# How to cut shapes

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**Abstract**

The fold and cut problem asks when a given polygon can be cut out of a sheet of paper with a single straight cut (after folding the paper some number of times). Around 20 years ago Demaine, Demaine, and Lubiw showed that actually all polygons can be cut out this way, and we will interactively explore one of their solutions. Paper and scissors will be provided! Join me for a fun afternoon of chopping up paper.

## 1 Who am I? (5 mins)

Hi everyone! I'm a 3rd year grad student in the maths programme here. Nice to meet you. Before talking about maths, I figured I should introduce myself a little.

I'm an international student from New Zealand and Japan. I spent about three quarters of my childhood in New Zealand and the other quarter in Japan. I'm fortunate enough to be bilingual! Somehow most of the grad students in the department seem to have exactly two non-mathematical interests, and one of them is always climbing. However I think my interests are quite varied, but largely encompass art. As I was growing up my main hobby was drawing. I basically just used pencil and paper, but when I started undergrad (at the University of Auckland in New Zealand) I expanded to painting. After starting grad school here at Stanford I've also started doing fibre arts - cross stitching, felting and crochet, and recently I've started sculpture.

In some sense the reason I'm doing maths is actually because of my love of visual art. Everyone around me encouraged me to pursue STEM as I was finishing high school because I had good grades in all of my science classes, but I started to lament the lack of art-vibes in my career choice. Somehow the sciences that are grounded in reality felt like they had less space for artistic creativity so I was pushed towards maths, and within maths I ended up doing topology.

Topology is the study of wobbly shapes. In my research today my goal is to understand two dimensional surfaces that are embedded inside four dimensional blobs. Concretely my research looks like me sitting in front of a notebook with about 10 colourful pens and drawing a bunch of pictures, trying to understand what's going on!

Today I'll be presenting the "fold and cut theorem". This isn't a topology result, but it's nonetheless geometric and highly visual. It's the type of mathematical gem that I love to stumble upon on the internet and I hope you'll all enjoy it as well.

## 2 Motivation and introduction (10 mins)

Arts and crafts: in any arts and crafts setting, one always finds themselves cutting shapes out of paper.

1. *Cut circle out of paper.* The typical way to cut a shape out of paper is to cut through the outside of the shape. This method will perfectly produce the desired shape, but at the cost of destroying the outside paper.
2. *Cut circular hole in paper.* Maybe you want to cut a hole in a piece of paper, rather than cutting out a disk. In this context, the usual approach is to puncture the middle of the circle and cut out the hole, which perfectly preserves the outside paper at the cost of destroying the center.
3. What if you want to cut out a circle while preserving both the inside and outside of it? *Fold paper, then cut.* By folding the paper first, we can cleanly cut both the inside and outside of the paper!

One of the most powerful aspects of scissors is the ease of cutting curved shapes. However, many cutting tools that exist such as saws and guillotines are restricted to straight cuts. In a mathematical setting as well, straight cuts the easiest to conceptualise and study. Hereafter we'll only consider straight cuts.

- Earlier I folded a piece of paper in half to cut out a piece of paper. No matter how many times I fold a piece of paper, I'll never be able to cut out a circle with a single straight cut because a circle has curved edges.
- That being said, if a shape consists only of straight edges, maybe it's possible to cut it out with only a single straight cut. *Complicated and fun example.*

**Theorem 2.1** (Demaine, Demaine, Lubiw)**.** *Given a shape consisting of straight edges (i.e. given a finite collection of line segments) on a piece of paper, there is a way to fold the paper a finite number of times so that a single straight cut through the folded paper cuts out the desired shape.*

## 3 Interactive examples (10 mins)

I now have some tasks for everyone, in approximately increasing difficulty as follows:

**Exercise 3.1.**

1. *Fold a piece of paper and make a single straight cut so that it unfolds to a regular $n$-gon (for some $n$).*
2. *Fold a piece of paper and make a single straight cut so that it unfolds to a star. (Six-pointed star is probably easiest.)*
3. *Fold a piece of paper and make a single straight cut so that unfolds to a shape that isn't connected. For example, two square holes side by side.*
4. *Fold a piece of paper and make a single straight cut so that it unfolds to a shape which isn't simply connected. For example, a square with a square hole inside it. (Can you generalise this to cut out shapes with arbitrarily many holes?)*

## 4 A proof: 1-skeleton method (20 mins)

Loosely speaking, this shows that we can make shapes with lots of edges, which aren't necessarily connected or simply connected. This leads to the following conjecture:

*Any shape comprised of finitely many piece-wise linear edges (not necessarily connected or simply connected) can be cut out from a piece of paper.*

I suspect that everyone has noticed that all of the solutions so far heavily rely on symmetry. Even though we've established that topologically arbitrarily complex shapes can be created, we're very far from knowing how to cut out asymmetric things.

