Using (a little bit of) entropy to classify surface geometries

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Abstract

The uniformization theorem, dating back to the 19th century, provides a classification of surfaces up to conformal equivalence. Classical proofs rely on harmonic analysis techniques, but with the advent of Ricci flow, spicy new proofs have been created. In this talk we'll introduce the relevant notions from Riemannian geometry, define Ricci flow, and use it to prove (modulo details) the uniformization theorem for closed oriented surfaces. The large genus cases are the easiest to work with, but uniformization on the sphere turns out to require a clever trick. You guessed it, the trick is entropy.

1 Motivation

The theme for the student analysis seminar this quarter is entropy. It turns out that one pretty cool application of entropy is in the classification of surface geometries. At the end of the 19th century some people proved the *uniformization theorem* which states that all simply connected surfaces with a complex structure are conformally equivalent to either the open unit disk, the complex plane, or the Riemann sphere. The original proof involved the Dirichlet problem and Hilbert space techniques. Now over 100 years has passed, and we can obtain stronger results using better techniques - namely, Ricci flow. At one point in this classification, entropy shows up. The primary reference for this talk is *Chow-Knopf, The Ricci Flow: An Introduction.*

2 Geometry

In this talk, a *surface* is a two-dimensional oriented smooth real manifold. Before talking about Ricci flow on surfaces we need to define a few geometric objects which are associated to surfaces. In fact, all of these quantities are defined on smooth manifolds so I'l define them in this generality.

Definition 2.1. A *metric* g on a smooth manifold M is a smoothly varying inner product on each of the tangent spaces of M. The data of (M, g) is called a *Riemannian manifold.*

By equipping a manifold with a metric, we introduce the notion of isometry. This added rigidity beyond the topological data of the manifold is what puts us in the realm of geometry.

Suppose we have a positive smooth function $r : M \to \mathbb{R}_{>0}$. Then rg is another metric on M, since at any point x in M, $r(x)g_x$ is an inner product on T_xM. It's natural to ask what structure is preserved if we map from (M, g) to (M, rg). Clearly this isn't an isometry unless r is constantly 1, since lengths of vectors in T_xM will be scaled by $r(x)$. On the other hand, one can easily show that angles between vectors are preserved. Multiplying a metric by a positive smooth function is called a *conformal transformation* since it preserves angles although it doesn't preserve lengths.

Definition 2.2. Two metrics g and h on a manifold M are *conformally equivalent* if $g = rh$ for a positive function r. M equipped with the equivalence class of metrics $[g]$ is called a *conformal manifold*.

Definition 2.3. A connection ∇ is a certain differential operator on the tangent bundle, and more generally, on tensor products of the tangent and cotangent bundles. It's not "obvious" how one might parallel transport information from one tangent space to another, but a connection provides us with the machinery to do so. By the fundamental theorem of Riemannian geometry, given a Riemannian manifold (M, g), one can find a unique torsion free connection ∇ which preserves the metric. This is called the *Levi-Civita connection*. Hereafter every Riemannian manifold is implicitly equipped with this canonical connection.

Definition 2.4. Given a metric we might want to think about what it looks like in local coordinates of our manifold. In local coordinates the "obvious metric" on $\mathbb{R}^n$ is just $(\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2$. However, in general we can't find a local basis of our manifold in which metrics looks like this. This is because metrics have an intrinsic local invariant called *Riemann curvature tensor*, or just *curvature*. One can define the Riemann curvature of a metric g by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for vector fields X, Y, Z.

In tensor index notation, we define ${R^i}_{jkl}$ by

$$W_i {R^i}_{jkl} X^j Y^k Z^l = g_{im} {R^i}_{jkl} X^j Y^k Z^l W^m = g(R(X,Y)Z, W).$$

There are a few more related local invariants of metrics: One thing we can do is take the trace of the Riemann curvature tensor to obtain the *Ricci curvature*:

$$\mathrm{Ric}_{ij} = {R^k}_{ikj}.$$

We can also take the trace of the Ricci curvature to obtain the *scalar curvature*:

$$\mathrm{Sc} = {\mathrm{Ric}^i}_i.$$

People tend to be most familiar with the scalar curvature, since in the case of surfaces, the scalar curvature is just 2 times the Gauss curvature. (The Gauss curvature is the famous pizza invariant.)

