Asymptotic Curvature of Hypersurfaces in Minkowski Space

Shintaro Fushida-Hardy

Department of Mathematics
The University of Auckland

Supervisor: Professor Rod Gover

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Abstract

The asymptotic curvatures of certain space-like hypersurfaces in Minkowski space are investigated using conformal tractor calculus [BEG94]. Given a conformally compact manifold M and a choice of scale, it is shown that the corresponding scale tractor I determines the asymptotic curvature. In particular, M is asymptotically hyperbolic if $|I|^2$ tends to a positive constant at conformal infinity. Understanding the asymptotic curvatures of hypersurfaces then becomes the question of understanding how hypersurface scale tractors relate to ambient scale tractors.

Totally umbilic hypersurfaces in Lorentzian manifolds are initially considered. Given a choice of scale, the intrinsic scale tractor is shown to be the projection of the ambient scale tractor. It follows that totally umbilic space-like hypersurfaces in Minkowski space with non-zero mean curvature are asymptotically hyperbolic.

More generally, constant mean curvature hypersurfaces in Lorentzian manifolds are also considered. Using results from [GWss] and extending them to Lorentzian signature ambient spaces, a formula relating intrinsic hypersurface scale tractors and ambient scale tractors is obtained, showing that constant non-zero mean curvature space-like hypersurfaces in Minkowski space are asymptotically hyperbolic. Finally, the result is generalised to such hypersurfaces in arbitrary asymptotically flat spacetimes.

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Contents

Introduction

Many studies have pointed to the existence of constant mean curvature (CMC) initial datasets, particularly in the special case of Minkowski space [Lee18]. Studying such hypersurfaces sheds light on the geometry of the associated spacetimes.

This dissertation interprets and provides an affirmative answer to the question "are all space-like CMC hypersurfaces in Minkowski space asymptotically hyperbolic?"

In the preliminaries, vector bundles and pseudo-Riemannian manifolds are introduced. Connections on pseudo-Riemannian manifolds are then defined, enabling differential calculus. In the final section of the preliminaries, various notions of curvature are introduced.

In chapter two, conformal geometry is introduced. It is observed that the Levi-Civita connection for a given metric in the conformal class is not conformally invariant. The precise conformal rescaling of certain curvature tensors is determined, and conformal densities are defined. These are used in chapter three to construct tractor calculus, the conformally invariant calculus on conformal manifolds.

A compactification of Minkowski space is subsequently constructed, with a parallel and null scale tractor, whereby the asymptotic curvatures of space-like hypersurfaces are studied. Initially totally umbilic hypersurfaces are considered. Certain objects intrinsic to the hypersurface are found to agree with projections of the corresponding ambient objects. For example, it is shown that the projection of the ambient tractor connection is the intrinsic tractor connection. Following this, the more general case of CMC hypersurfaces is investigated.

A space-like CMC hypersurface in Minkowski space with non-zero mean curvature is proved to be asymptotically hyperbolic. More generally, the result also holds for space-like CMC non-zero mean curvature hypersurfaces in any asymptotically flat spacetime.

Chapter 1

Preliminaries

The reader is assumed to be familiar with differential calculus on $\mathbb{R}^n$ and linear algebra, as well as elementary notions from differential geometry including smooth manifolds, smooth functions, Lie groups, and exterior derivatives. Familiarity with *Einstein index notation* will be beneficial, since Penrose's *abstract index notation* is used extensively in this dissertation.

1.1 Vector Bundles and Tensor Fields

Definition 1.1.1. Let F, M, E be smooth manifolds, and $\pi : E \to M$ a smooth map. (E, π, M, F) is called a *fibre bundle* if for each $p \in M$ there is a neighbourhood $U \subset M$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ U & \end{tikzcd}$$

φ is called a *local trivialisation*, π the *projection*, F the *typical fibre*, M the *base space*, and E the *total space*. (E, π, M, F) is often written as $\pi : E \to M$ or E.

Definition 1.1.2. Let $\pi : E \to M$ be a fibre bundle. A *section* is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = id_M$. The space of all sections of the fibre bundle is denoted $\Gamma(E)$.

