Bidding with Securities: Auctions and Security Design

Peter M. DeMarzo, Ilan Kremer and Andrzej Skrzypacz

VIII. Proofs of Technical Lemmas (for the online appendix)

PROOF OF LEMMA 1: Using Assumption B and that $S(z)$ is between 0 and z, dominated convergence implies that the derivatives exist and are equal to $ES'(v) = \int S(z)\, h_v(z \mid v)\, dz$ and $ES''(v) = \int S(z)\, h_{vv}(z \mid v)\, dz$. Then, for any z^*,

$$ES'(v) = \int S(z)\, h_v(z \mid v)\, dz = \int \left[S(z) - S(z^*) \right] \left[\frac{h_v(z \mid v)}{h(z \mid v)} \right] h(z \mid v)\, dz,$$

since $\int S(z^*)\, h_v(z \mid v)\, dz = S(z^*) \frac{\partial}{\partial v} \int h(z \mid v)\, dz = 0$.

From SMLRP, [h_v / h] is increasing in z. Therefore, we can choose z^* so that

$$\left[\frac{h_v(z \mid v)}{h(z \mid v)} \right] \gtrless 0 \quad \text{if and only if} \quad z \gtrless z^*.$$

Then, since S is weakly increasing,

$$\left[S(z) - S(z^*) \right] \left[\frac{h_v(z \mid v)}{h(z \mid v)} \right] \geq 0,$$

and the inequality is strict for z such that $S(z) \neq S(z^*)$. This set has positive measure since $S \neq 0$ and Z has full support conditional on v. Hence, $ES'(v) > 0$. The proof of $ES'(v) < 1$ is identical, substituting $Z - S(Z)$ for S. ◆

PROOF OF LEMMA 2: The proof that $s(v)$, which solves $ES(s,v) = v$, is the unique weakly undominated strategy is standard. Differentiating $ES(s(v),v) = v$ yields,

$$s'(v) = \frac{\frac{\partial}{\partial v}\left[v - ES(s,v) \right]}{\frac{\partial}{\partial s} ES(s,v)}.$$

Thus s increasing in v follows, since ES is increasing in s, and from Lemma 1, $\frac{\partial}{\partial v}\left[v - ES(s,v) \right] > 0$ as long as $S(s,Z) \neq Z$ (which is not possible in equilibrium since $X > 0$). ◆

PROOF OF LEMMA 3: Let $P(s)$ be the probability of winning with a bid of s, and $\pi(s, v) = \log(P(s)) + \log(v - ES(s, v))$. Then

$$s(v) \in \arg\max_s P(s)(v - ES(s,v)) = \arg\max_s \pi(s,v).$$

By Assumption C the objective in the second expression is strictly supermodular, and so by Donald M. Topkis (1978), any selection $s(v)$ is weakly increasing in v. If $s(v)$ were constant on an interval, then the highest type in that interval can increase his bid marginally and increase his probability of winning, and thus his payoff, by a discrete amount. Thus, $s(v)$ is increasing. This implies $P(s(v)) = F_n(v) \equiv F(v)^{n-1}$.

Continuity of s follows since otherwise a type just above a discontinuity could gain by lowering his bid. For differentiability, note that we can rewrite the bidder's optimality condition as

$$v \in \arg\max_{v'} F_n(v')(v - ES(s(v'), v)) .$$

Letting $u(s, v) = v - ES(s,v)$, this implies that for any $v' > v$,

$$F_n(v)u(s(v),v) \geq F_n(v')u(s(v'),v) = F_n(v')\left[u(s(v),v) + u_1(s^*,v)(s(v') - s(v))\right],$$

for some s^* between $s(v)$ and $s(v')$. Since $u_1 < 0$, this can be rewritten as

$$\frac{F_n(v') - F_n(v)}{v' - v} \times \frac{u(s(v),v)}{-F_n(v')u_1(s^*,v)} \leq \frac{s(v') - s(v)}{v' - v}.$$

Changing the roles of v and v' yields, for some s^{**} between $s(v)$ and $s(v')$,

$$\frac{F_n(v') - F_n(v)}{v' - v} \times \frac{u(s(v'),v')}{-F_n(v)u_1(s^{**},v')} \geq \frac{s(v') - s(v)}{v' - v}.$$

Taking limits establishes the differential equation for s.

