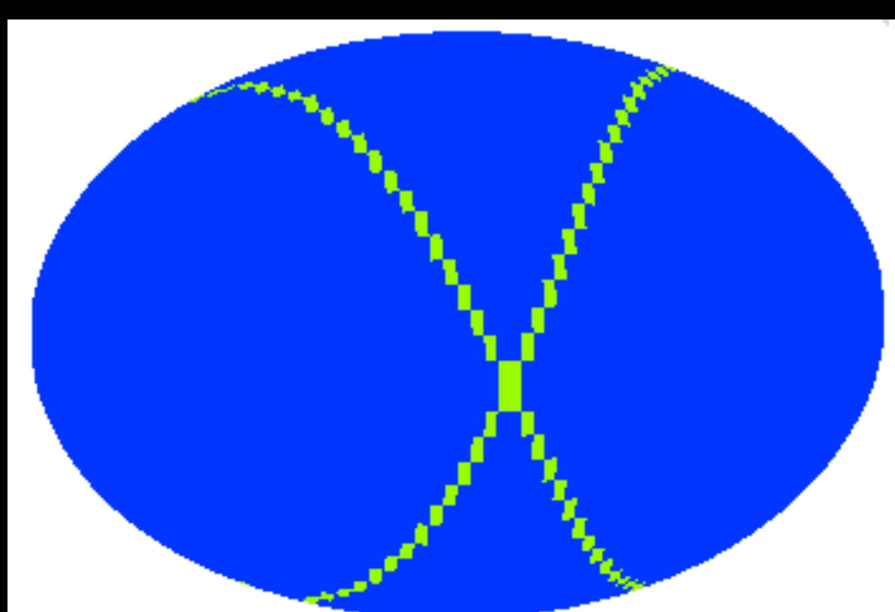

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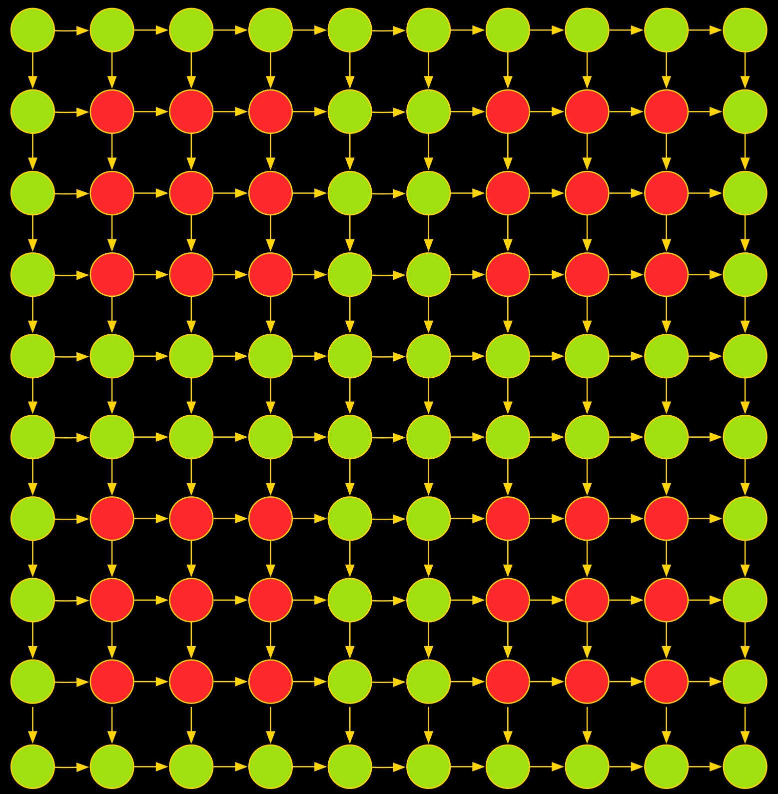