Definition 1.1.3. Let V be a finite dimensional vector space over $\mathbb{R}$. A *vector bundle* is a fibre bundle (E, π, M, V) such that for each $p \in M$, there is neighbourhood U of p and a local trivialisation $\varphi : \pi^{-1}(U) \to U \times V$ such that for every $x \in U$ $\varphi_2 : E_x \to V$ is a vector space isomorphism, where $\varphi = (\varphi_1, \varphi_2)$. The dimension of V is the *rank* of the vector bundle.

Definition 1.1.4. A *principal G-bundle*, where G denotes a Lie group, is a fibre bundle $\pi : E \to M$ together with a smooth right action $E \times G \to E$ such that G preserves the fibres of E, and acts freely and transitively on them.

Definition 1.1.5. Suppose $\pi : P \to M$ is a principal G-bundle. Let $\rho : G \times V \to V$ be a finite dimensional representation. The *associated vector bundle* is

$$\widehat{\pi} : P \times_\rho V \to M$$

where $P \times_\rho V := {}^{P \times V}\big/_{\sim}$, and $\sim$ is the equivalence relation $(p, v) \sim (pg, \rho(g^{-1}, v))$. $\widehat{\pi}$ is defined by $\widehat{\pi} : [(p, v)] \mapsto \pi(p)$.

Definition 1.1.6. Given vector bundles $(\mathcal{V}, \pi, M, V)$ and $(\mathcal{W}, \varpi, M, W)$, a *vector bundle homomorphism* (VB-morphism) is a smooth map $\varphi : \mathcal{V} \to \mathcal{W}$, which is linear on each fibre, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathcal{W} \arrow[dl, "\varpi"] \\
M &
\end{tikzcd}$$

An invertible VB-morphism whose inverse is also a VB-morphism is called a *VB-isomorphism*. If a VB-isomorphism exists between two vector bundles, they are said to be *isomorphic*.

Definition 1.1.7. A *short exact sequence of vector bundles* is a diagram

$$\emptyset \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow \emptyset$$

such that the sequence is exact and each map is a VB-morphism. $\emptyset$ denotes the vector bundle with fibre $\{0\}$, i.e. the "zero bundle". Similarly the kernel of each map is the preimage of $\emptyset$ (as $\emptyset$ is a subbundle of any vector bundle). There is a subbundle $\mathcal{V}'$ of $\mathcal{V}$ such that $\mathcal{U} \cong \mathcal{V}'$ and $\mathcal{W} \cong {}^{\mathcal{V}}\big/_{\mathcal{V}'}$. Although all short exact sequences of vector bundles split, situations arise in subsequent chapters in which splitting is choice dependent (there being no canonical choice). The semidirect sum notation is then adopted, being

$$\mathcal{V} = \mathcal{W} \niplus \mathcal{U}.$$

Definition 1.1.8 (Tangent bundle)**.** Let M be a smooth manifold. Let $p \in M$. The *tangent space at p* is the set of all derivations at p, denoted T_pM. (A derivation is a linear map $X \in \hom(C^\infty(M), \mathbb{R})$ satisfying the product rule.) The *tangent bundle*, denoted TM or $\mathcal{E}^a$, is the vector bundle with fibre $E_p = T_pM$, equipped with the natural smooth structure.

The tangent bundle exists uniquely on every smooth manifold. Further construction details can be found in [Lee00] (chapter 3).

The cotangent bundle of M is the dual of the tangent bundle, denoted T^*M or $\mathcal{E}_a$. The fibre of $\mathcal{E}_a$ at each p in M is the dual vector space to T_pM. Tensor bundles are constructed by taking tensor products of vector bundles, for example,

$$\mathcal{E}_{ab}{}^c := T^*M \otimes T^*M \otimes TM.$$

The indices above are "abstract", being labels for keeping track of object type and contractions without invoking a frame. Abstract index notation is discussed in [CG18] (chapter 0). Indices starting in the middle of the alphabet $(i, j, \cdots)$ are used when they correspond to a frame. Einstein summation convention is then used unless stated otherwise.

Sections of the tangent bundle are called *vector fields* or *contravariant vector fields*. Sections of the cotangent bundle are called *covariant vector fields* or *1 forms*. Sections of tensor product bundles are called *tensor fields* or simply *tensors*. Consistent with established notation (see for example [CG18]), $\mathcal{E}$ is used to denote the trivial line bundle, so $\Gamma(\mathcal{E}) = C^\infty(M)$.