**Exercise 4.1.** *1. Fold a piece of paper and make a single cut so that it unfolds to a given arbitrary (asymmetric) triangle.*

*Demonstration: fold paper, cut out arbitrary triangle.*

In fact, this gives us a general approach that might work on all "generalised polygons".

(a) Determine 1-skeleton of the polygon.

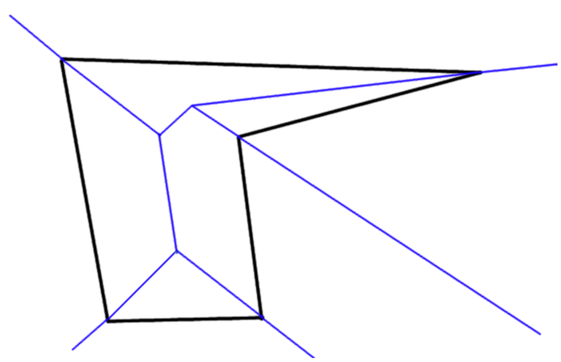
(b) Draw all edge-orthogonals, reflecting through the 1-skeleton where necessary.

(c) Fold along these lines!

The best way to explain this procedure is through an example. Can anyone give me an example of a not-too-complicated but slightly non-trivial shape?

(a) 1-skeleton: Draw the shape. Consider the time-evolution of the shape, given by expanding/contracting the shape with a constant orthogonal speed. The path traced by vertices determines the 1-skeleton.

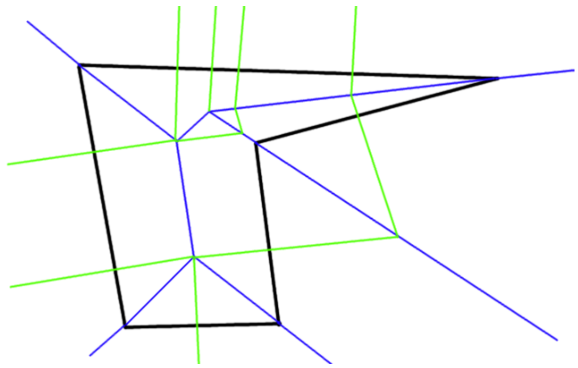
*Demonstration: draw example on board.*



(b) egde-orthogonals: The 1-skeleton is intuitively useful, because they correspond to angle bisectors. This means they work to bring all of the edges of our polygon onto each other. However, it isn't enough: there is no way to fold along these lines in general. To remedy this, we add more crease-guides. Intuitively the locations where "folding the paper into something flat" fails are the vertices of the 1-skeleton. Drawing lines from the 1-skeleton which are orthogonal to the polygon edges corresponds to adding more angle bisectors (as the orthogonal bisects the angle $\pi$).

Unfortunately the orthogonals can't always be drawn in a naive way, as the 1-skeleton can get in the way. When this happens, to ensure that our shape folds properly, we reflect through the 1-skeleton.

*Demonstration: drawing orthogonals.*



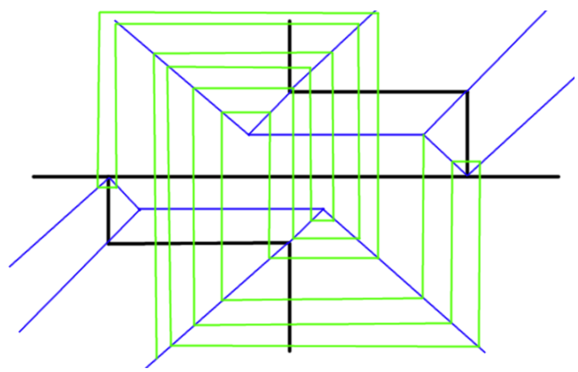
To determine whether or not this solves the problem, we need to verify two things:

(i) The above procedure always gives finitely many crease-guides.

(ii) There exists a flat folding of the paper along creases determined by only the 1-skeleton and orthogonals.

It turns out that this doesn't work! (i) can fail.

*Demonstration: crease-guides not necessarily finite.*



**Conjecture 4.2.** *Given a random graph, the orthogonals of its 1-skeleton are dense with probability 0.*

If the conjecture is true, then tiny perturbations of our desired polygon should always be fold-and-cut-able. However, the 1-skeleton method will not work in full generality.

**Exercise 4.3.** *1. Use the 1-skeleton method to determine a folding of some shape that you like.*

# 5 Disk-packing method

Although the 1-skeleton approach doesn't actually work in full generality, there is a slightly more impractical algorithm which is guaranteed to work. The idea is essentially to first decompose the shape into a collection of triangles, and then simultaneously cut out all of the triangles. (By slightly "shrinking" the shape first, the only edges of the triangles that end up getting cut are the edges belonging to the shape of interest.)