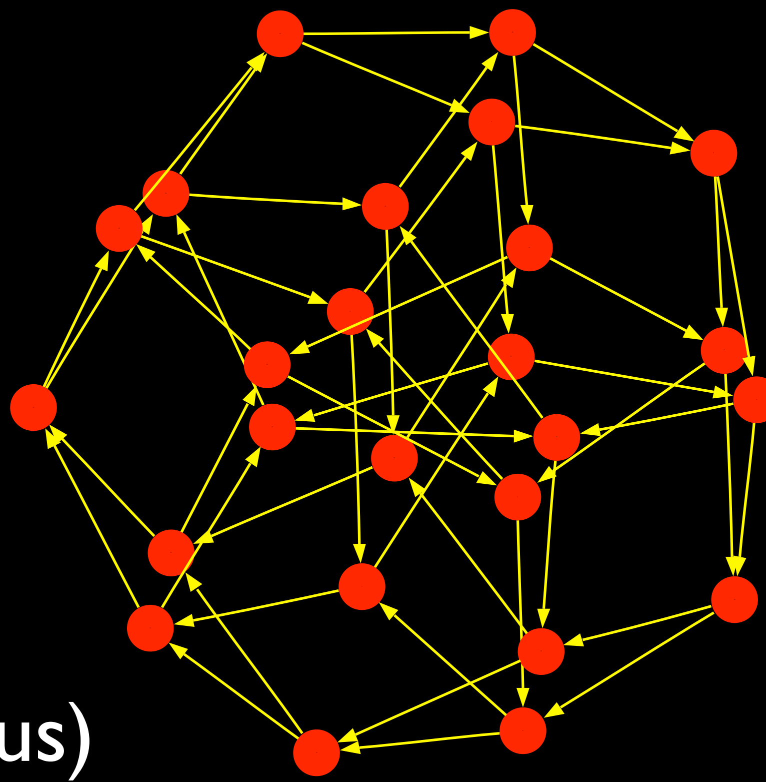

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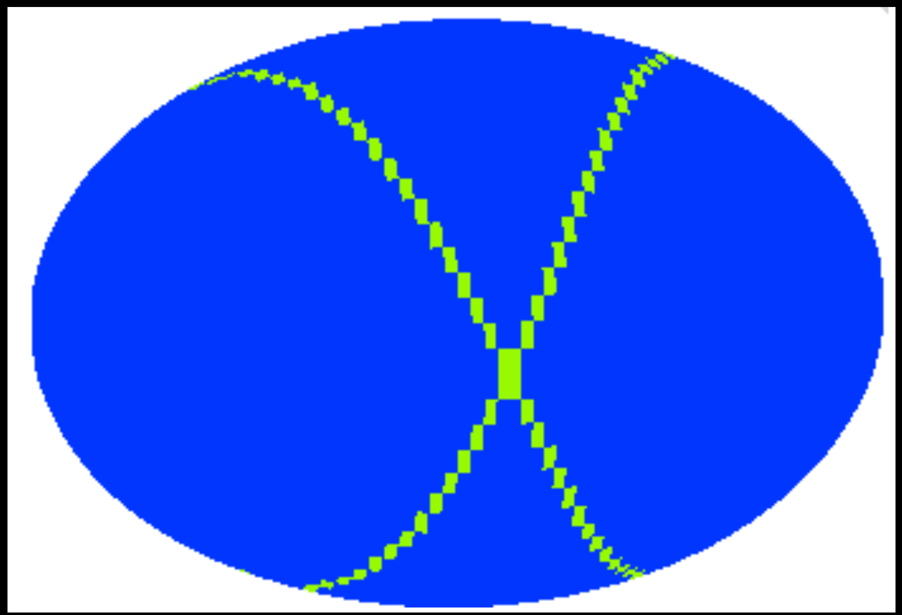