Definition 1.1.9. Let $T_{ab\cdots e}$ be a tensor field. The *symmetric part* of T is denoted $T_{(ab\cdots e)}$, and the *skew part* of T, $T_{[ab\cdots e]}$.

For example, $T^{[ab]c}{}_d = \frac{1}{2}(T^{abc}{}_d - T^{bac}{}_d)$, and $S_{(abc)} = \frac{1}{6}(S_{abc} + S_{acb} + \cdots + S_{cba})$. Similarly, the space of symmetric two-tensor fields is written $\Gamma(\mathcal{E}_{(ab)})$. However, Λ^n is used to denote the top exterior power.

Definition 1.1.10. The Kulkarni-Nomizu product is denoted by $\owedge$, wherein

$$f_{ab} \owedge g_{cd} = f_{ac}g_{bd} + f_{bd}g_{ac} - f_{ad}g_{bc} - f_{bc}g_{ad}.$$

1.2 Pseudo-Riemannian Manifolds

Definition 1.2.1. Let M be a smooth manifold, and $g_{ab} \in \Gamma(\mathcal{E}_{(ab)})$. If $g_p : T_pM \to \mathrm{Hom}(T_pM, \mathbb{R})$ defined by $g_p(v) \mapsto (g_{ab}v^b)_p$ is an isomorphism, g is said to be *non-degenerate*.

Definition 1.2.2. Let M be a smooth manifold, and $g_{ab} \in \Gamma(\mathcal{E}_{(ab)})$. If g is non-degenerate, it is termed a *pseudo-Riemannian metric tensor* (or simply a *metric*). The manifold equipped with g, written (M, g), is called a *pseudo-Riemannian manifold*.

Definition 1.2.3. Let g be a metric and $x \in M$. The *signature* (p, q) of g is the number of positive and negative eigenvalues of g_x, respectively.

The signature is independent of x [O'N83]. On a manifold of dimension n, a metric of signature $(n, 0)$ is termed *Riemannian*, a metric of signature $(n-1, 1)$ being *Lorentzian*.

Since metrics are non-degenerate, they can be used to raise, lower, and contract indices. Although the trace of an endomorphism is independent of a metric, only the metric-trace is used in this dissertation. Therefore the terms *contraction* and *trace* are used interchangeably.

Definition 1.2.4. Let T be a 2-tensor on an n-manifold. The *trace free part* of T, denoted $\overset{\circ}{T}$, is the tensor obtained by subtracting its trace. For example,

$$\overset{\circ}{T}_{ab} = T_{ab} - \frac{1}{n}T^c{}_c g_{ab} = T_{ab} - \frac{1}{n}g^{cd}T_{cd}g_{ab}.$$

1.3 Levi-Civita Connection

Definition 1.3.1. Given a vector bundle $\pi : E \to M$, a *connection* over E is a map $\nabla : \Gamma(TM)\times\Gamma(E) \to \Gamma(E)$, with $\nabla(X, Y)$ written $\nabla_X Y$, satisfying the following properties:

- For all $f, g \in \Gamma(\mathcal{E})$, $A, B \in \Gamma(TM)$, and $Y \in \Gamma(E)$, $\nabla_{fA+gB}Y = f\nabla_A Y + g\nabla_B Y$.
- For all $U, V \in \Gamma(E)$, $\nabla_X(U + V) = \nabla_X U + \nabla_X V$.
- For all $f \in \Gamma(\mathcal{E})$, $X \in \Gamma(TM)$, and $Y \in \Gamma(E)$, $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$.

An *affine connection* is a connection on TM. If a connection is introduced without a specified vector bundle, the connection is assumed to be affine.

Definition 1.3.2. A connection on a pseudo-Riemannian manifold is *compatible with the metric* if for any $X, Y, Z \in \Gamma(TM)$,

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

In abstract index notation, this is equivalent to $\nabla_a g_{bc} = 0$. Generally, any tensor T is said to be *parallel* with respect to a connection if ∇T vanishes.

Definition 1.3.3. The *torsion* $T : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ of a connection is defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

for all $X, Y \in \Gamma(TM)$. The connection ∇ is termed *torsion free* if $T = 0$.

