CUTTING SQUARES INTO TRIANGLES

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Abstract. Given a positive integer n, can we cut a square into n equal area triangles? This turns out to be a deep problem, which we study with tools from algebra and graph theory.

1. Introduction: equidissections of polygons

10 minutes

Question. Let T be a triangle, and n a positive integer. Can T be cut up into n equal area triangles?

Answer. Yes! Choose any edge of the triangle, and cut it into n equal segments. Lines from the endpoints of these segments to the opposite corner of the triangle form n equal area triangles.

This procedure is called an *equidissection*.

Definition 1.1. Let P be a polygon. An *equidissection* of P is a decomposition of P into (some number of) equal area triangles. Alternatively, an *equidissection* is a *triangulation* of P consisting of equal area triangles.

Question. Let S be a square, and n a positive integer. Can S be cut up into n equal area triangles?

Answer. If n is even, yes: cut the square in half, and apply the previous result to both halves. If n is odd, things get difficult!

Theorem 1.2 (Monsky)**.** *Let S be a square, and n an odd number. There are no equidissections of S into n triangles.*

This proof is surprisingly involved. We start with a foray into graph theory.

2. Ingredient 1: graph theory

15 minutes We apply something called *Sperner's lemma*. This essentially says the following:

> Let P be a polygon with triangulation T. Colour every vertex of the triangulation either red, blue, or green. Suppose vertices on any given *edge of P* are restricted to two colours. Then at least one triangle in the triangulation consists of vertices with all three colours.

We will prove the following:

Lemma 2.1. Let S be a square with triangulation T. Suppose

1. (1) Vertices of T along the bottom side of S are red or blue. (The bottom-left corner of S is red, the bottom-right corner is blue.)
2. (2) Vertices of T along the left side of S are red or green.
3. (3) Vertices of T along the right side of S are blue or green.
4. (4) Vertices of T along the top side of S are blue or green.

Then there is at least one triangle in T whose vertices exhibit all three colours.

Proof. Proof is conveyed visually, as are the conditions above.

1. (1) Define a "sub-dual graph" as follows:
   - The vertices are the faces of T.
   - There is an edge between vertices exactly when it crosses over a red-blue edge.
2. (2) The handshaking lemma: $\sum_v \deg(v) = 2|E|$. It follows that there are an *even number* of vertices in the above graph with odd degree.
3. (3) Consider the vertex corresponding to "outside the square". All of its edges pass through the bottom side of the square. Since the bottom-left corner is red while the bottom-right corner is blue, there must be an odd number of red-blue edges along the bottom side of the square. It follows that the outer vertex has *odd degree*.
4. (4) In summary, there are an even number of vertices with odd degree. We also know there is at least one vertex of odd degree. Therefore there is at least one other vertex of odd degree!
5. (5) The degrees are necessarily $0, 1$, or 2. Therefore there is a degree-1 vertex. This vertex corresponds to a triangle whose vertices are all three colours.

□

We've established a random graph theoretic fact about colouring triangulations. The strength of this result comes from choosing a suitable method of colouring vertices of a triangulation.

3. Ingredient 2: algebra

10 minutes To colour the vertices of a triangulation of a square, we'll devise a systematic way to colour every point in $\mathbb{R}^2$. This will use the *2-adic norm*. The next part of our talk concerns the definition of the *2-adic norm* and its properties.

Definition 3.1. Let $a \in \mathbb{Q}$. There is a unique way to write a as

$$a = 2^n \frac{p}{q}$$

where p/q is a reduced fraction and 2 doesn't divide either p nor q. The *2-adic norm* is defined to be

$$|a|_2 = 2^{-n}.$$

Proposition 3.2. The 2-adic norm satisfies the following properties:

(1) Non-negative: $|a|_2 \geq 0$.
(2) Definite: $|a|_2 = 0 \Leftrightarrow a = 0$.
(3) Multiplicative: $|ab|_2 = |a|_2|b|_2$.
(4) Ultrametric: $|a + b|_2 \leq \max(|a|_2, |b|_2)$.

Proof. Ask room to prove them one by one as I write them! The first three are elementary. The ultrametric inequality comes from the fact that 2 must divide $a + b$ at least as many times as it divides each of a and b. □

Proposition 3.3. The 2-adic norm extends from $\mathbb{Q}$ to $\mathbb{R}$.

Proof. This is the only step we won't prove in this talk. The difficulty is that an arbitrary real number can't be expressed as $2^{-n}p/q$. For those interested, this is a general result about extension of generalised "absolute values" from fields to field extensions. (I once read a proof in Serge Lang's algebra.) For those interested: Lang was an avid activist as well as mathematician, but for both good and bad things. □

4. Monsky's theorem

15 minutes

Theorem 4.1. *A square cannot be equidissected into and odd number of pieces.*

The proof proceeds in three steps.

