450D Political Methodology IV
TA Section 6

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Goal Today

- Transformation of Variables
- Hierarchical models

Road map to those goals:

1. Transformation of Variables (v2)
   - Motivation
   - Process
   - Derivation
   - Application: Inverse transform sampling
2. Multilevel models
   - Regression model examples in Stan
   - Common error messages
   - REML and `lmer`

Transformation of Variables

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$$f_x(x) = \begin{cases} 1, & \text{for } \ 0 < x < 1 \\ 0, & \text{for } \ x < 0, x > 1 \end{cases}$$

Suppose we then want to apply a transformation $g(x)$ to those values, defined as:

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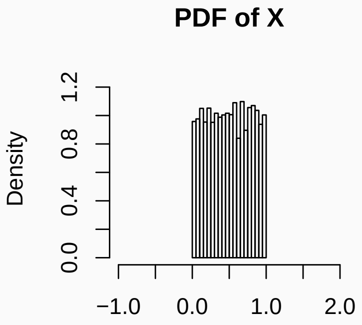
Q: What is the probability density function of the transformed values, that is, what is $f_y(y)$?

Simulation

```
m <- 10000
x <- runif(m, 0, 1)
f.y <- function(x, theta) -log(x)/theta

par(mfrow=c(1,2))
hist(x, freq=FALSE, main="PDF of X", ylim=c(0,1.25), xlim=c(-1,2))
f.y(x, theta=1) %>% hist(freq=FALSE, main="PDF of Y", xlab="y", ylim=c(0,1))
```



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$$\begin{aligned} y = g(x) &= \frac{1}{\lambda} - \log(x) \\ \lambda \cdot y &= -\log(x) \\ \exp(-\lambda \cdot y) &= x \\ &= g^{-1}(y) \end{aligned}$$

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$$\begin{aligned}\frac{d}{dy}g^{-1}(y) &= \frac{d}{dy}\exp(-\lambda \cdot y) \\ &= \exp(-\lambda \cdot y) \cdot -\lambda\end{aligned}$$

Process

6. Take the derivative of $g^{-1}(y)$ using the chain rule

$$\begin{aligned}\frac{d}{dy}g^{-1}(y) &= \frac{d}{dy}\exp(-\lambda \cdot y)\\ &= \exp(-\lambda \cdot y) \cdot -\lambda\end{aligned}$$

7. Solve for $f_x(g^{-1}(x))$

$$\begin{aligned}f_x(x) &= 1 \quad \text{for } 0 < x < 1\\ f_x(g^{-1}(x)) &= 1 \quad \text{for } 0 < g^{-1}(x) < 1\\ &\qquad\quad 0 < g^{-1}(x) < 1\\ &\qquad\quad 0 < \exp(-\lambda \cdot y) < 1\end{aligned}$$

Process

8. Combine these pieces of information

$$f_y(y) = f_x\{g^{-1}(y)\}\left|\frac{d}{dy}g^{-1}(y)\right|$$

so

$$f_y(y) = \begin{cases} 1 \times |\exp(-\lambda \cdot y) \cdot -\lambda|, & \text{for } \; 0 < y < \infty \\ 0 \times |\exp(-\lambda \cdot y) \cdot -\lambda|, & \text{for } \; y < 0 \end{cases}$$

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It turns out that this is the PDF of the exponential distribution.

$$Exponential(\theta) = \lambda \cdot \exp(-\lambda \cdot x)$$

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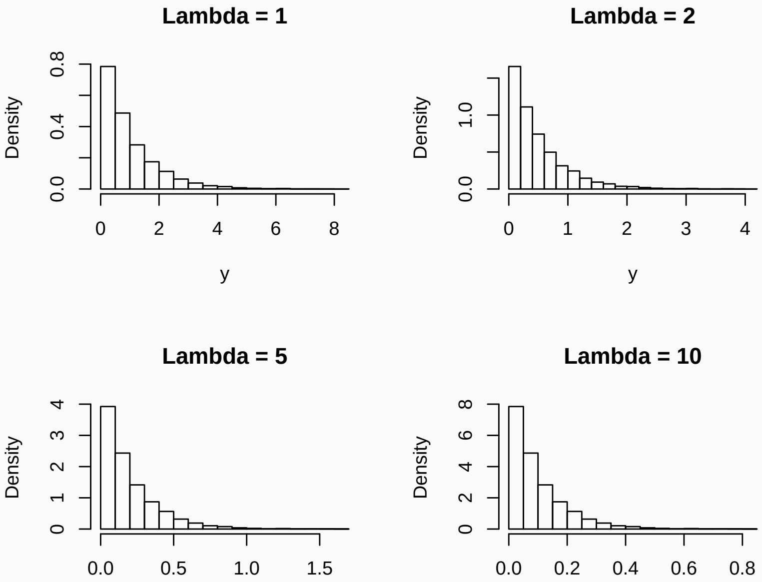

Lambda = 1
Density
0.0
0.4
0.8
0
2
4
6
8
y
Lambda = 2
Density
0.0
1.0
0
1
2
3
4
y
Lambda = 5
Density
0
1
2
3
4
0.0
0.5
1.0
1.5
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Density
0
2
4
6
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$$\begin{aligned} CDF_y(y) &= F_y(y) \\ &= P(Y \leq y) \\ &= P\left(g(x) \leq y\right) \\ &= P(x \leq g^{-1}(y)) \\ &= F_x(g^{-1}(y)) \\ &= CDF_x(g^{-1}(y)) \end{aligned}$$

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And then you need the absolute value for some reason.

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Intuitively, we are unlikely to choose a number in the far end of tails because there is very little area in them which would require choosing a number very close to zero or one.

Process

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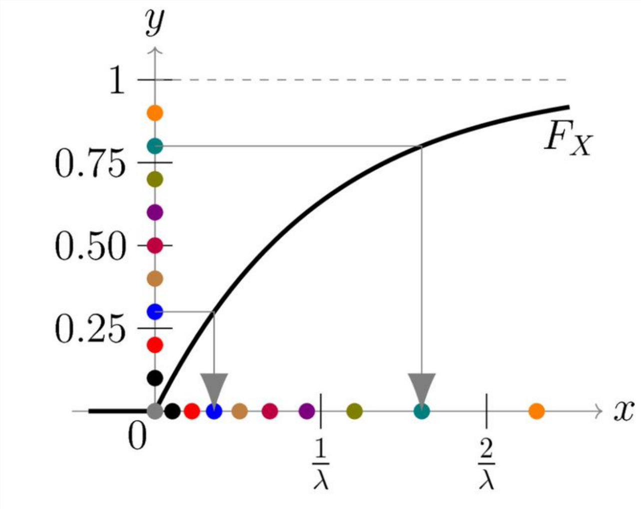
\end{aligned}
$$

Plug our values of U into $F^{-1}(y)$, and get output $\sim F(y)$.



Notice that even though our y axis is uniformly distributed, our x axis has many small values and a few large values, the exponential distribution.

Derivation

Claims: if U is a uniform random variable on (0,1), then $F^{-1}(U) \sim F$.

$$\begin{aligned} Pr(F^{-1}(U) \leq x) &= Pr(F(F^{-1}(U)) \leq F(x)) \\ &= Pr(U \leq F(x)) \end{aligned}$$

By properties of the uniform dist on (0,1), $Pr(U \leq x) = x$

$$= F(x)$$

Multilevel models

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*on the whiteboard