With enough patience one can derive a bunch of identities for various curvature related quantities. Many of the derivations aren't particularly enlightening, so I'll simply list the ones that we'll use during this talk.

- $\mathrm{Ric}[cg] = \mathrm{Ric}[g]$ for any positive constant c.
- On a 2-manifold, $\mathrm{Ric} = \frac{1}{2} \mathrm{Sc}\, g$.
- If g and h are conformally related metrics on a surface with $g = e^u h$ for some function u, then their scalar curvatures are related by $\mathrm{Sc}_g = e^{-u}(\mathrm{Sc}_h - \Delta_h u)$.
- On a surface, $[\Delta, \nabla] = \frac{1}{2} \mathrm{Sc}\, \nabla$.

3 Ricci flow

Now that we've defined the Ricci curvature we can actually make sense of Ricci flow!

Definition 3.1. The *Ricci flow* is

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}$$
$$g(0) = g_0.$$

Since the metric and Ricci curvature are both 2-tensors, this equation makes sense. This is essentially the heat equation but with curvature instead of heat. The negative sign convention is the same as in the heat equation, to guarantee existence of short positive time solutions (although in general we don't have negative time solutions).

Example. Consider the sphere $\mathbb{S}^n$ with the usual metric g_0. The sphere has a constant positive curvature everywhere - in fact, the Ricci curvature satisfies

$$\mathrm{Ric} = (n-1)g_0.$$

We wish to determine the time evolution of this metric under the Ricci flow. We'll make an educated guess that

$$g(t) = r(t)^2 g_0$$

is a solution to the Ricci flow. But g is a solution if and only if:

$$2r\frac{\mathrm{d}r}{\mathrm{d}t}g_0 = \frac{\partial}{\partial t}g = -2\,\mathrm{Ric}[g] = -2\,\mathrm{Ric}[g_0] = -2(n-1)g_0.$$

Thus $g(t)$ is a solution if and only if r solves the ODE

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{n-1}{r}.$$

Combining this with the initial condition gives $r(t)^2 = r_0^2 - 2(n-1)t$.

Proposition 3.2. *Let M be an n-sphere equipped with a round metric of radius r_0. Ricci flow uniformly shrinks M to a point in time $T = r_0^2/(2n-2)$.*

The point of this talk is to give some sort of classification of surfaces using Ricci flow - so if surfaces shrink, this is bad news. We want to modify Ricci flow in some way to prevent shrinking. To do this, we need to determine how the Riemannian volume form evolves in time.

Proposition 3.3. *The evolution of the Riemannian volume form is given by*

$$\frac{\partial}{\partial t}\mathrm{d}\mu = -\,\mathrm{Sc}\,\mathrm{d}\mu.$$

Proof. In local coordinates the Riemannian volume form is given by

$$\mathrm{d}\mu = \sqrt{\det g}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

Moreover, by a formula on wikipedia, given a matrix A which depends on time, we have

$$(\det A)^{-1}\frac{\partial}{\partial t}\det A = \mathrm{tr}(A^{-1}\frac{\partial}{\partial t}A).$$

Substituting g for A, the second formula becomes

$$(\det g)^{-1}\frac{\partial}{\partial t}\det g = -2\,\mathrm{Sc},$$

so

$$\frac{\partial}{\partial t}\sqrt{\det g} = \frac{1}{2}\sqrt{\det g}\Big((\det g)^{-1}\frac{\partial}{\partial t}\det g\Big) = -\,\mathrm{Sc}\,\sqrt{\det g}.$$

□

Therefore we can control the shrinking effect of Ricci flow by some sort of "normalization" using the scalar curvature.

Definition 3.4. Let r denote the *average scalar curvature* of a Riemannian manifold M,

$$r = \frac{\int_M \mathrm{Sc}\,\mathrm{d}\mu}{\int_M \mathrm{d}\mu}.$$

Then the *normalized Ricci flow* is

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric} + \frac{2}{n} r g$$
$$g(0) = g_0.$$

We can easily verify that $\frac{\partial}{\partial t}\mathrm{d}\mu = 0$ under this normalized flow by following the same calculation as in the previous proposition.

A natural question one might now ask is: how do the solutions of the normalized Ricci flow differ to the solutions of the usual Ricci flow?