Definition 1.3.4. Let $\{E_1, \cdots, E_n\}$ be a local coordinate frame for TM and ∇ a connection. The functions Γ defined by

$$\nabla_i E_j = \Gamma^k{}_{ij} E_k$$

are called the *connection coefficients* of ∇, and uniquely determine the connection. (The summation convention is not being used.)

Theorem 1.3.5 (Fundamental theorem of Riemannian geometry)**.** *Let (M, g) be a pseudo-Riemannian manifold. Then there exists a unique connection on $\mathcal{E}^a$ which is torsion free and metric compatible. This connection is called the Levi-Civita connection.*

Proof. A proof can be found in [O'N83] (chapter 3). □

The proof cited above is a derivation of the *Koszul formula*:

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk} \right). \tag{1.1}$$

To be the Levi-Civita connection, the connection coefficients must satisfy the Koszul formula. Connection coefficients of the Levi-Civita connection are called *Christoffel symbols*.

1.4 Curvature

Definition 1.4.1. Let ∇ denote the Levi-Civita connection on a pseudo-Riemannian manifold. The *Riemann curvature* is defined by

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

for all $X, Y, Z \in \Gamma(\mathcal{E})$.

A pseudo-Riemannian manifold is termed *flat* if its Riemann curvature vanishes. This definition can be generalised to other connections, in which case the connection is termed flat if its associated curvature form vanishes.

Proposition 1.4.2. *The Riemann curvature is a tensor field.*

Proof. See [Lee97] (chapter 7). □

In abstract index notation, $(R(X, Y)Z)^d$ is written $X^a Y^b R_{abc}{}^d Z^c$. This agrees with index conventions in [Lee97], but differs from those in, for example, [GW15].

Proposition 1.4.3. *The Riemann curvature satisfies the following symmetries.*

- *Skew symmetries:* $R_{abcd} = -R_{bacd} = -R_{abdc}$.
- *Interchange symmetry:* $R_{abcd} = R_{cdab}$.
- *Algebraic Bianchi identity:* $R_{(abc)d} = 0$.
- *Differential Bianchi identity:* $\nabla_{(a} R_{bc)de} = 0$.

Proof. A proof can be found in [Lee97] (chapter 7). □

Definition 1.4.4. The *Ricci curvature* and *Scalar curvature* of a pseudo-Riemannian manifold are defined by

$$\mathrm{Ric}_{ab} = R_{ab} = R_{cab}{}^{c}, \quad \mathrm{Sc} = R = R_a{}^a.$$

Because of the index convention in 1.4.1, the Ricci curvature agrees with [GW15, CG18].

Definition 1.4.5. The *sectional curvature* $K(X, Y)$ at a point $p \in M$ is defined by

$$-2g(R(X, Y)Y, X) = K(X, Y)g(X, Y) \owedge g(X, Y)$$

for any two linearly independent $X, Y \in T_pM$.

Remark. *This is well defined since the Riemann curvature is a tensor field. On a 2-manifold, this definition corresponds to Gauss's Theorema Egregium.*

A geometry with constant $K = 0$ is called a *Euclidean geometry*; $K = -1$, *hyperbolic geometry*; $K = 1$, *elliptic geometry*.

Proposition 1.4.6. *The Ricci curvature on a manifold is symmetric.*

Proof. By the algebraic Bianchi identity, $R_{abcd} + R_{bcad} + R_{cabd} = 0$. Using skew symmetry of the first two indices of R_{abcd} and contracting by g^{ad} gives

$$R_{bc} - R_{cb} + R_{bca}{}^{a} = 0.$$

Since the last two indices of R_{abcd} are also skew, $R_{bca}{}^{a} = 0$. □

Proposition 1.4.7. *The Ricci and Scalar curvatures on a manifold are related by the* <u>*contracted Bianchi identity:*</u>

$$\nabla_a R^a{}_b = \frac{1}{2}\nabla_b R.$$

Proof. This is proved using the differential Bianchi identity:

$$\nabla_a R_{bcde} - \nabla_c R_{bade} + \nabla_b R_{cade} = 0.$$

Contracting the expression by g^{be} gives

$$\nabla_a R_{cd} - \nabla_c R_{ad} - \nabla_b R_{caed} g^{be} = 0,$$

by metric compatibility. Recognising that R_{ab} is symmetric, a further contraction yields

$$\nabla_a R - \nabla_c R^c{}_a - \nabla_b R^b{}_a = 0.$$

Rearranging this equation gives the proposition result. □

Chapter 2

Conformal Manifolds and Conformal Invariants

2.1 Conformal Metrics

Definition 2.1.1. Let M be a smooth manifold and g a pseudo-Riemannian metric on M. $\widetilde{g}$ is *conformally related* to g if there exists $\omega \in \Gamma(\mathcal{E})$ such that $\widetilde{g} = e^{2\omega} g$.