(1) We consider the square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$. We use the 2-adic norm to devise a method of colouring every point in the square with one of three colours: red, blue, and green.
(2) We apply Sperner's lemma to see that any triangulation of the square must contain a triangle whose vertices are all three colours; red, blue, and green.
(3) We use properties of the 2-adic norm to "compute" the area of such a triangle.

Step 1. Consider the unit square $[0, 1]^2 \subset \mathbb{R}^2$. We colour every point (a, b) in the square with the following rules:

(1) If $|a|_2 < 1$ and $|b|_2 < 1$, then (a, b) is red.
(2) If $|a|_2 \geq \max(1, |b|_2)$, then (a, b) is blue.
(3) Otherwise, (a, b) is green.

Example. The vertex $(0, 0)$ has norms $|a|_2 = |0|_2 = 0$ and $|b|_2 = |0|_2 = 0$. These are both less than 1, so $(0, 0)$ is coloured red.

Example. The vertex $(1, 0)$ has norms $|a|_2 = |1|_2 = 1$ and $|b|_2 = 0$. This is because

$$1 = 2^0 \frac{1}{1}; \quad |1|_2 = 2^{-0} = 1.$$

It follows that the vertex is blue.

Example. The vertex $(1, 1)$ is blue, and $(0, 1)$ is green.

Example. Any straight line through the square has at most 2 colours. This can be shown by writing the line as $(a + tr, b + ts)$ and applying the ultrametric inequality.

Step 2. Choose any triangulation of $[0,1]^2$. The colouring we chose above perfectly corresponds to our version of Sperner's lemma. Therefore the triangulation contains a triangle whose vertices are all three colours; red, blue, and green.

Step 3. Let Δ be the triangle whose vertices are all three colours. To make our subsequent algebra easier, we'll translate our square so that the *red vertex* of Δ becomes $(0,0)$.

Lemma 4.2. Translating the entire square to send a red point to $(0,0)$ leaves the colouring unchanged.

Proof. Suppose the point (x,y) is sent to $(x-a, y-b)$ where (a,b) is red. One can use the ultrametric inequality to verify that the colour of $(x-a, y-b)$ is the same as that of (x,y).

Example case: if (x,y) is red, then $|x|_2 < 1$, $|y|_2 < 1$. But the ultrametric inequality ensures that $|x-a|_2 \leq \max(|x|_2, |a|_2) < 1$, and similarly $|y-b|_2 < 1$. Therefore $(x-a, y-b)$ is red. □

In summary, our triangle Δ can now be written with vertices:

$$v_r = (0,0), \quad v_b = (a,b), \quad v_g = (c,d).$$

Ask room: What is the area of this triangle? The area of the triangle is

$$A(\Delta) = \frac{|ad-bc|}{2}.$$

This comes from linear algebra, because the area of a parallelogram is the determinant of the matrix whose columns are the vectors defining the edges of the parallelogram.

As one might expect, given our information, we aren't able to actually compute $A(\Delta)$. However, we can find a lower bound on its 2-adic norm!

Lemma 4.3. $|A(\Delta)|_2 \geq 2$.

Proof.

$$\begin{aligned} |A(\Delta)|_2 &= \left| \frac{|ad-bc|}{2} \right|_2 \\ &= \left| \frac{1}{2} \right|_2 |ad-bc|_2 \\ &\geq 2|ad|_2 \\ &= 2|a|_2|d|_2 \geq 2. \end{aligned}$$

In the last step, we use that $|a|_2$ and $|d|_2$ are both at least 1 (by the colouring).

There is one elusive step: by the ultrametric inequality, we know that

$$|ad|_2 = |(ad-bc)+bc|_2 \leq \max(|ad-bc|_2, |bc|_2).$$

On the other hand, in our colouring, (c,d) is green which forces $|c|_2 < |d|_2$, and (a,b) is bluewhich forces $|b|_2 \leq |a|_2$. Therefore

$$|ad|_2 = |a|_2|d|_2 > |b|_2|c|_2 = |bc|_2.$$

Therefore

$$|bc|_2 < |ad|_2 \leq \max(|ad - bc|_2, |bc|_2).$$

This forces $|ad|_2 \leq |ad - bc|_2$. $\square$

Lemma 4.4. For n odd, $|1/n|_2 = 1$.

Proof. Ask room. This is because $1/n = 2^0 \cdot 1/n$, so $|1/n|_2 = 2^{-0} = 1$. $\square$

Theorem 4.5 (Monsky)**.** *there are no equidissections of a square into an odd number of triangles.*

Proof. Consider *any* triangulation of a square. The earlier work with Sperner's lemma and the 2-adic colouring demonstrates that there exists a triangle Δ in our triangulation with $|A(\Delta)|_2 \geq 2$. However, an odd-numbered equidissection necessarily consists of triangles whose areas are all $1/n$ for n odd. This means every triangle must have area with 2-norm 1. $\square$