Proposition 3.5. *Solutions to the Ricci flow are in bijective correspondence with solutions to the normalized Ricci flow. Solutions to each flow differ by a rescaling of space and reparametrization of time.*

4 A priori estimates

We're now ready to state a version of the uniformization theorem and provide a proof outline. Recall that the original statement applied only to simply connected complex manifolds. Our statement applies to surfaces with arbitrary genus, and there's no need to use complex geometry.

Theorem 4.1 (Uniformization theorem)**.** *Let (Σ, g) be a closed Riemannian surface. Then g is conformally equivalent to a metric with constant curvature.*

Remark.

1. *Suppose g is conformally equivalent to two metrics with constant curvature. By the Gauss-Bonnet theorem, the curvatures of these metrics have the same sign. This gives a notion of uniqueness for the constant curvature metric in the above theorem.*
2. *Every orientable 2-manifold admits a smooth structure and hence a metric (Rado). Therefore the uniformization theorem gives a classification of all closed orientable 2-manifolds.*

A modern proof of the above theorem uses Ricci flow:

Theorem 4.2 (Uniformization II)**.** *Let (Σ, g_0) be a closed Riemannian surface. There exists a unique solution to the normalized Ricci flow*

$$\frac{\partial}{\partial t} g = (r - \mathrm{Sc}) g$$
$$g(0) = g_0.$$

Moreover, the solution exists for all time, and converges exponentially in any C^k norm to a smooth metric with constant curvature.

Remark.
- *By the Gauss-Bonnet theorem,*

$$r\,Area(\Sigma) = \pi \chi(\Sigma).$$

Since the normalised Ricci flow preserves area, r is constant.

- *One of the benefits of proving the uniformization theorem using normalised Ricci flow is that the solution $g(t)$ gives a "homotopy" from g_0 to the constant curvature metric.*
- *You might look at this formulation and say, hey, that's not Ricci flow! It uses the Scalar curvature! However, in dimension 2, the Ricci curvature is given entirely by the Scalar curvature:* $\mathrm{Ric} = \frac{1}{2}\,\mathrm{Sc}\, g$*. This is fantastic news because it means that given any solutions to the normalised Ricci flow on a surface, at time t, it'll always be conformally related to the initial metric.*

To prove uniformization, initially one derives a priori estimates that hold for arbitrary closed surfaces. Do there exist uniform upper and lower bounds on the evolution of scalar curvature? Suppose g is a solution to normalised Ricci flow on a surface. Then choosing any metric h in the conformal class of g, there exists some smooth function u such that $g = e^u h$. Thus by a result from earlier on, we have

$$\mathrm{Sc}_g = e^{-u}(\mathrm{Sc}_h - \Delta_h u).$$

We also know that

$$\frac{\partial}{\partial t} g = (r - \mathrm{Sc})g \implies \frac{\partial}{\partial t} u = (r - \mathrm{Sc}).$$

The time evolution of Scalar curvature is then given by differentiating both sides of the equation:

$$\frac{\partial}{\partial t}\mathrm{Sc} = \Delta\,\mathrm{Sc} + \mathrm{Sc}(\mathrm{Sc} - r).$$

This is a famous equation shape - it's a reaction-diffusion equation. The question of whether or not normalised Ricci flow on surfaces converges depends on which term in this equation dominates. If the Laplacian term dominates, the scalar curvature is being diffused, suggesting that solutions tend to constant curvature metrics. On the other hand, if the reaction term dominates, then scalar curvature is being concentrated.

A great thing about the reaction diffusion equation is that we can use *maximum principles* to obtain some estimates. One of the relevant maximum principles is as follows:

Proposition 4.3. *Let M be a closed manifold, F locally Lipschitz. Suppose u satisfies*

$$\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + F(u).$$

Suppose there exists $C \in \mathbb{R}$ such that $u(x, 0) \leq C$ for all $x \in M$. Let φ be a solution to the associated ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi = F(\varphi), \quad \varphi(0) = C.$$

Then $u(x, t) \leq \varphi(t)$ for all t.