The above can be confirmed to be an equivalence relation. On any tangent space two conformally related metrics give rise to the same angle between vectors, although the norms may differ. The equivalence class of g, denoted $[g]$, is called the *conformal class* of g.

Definition 2.1.2. A smooth manifold M equipped with a conformal class $[g]$ is called a *conformal manifold*.

In pseudo-Riemannian geometry, given any metric, the Levi-Civita connection exists. Since conformal manifolds are less rigid than pseudo-Riemannian manifolds, the question arises as to whether a conformal manifold is equipped with a distinguished connection.

2.2 Rescaling the Levi-Civita Connection

Let $(M, [g])$ be a conformal manifold, and choose $g, \widetilde{g} \in [g]$ such that $\widetilde{g} = e^{2\omega} g$ for some $\omega \in \Gamma(\mathcal{E})$. Let $\widetilde{\Gamma}$ denote the Christoffel symbols of the Levi-Civita connection associated to $\widetilde{g}$. Setting $\Upsilon_a := \nabla_a \omega$, the Kozsul formula (1.1) gives:

$$\widetilde{\Gamma}^a{}_{bc} = \Gamma^a{}_{bc} + \Upsilon_b \delta^a_c + \Upsilon_c \delta^a_b - \Upsilon^d g_{bc}.$$

This shows that in general the Christoffel symbols of the Levi-Civita connection change when the metric is rescaled, i.e. there is generally no well defined Levi-Civita connection on a conformal manifold. For a tensor or differential operator to be well defined on a conformal manifold, its action must be independent of the choice of metric from the conformal class.

Proposition 2.2.1. *Let* $(M,[g])$ *be a conformal manifold, and choose* $g, \widetilde{g} \in [g]$ *such that* $\widetilde{g} = e^{2\omega} g$ *for some* $\omega \in \Gamma(\mathcal{E})$*. Let* $V^a \in \Gamma(\mathcal{E}^a)$ *and* $\mu_a \in \Gamma(\mathcal{E}_a)$*. If* ∇ *denotes the Levi-Civita connection associated to* g*, and* $\widetilde{\nabla}$ *the Levi-Civita connection associated to* $\widetilde{g}$*, then*

$$\widetilde{\nabla}_a V^b = \nabla_a V^b + \Upsilon_a V^b + \Upsilon_c V^c \delta_a^b - \Upsilon^b V_a \tag{2.1}$$

$$\widetilde{\nabla}_a \mu_b = \nabla_a \mu_b - \Upsilon_a \mu_b - \Upsilon_a \mu_b + \Upsilon^c \mu_c g_{ab}. \tag{2.2}$$

Proof. For contravariant vector fields,

$$\begin{aligned} \widetilde{\nabla}_a V^b &= V^c \widetilde{\Gamma}^b{}_{ac} + \partial_a V^b \\ &= V^c(\Gamma^a{}_{bc} + \Upsilon_b \delta_c^a + \Upsilon_c \delta_b^a - \Upsilon^a g_{bc}) + \partial_a V^b \\ &= \nabla_a V^b + \Upsilon_a V^b + \Upsilon_c V^c \delta_a^b - \Upsilon^b V_a. \end{aligned}$$

The calculation for covariant vector fields is similar. □

Because connections satisfy the product rule and locally any tensor field is simple, 2.2.1 determines the effect of conformal rescaling on the Levi-Civita connection for general tensor fields.

2.3 Rescaling Curvature Tensors

Definition 2.3.1. The trace free part of the Riemann curvature tensor is called the *Weyl tensor*.