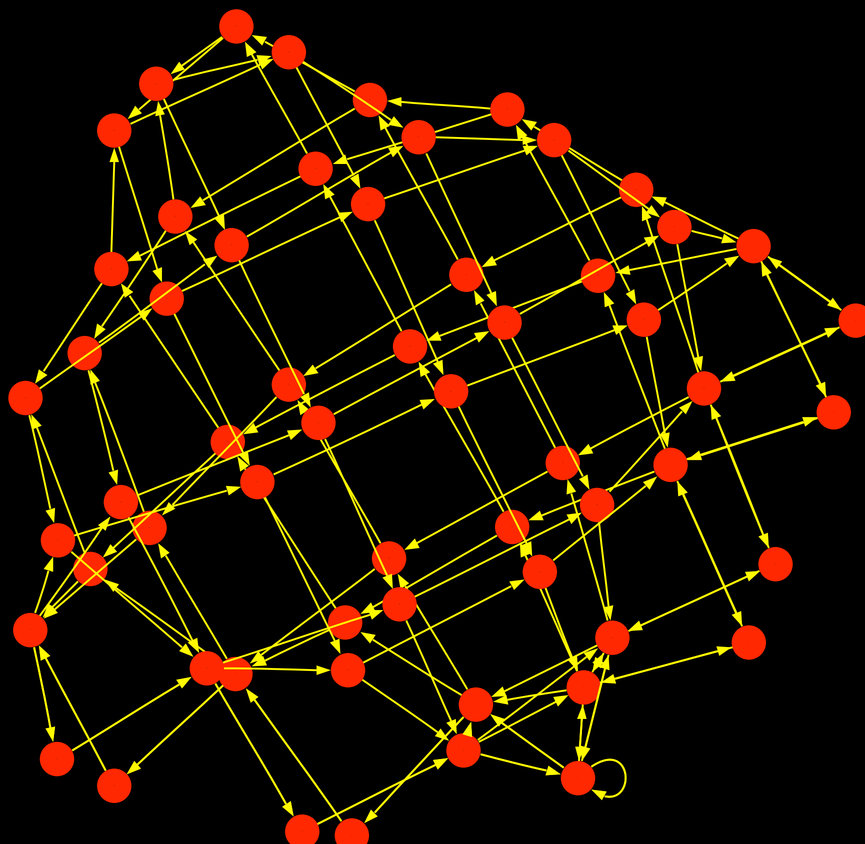

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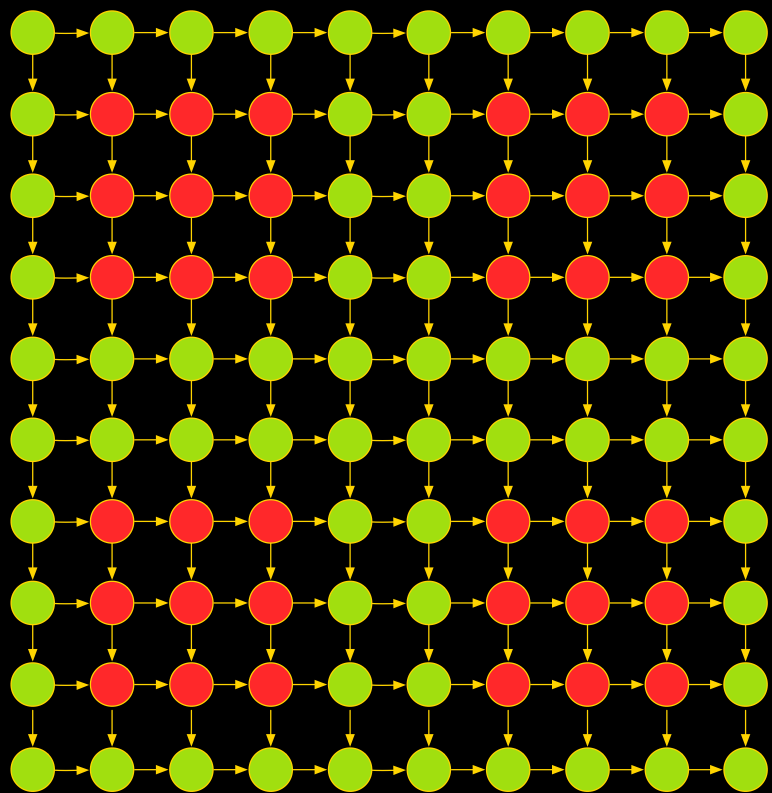