Reversing all of the inequalities above gives an analogous "maximum principle" which can be used to find upper bounds for solutions to reaction diffusion equations. Apparently the maximum principle is essentially proven using the first and second derivative tests. Skipping the calculations, we obtain the following estimates:

Proposition 4.4. *For any solution g of the normalised Ricci flow on a compact surface, there exists a constant C depending only on the initial conditions such that the following hold:*

1. *If $r < 0$, then $r - Ce^{rt} \leq \mathrm{Sc} \leq r + Ce^{rt}$.*
2. *If $r = 0$, then $-\dfrac{C}{1 + Ct} \leq \mathrm{Sc} \leq C$.*
3. *If $r > 0$, then $-Ce^{-rt} \leq \mathrm{Sc} \leq r + Ce^{rt}$.*

Technically the lower bounds are easy but the upper bounds require some more work. The upper bonds are achieved by introducing a "curvature potential" $\Delta f = \mathrm{Sc} - r$.

These bounds show that we can never develop singularities in finite time. Combining short time existence with these bounds give the existence of solutions for all time.

$$\text{short time existence} + \text{long time estimates} = \text{long time existence}$$

As remarked earlier, Ricci flow is designed to guarantee short time existence in the same way that the heat equation is. Therefore we obtain the following big preliminary result.

Theorem 4.5. *If (Σ, g_0) is a closed Riemannian surface, a unique solution $g(t)$ of the normalised Ricci flow exists for all time.*

It remains to prove the C^k convergence to smooth metrics with constant curvatures.

5 Convergence when $r \leq 0$

In the actual statement of the uniformization theorem, we also require that the solution of the normalised Ricci flow converges exponentially in any C^k norm to a smooth constant-curvature metric. We break this up into three cases.

Proposition 5.1 (Uniformization II, genus at least 2)**.** *Let (Σ, g_0) be a closed Riemannian surface with $r < 0$ (i.e. genus at least 2). There exists a unique solution to the normalized Ricci flow. Moreover, the solution exists for all time, and converges exponentially in any C^k norm to a smooth metric with constant curvature.*

What do we know?

- A unique solution to the normalized Ricci flow exists for all time.
- By our estimates, $|\mathrm{Sc} - r| \leq Ce^{rt}$ for all t, so $\mathrm{Sc} \to r$ exponentially. I.e. g converges exponentially to a constant curvature metric.
- It remains to show that all derivatives of Sc decay exponentially.

Proposition 5.2. *Let g be a solution to the normalized Ricci flow on a surface Σ with $r < 0$. Then for each k there exists C_k such that for all t,*

$$\sup_{x \in \Sigma} |\nabla^k \mathrm{Sc}(x,t)|^2 \leq C_k e^{rt/2}.$$

Proof. This is a standard proof by induction where you literally just compute things and the result falls out.

Base case: recall that

$$\frac{\partial}{\partial t} \mathrm{Sc} = \Delta \mathrm{Sc} + \mathrm{Sc}(\mathrm{Sc} - r).$$

There is also a commutation relation

$$[\Delta, \nabla] = \frac{1}{2} \mathrm{Sc} \nabla.$$

Combining these we have

$$\frac{\partial}{\partial t}(\nabla \mathrm{Sc}) = \nabla(\Delta \mathrm{Sc} + \mathrm{Sc}(\mathrm{Sc} - r)) = \Delta \nabla \mathrm{Sc} + \frac{3}{2} \mathrm{Sc} \nabla \mathrm{Sc} - r \nabla \mathrm{Sc},$$

and then by applying the product rule a bunch of times

$$\frac{\partial}{\partial t} |\nabla \mathrm{Sc}|^2 = \Delta |\nabla \mathrm{Sc}|^2 - 2|\nabla\nabla \mathrm{Sc}|^2 + (4\,\mathrm{Sc} - 3r)|\nabla \mathrm{Sc}|^2.$$

But we already have a bound on $\mathrm{Sc} - r$, namely $|\mathrm{Sc} - r| \leq Ce^{rt}$, so

$$\frac{\partial}{\partial t}|\nabla \mathrm{Sc}|^2 \leq \Delta|\nabla \mathrm{Sc}|^2 - 2|\nabla\nabla \mathrm{Sc}|^2 + (r + 4Ce^{rt})|\nabla \mathrm{Sc}|^2.$$

For all t sufficiently large, this implies

$$\frac{\partial}{\partial t}|\nabla \mathrm{Sc}|^2 \leq \Delta|\nabla \mathrm{Sc}|^2 + \frac{r}{2}|\nabla \mathrm{Sc}|^2.$$

The maximum principle applies! At $t = 0$, Sc is bounded uniformly by some constant C_1 since Σ is compact. The solution to the associated ODE is just $C_1 e^{rt/2}$. Hence

$$|\nabla \mathrm{Sc}|^2 \leq C_1 e^{rt/2}.$$

Inductive step: much the same. At the end we apply the maximum principle again. □