Remark. *Later in 2.3.4 it is shown that the Weyl tensor is conformally invariant.*

The Weyl tensor can be determined using an ansatz. Suppose $T_{ab} \in \Gamma(\mathcal{E}_{ab})$ is a symmetric tensor. Consider

$$S_{abcd} = g_{d[a}T_{b]c} - g_{c[a}T_{b]d}.$$

By construction S_{abcd} satisfies the first Bianchi identity along with the skew and interchange symmetries of the Riemann curvature tensor. Therefore, declare

$$W_{abcd} = R_{abcd} + \lambda S_{abcd} \tag{2.3}$$

to determine S_{abcd} and W_{abcd}. Contraction by g^{ad} gives

$$R_{bc} + \frac{\lambda}{2}((n-2)S_{bc} + g_{bc}S) = 0. \tag{2.4}$$

A further contraction by g^{bc} gives

$$S = -\frac{1}{\lambda(n-1)}R. \tag{2.5}$$

Substitution of (2.5) into (2.4) gives

$$S_{bc} = -\frac{2}{\lambda(n-2)}\left(R_{bc} - \frac{R}{2(n-1)}g_{bc}\right). \tag{2.6}$$

Finally, substitution of (2.6) into (2.3) gives

$$\begin{aligned} W_{abcd} = R_{abcd} &- \frac{1}{n-2}\left(-g_{bd}R_{ac} + g_{bc}R_{ad} + g_{ad}R_{bc} - g_{ac}R_{bd}\right) \\ &- \frac{1}{(n-1)(n-2)}\left(g_{ac}g_{bd}R - g_{ad}g_{bc}R\right). \end{aligned} \tag{2.7}$$

Definition 2.3.2. The *Schouten tensor* is defined by

$$P_{ab} \coloneqq \frac{1}{n-2}\left(R_{bc} - \frac{R}{2(n-1)}g_{bc}\right).$$

Proposition 2.3.3. *The Riemann curvature can be written in terms of the Weyl and Schouten tensors as*

$$R_{abcd} = W_{abcd} - g_{ab} \owedge P_{cd}.$$

Proof. This is immediate from (2.7) and 2.3.2. □

Theorem 2.3.4. *Let $(M, [g])$ be a conformal manifold, and choose $g, \widetilde{g} \in [g]$ such that $\widetilde{g} = e^{2\omega}g$ for some $\omega \in \Gamma(\mathcal{E})$. Define*

$$\Lambda_{ab} \coloneqq -\Upsilon_{ab} + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon^2 g_{ab}.$$

The following hold:

1. $\widetilde{R}_{abc}{}^d = R_{abc}{}^d - \Lambda_{ab} \owedge \delta_c^d$
2. $\widetilde{P}_{ab} = P_{ab} + \Lambda_{ab}$
3. $\widetilde{W}_{abc}{}^d = W_{abc}{}^d$

Proof. See appendix A.1. □

In the pseudo-Riemannian setting, a manifold is termed *flat* if the Riemann curvature tensor vanishes. Although the Riemann curvature is not well defined on a conformal manifold, the Weyl tensor is, being independent of scale choice. A manifold of dimension at least 4 is termed *conformally flat* if the Weyl tensor vanishes.

2.4 Conformal Densities

This section introduces the notion of conformal densities to aid in the construction of conformally invariant operators.

Definition 2.4.1. A differential operator P is *conformally covariant with biweight* (u, v), if under a conformal change of metric $\widetilde{g} = e^{2\omega} g$,

$$\widetilde{P} \circ e^{u\omega} = e^{v\omega} \circ P.$$

Definition 2.4.2. Let $T_{ab} = \nabla_a \nabla_b - P_{ab}$. The *almost Einstein operator*, $A_{ab} : \Gamma(\mathcal{E}) \to \Gamma(\mathring{\mathcal{E}}_{(ab)})$, is defined by

$$A_{ab} = \mathring{T}_{(ab)}.$$

Proposition 2.4.3. *A_{ab} is conformally covariant with biweight* $(1, 1)$.

Proof. This follows from a computation using 2.2.1 and 2.3.4. □

Because conformally covariant operators are not yet well defined on conformal manifolds, bundles of *conformal densities* which are specific associated bundles, are defined below.