For the boundary condition, note that $P(s(v_L)) = 0$, and since all types earn non-negative profits $ES(s(v_L), v_L) \leq v_L$. But if the inequality were strict, the lowest type could raise his bid and earn positive profits with positive probability.

Having established uniqueness, it remains to verify existence by establishing the sufficiency of the bidder's first order condition. Consider any s' such that $s(v_L) < s' < s(v)$. There exists $v_L < v' < v$ such that $s(v') = s'$. Thus, by Assumption C,

$$\pi_s(s', v) > \pi_s(s', v') = 0.$$

A similar argument shows that for $s(v) < s' < s(v_H)$, $\pi_s(s', v) < 0$. Hence, π is quasiconcave in s and the first order condition is sufficient. ◆

PROOF OF LEMMA 4: Using the revelation principle, note that if type v reports v' he will win with probability $F^{n-1}(v')$. His expected payoff conditional on winning is equal to $(v - T(v, v'))$, where $T(v, v')$ is the expected payment by type v when he reports v'. Thus, type v will choose v' to maximize $F^{n-1}(v')\left(v - T(v,v')\right)$. Thus, we need to establish the correct form for T.

Letting V_{-i}^* be the highest type excluding i, bidder i wins with report v' if $V_{-i}^* < v'$. Let $\tilde{S}_{v'} \in \boldsymbol{S}$ be the random security that he will pay if he wins. Then define

$$\hat{S}_{v'}(z) = E\left[\tilde{S}_{v'}(z) \,\middle|\, V_{-i}^* \leq v'\right], \tag{8}$$

a security in the convex hull of $\boldsymbol{S}$ (which does not depend on i by symmetry). This is the "expected security" paid with a report of v'. Using the assumption that types are independent and that Z_i and V_{-i} are independent given V_i (private values), we obtain:

$$T(v,v') = E\left[\tilde{S}_{v'}(Z_i) \,\middle|\, V_i = v, V_{-i}^* \le v'\right] = E\left[E\left[\tilde{S}_{v'}(Z_i) \,\middle|\, Z_i, V_i = v, V_{-i}^* \le v'\right] \,\middle|\, V_i = v, V_{-i}^* \le v'\right]$$
$$= E\left[E\left[\tilde{S}_{v'}(Z_i) \,\middle|\, Z_i, V_{-i}^* \le v'\right] \,\middle|\, V_i = v, V_{-i}^* \le v'\right] = E\left[\hat{S}_{v'}(Z_i) \,\middle|\, V_i = v, V_{-i}^* \le v'\right]$$
$$= E\hat{S}_{v'}(v)$$

This completes the proof. ◆

PROOF OF LEMMA 5: Let $G(z) = S_1(z) - S_2(z)$. Then if $EG(v^*) = 0$,

$$EG'(v^*) = \int G(z)\, h_v(z \mid v^*)\, dz = \int G(z) \left[\frac{h_v(z \mid v^*)}{h(z \mid v^*)}\right] h(z \mid v^*)\, dz = \int G(z) \left[\frac{h_v(z \mid v^*)}{h(z \mid v^*)} - \frac{h_v(z^* \mid v^*)}{h(z^* \mid v^*)}\right] h(z \mid v^*)\, dz$$

where the last equality follows since

$$\int G(z) \frac{h_v(z^* \mid v^*)}{h(z^* \mid v^*)} h(z \mid v^*)\, dz = \frac{h_v(z^* \mid v^*)}{h(z^* \mid v^*)} EG(v^*) = 0 .$$

From SMLRP, [h_v / h] is increasing in z. Therefore,

$$G(z) \left[\frac{h_v(z \mid v^*)}{h(z \mid v^*)} - \frac{h_v(z^* \mid v^*)}{h(z^* \mid v^*)}\right] \ge 0 ,$$

and the inequality is strict on the set $\{z : S_1(z) \ne S_2(z)\}$. Thus, $EG'(v^*) > 0$. ◆

PROOF OF LEMMA 6: For debt securities, consider any feasible security S_2. If $S_2(z) > \min(d, z)$, then $z > d$ and so $S_2(z') > \min(d, z')$ for all $z' > z$. Hence $\min(d, z)$ crosses S_2 from above.

For levered equity, note that a convex combination of these securities for different levels of leverage is a security $S_2(z)$ that is convex in z with maximum slope α. Thus, any levered equity security crosses S_2 from below. A similar argument applies to call options, and to convertible debt when indexed by the equity share α. ◆