Proposition 5.3 (Uniformization II, genus 1)**.** *Let (Σ, g_0) be a closed Riemannian surface with $r = 0$ (i.e. genus 1). There exists a unique solution to the normalized Ricci flow. Moreover, the solution exists for all time, and converges exponentially in any C^k norm to a smooth metric with constant curvature.*

Proof. Similar to the genus-at-least-2 case. Lots of maximum principle. □

6 Convergence when $r > 0$

Proposition 6.1 (Uniformization II, genus 0)**.** *Let (Σ, g_0) be a closed Riemannian surface with $r > 0$ (i.e. genus 0). There exists a unique solution to the normalized Ricci flow. Moreover, the solution exists for all time, and converges exponentially in any C^k norm to a smooth metric with constant curvature.*

What do we know? We have the bounds:

$$-Ce^{-rt} \leq \mathrm{Sc} \leq r + Ce^{rt}.$$

Not good! Right hand side grows exponentially. Things get ridiculously difficult so I won't delve too deep. First we wish to find a uniform upper bound for Sc. Hello entropy:

Definition 6.2. Suppose (M, g) is a Riemannian manifold with strictly positive scalar curvature. Its *entropy* is defined by

$$N(g) = \int_M \mathrm{Sc} \log \mathrm{Sc} \, \mathrm{d}\mu.$$

We've defined entropy and we have a notion of time, so does it decrease with time?

Proposition 6.3. *If g is a solution of the normalized Ricci flow on a surface Σ, and* Sc *is strictly positive at time 0, then*

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\int_\Sigma \frac{|\nabla \mathrm{Sc} + \mathrm{Sc}\,\nabla f|^2}{\mathrm{Sc}} \mathrm{d}\mu - 2\int_\Sigma |M|^2 \mathrm{d}\mu.$$

Here f denotes the curvature potential $\Delta f = \mathrm{Sc} - r$, *and M its trace-free Hessian.*

This shows that entropy is strictly decreasing unless Sc is identically equal to r, i.e. entropy is strictly decreasing unless g has constant curvature. In general the sign of Sc might dip below zero, but the definition of entropy can be modified to accommodate this.

In case anyone asks:

$$\widehat{N}(g, s) = \int_\Sigma (\mathrm{Sc} - s) \log(\mathrm{Sc} - s) \mathrm{d}\mu,$$

where s is a solution to $\frac{\mathrm{d}}{\mathrm{d}t} s = s(s - r)$. By the maximum principle, $\mathrm{Sc} - s$ is positive.

Proposition 6.4. *Let g be a solution of the normalized Ricci flow on Σ, for $r > 0$. There exists a constant depending only on g_0 such that for all t*

$$N(g(t)) \leq C.$$

Proposition 6.5. *Let $T > 0$, and x_1, t_1 such that*

$$\mathrm{Sc}(x_1, t_1) = \max_{\Sigma \times [0,T]} \mathrm{Sc} = \kappa.$$

There is a constant c (depending only on g_0) such that

$$C \geq N(g(t_1)) \geq c \log \frac{\kappa}{2}.$$

In this way we can find a uniform bound for the scalar curvature! This is the first step in proving uniformization for the final case with genus 0. Unfortunately things got too involved from here, but this convinced me that entropy is a real thing.

7 Uniformization in higher dimensions?

Ricci flow has been used to "uniformize" Riemannian surfaces. The natural direction is to investigate higher dimensional surfaces. Recall that the uniformization theorem gives a classification of metrics, but it also implies a topological classification.

- Can we classify 3-manifolds using Ricci flow? Yes, using Ricci flow, Perelman proved the *geometrization conjecture* which is pretty much the exact analogue of the uniformization theorem for three manifolds. It states that every three manifold can be canonically decomposed into "primes" each having a unique geometry.
- Can we classify n-manifolds (for $n \geq 4$) using Ricci flow? Mathematically impossible. If we classify geometry we automatically classify topology. But given any finitely presented group we can find an n-manifold whose fundamental group is that manifold. By the unsolvability of the word problem, there doesn't exist a topological classification. (Surgery theory gives a classification which is weaker than homeomorphism.)
- What else can we do with Ricci flow?