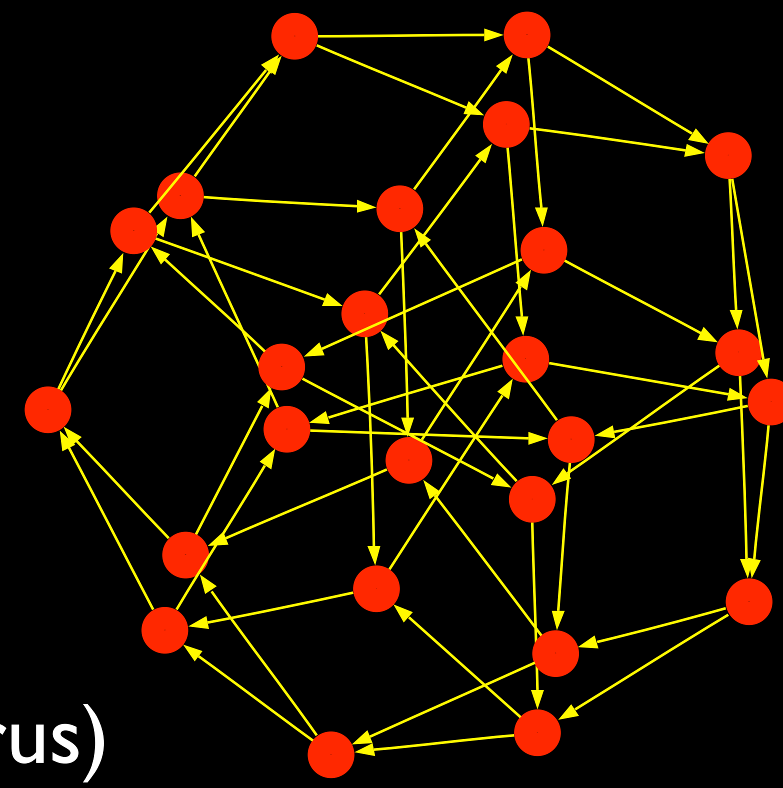

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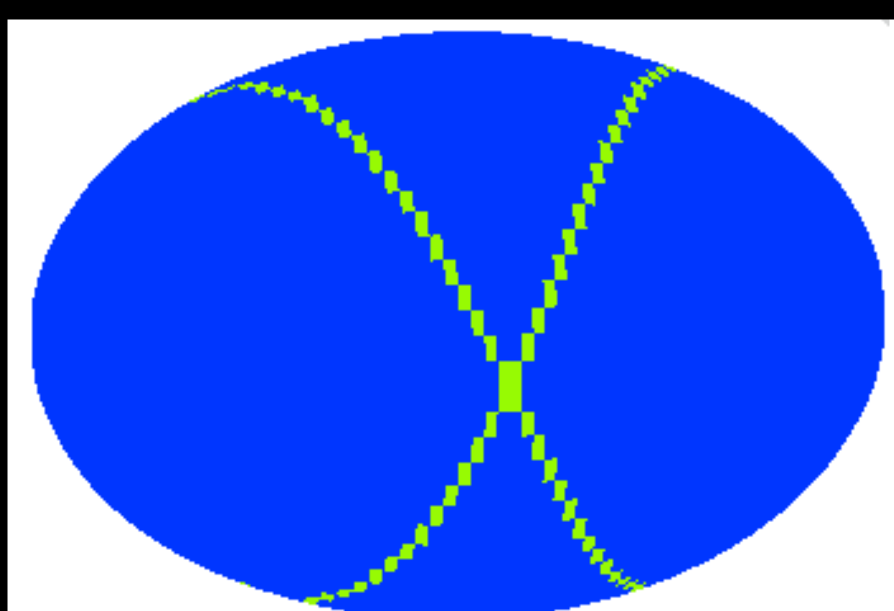

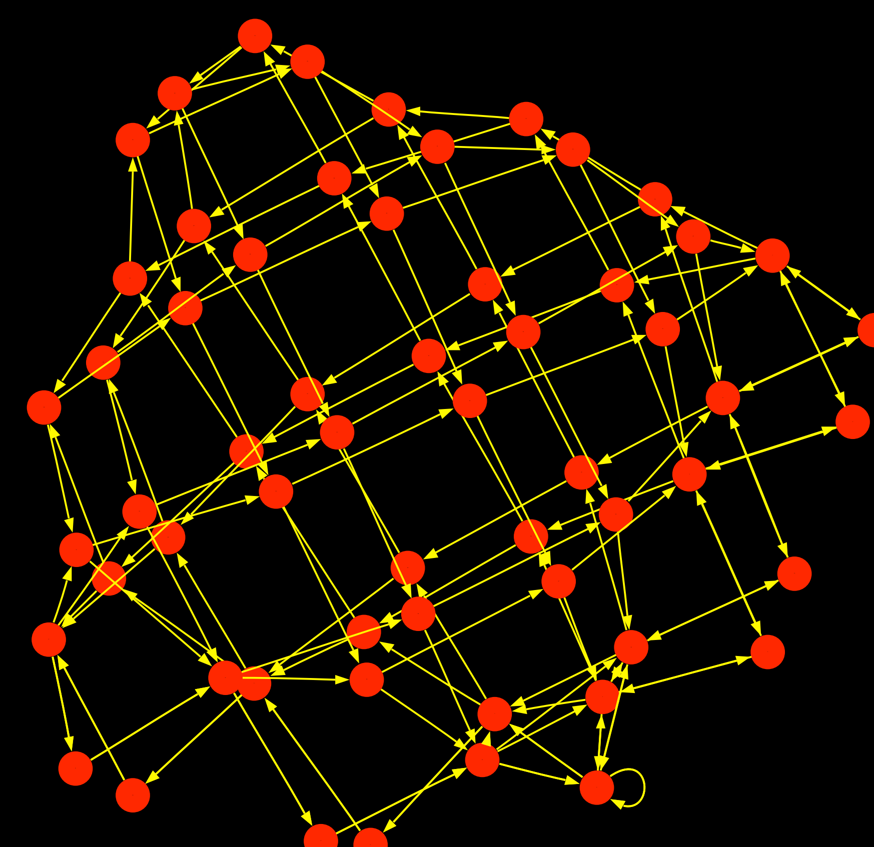

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