Definition 2.4.4 (Conformal densities)**.** Let $(M, [g])$ be a conformal manifold. Define $\mathcal{Q}$ as the ray subbundle of $\mathcal{E}_{(ab)}$ such that the fibre at p is $\{g_p : g \in [g]\}$. Given g_p in the fibre $\mathcal{Q}_p$, define a principal group action $\varrho : \mathcal{Q}_p \times \mathbb{R}_+ \to \mathcal{Q}_p$ by $\varrho^s(g_p) = s^2 g_p$, and a representation $\rho : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by $\rho^w(r) = r^{-w}$. The *bundle of conformal densities of weight* w is the associated bundle with total space

$$\mathcal{E}[w] = \mathcal{Q} \times_{\rho^w} \mathbb{R}.$$

A smooth section of $\mathcal{E}[w]$ can be identified with a smooth function $f : \mathcal{Q} \to \mathbb{R}$ such that $f(s^2 g_p) = s^w f(g_p)$. In particular, if $\widetilde{g} = e^{2\omega} g$, then pulling back f along g and $\widetilde{g}$ (which are sections of $\mathcal{Q}$) results in $f^{\widetilde{g}} = e^{\omega w} f^g$.

Given any unweighted vector bundle $\mathcal{B}$, $\mathcal{B}[w]$ is used to denote $\mathcal{B} \otimes \mathcal{E}[w]$. (i.e. $\mathcal{B}$ is weighted with weight w.) Conformally covariant operators with biweight (u, v) correspond to well defined operators on bundles with weight u to bundles with weight v.

Since each bundle $[w]$ is trivial, and inherits an orientation from $\mathbb{R}$, there is a notion of positive densities. The subbundle of positive densities is denoted $\mathcal{E}_+[w]$.

Definition 2.4.5 (Conformal metric)**.** The inclusion map $\boldsymbol{g} : \mathcal{Q} \to \mathcal{E}_{(ab)}$ is homogeneous of degree two. Therefore it may be identified with a section $\boldsymbol{g} \in \Gamma(\mathcal{E}_{(ab)}[2])$ which is called the *conformal metric*.

Given any $\sigma \in \Gamma(\mathcal{E}_+[1])$, $\sigma^{-2}\boldsymbol{g}$ is a pseudo-Riemannian metric on M. σ (or g) is termed a *choice of scale*. Since $\mathcal{E}[1]$ is a trivial line bundle, any two sections are related by a smooth function. Therefore, given a conformal manifold $(M, [g])$, every $g \in [g]$ can be obtained by $\sigma^{-2}\boldsymbol{g}$ for some $\sigma \in \Gamma(\mathcal{E}_+[1])$. On the other hand, given any $\sigma \in \Gamma(\mathcal{E}_+[1])$, $\sigma^{-2}\boldsymbol{g}$ necessarily belongs to $[g]$. Thus $(M, [g])$ and $(M, \boldsymbol{g})$ are equivalent descriptions of conformal manifolds, the latter being used hereafter. Notation will be abused by writing $g \in \boldsymbol{g}$.

With a conformal class, indices cannot be raised or lowered, since the musical isomorphisms would disagree given two distinct choices of scale. However, an advantage of $\boldsymbol{g}$ is that raising, lowering, and contracting indices is now well defined. (This is because $\mathcal{E}[w_1] \otimes \mathcal{E}[w_2] = \mathcal{E}[w_1 + w_2]$.) For example, if $v^a \in \Gamma(\mathcal{E}^a[w_1]), \omega_a \in \Gamma(\mathcal{E}_a[w_2])$, then

$$\boldsymbol{g}_{ab}v^a = v_b \in \Gamma(\mathcal{E}_b[w_1 + 2]), \quad \boldsymbol{g}^{ab}\omega_a = \omega^b \in \Gamma(\mathcal{E}^b[w_2 - 2]).$$

Moreover, as explained in [Cur16] (chapter 2), $\boldsymbol{g}$ determines an isomorphism

$$\otimes^n \boldsymbol{g} : (\Lambda^n)^2 \xrightarrow{\simeq} \mathcal{E}[2n]. \tag{2.8}$$

Although tracking weights during calculations may seem an unwanted complication, calculations become self-checking, since indices and weights must be the same in every term.

Let $(M, \boldsymbol{g})$ be a conformal manifold. Given a scale $g = \sigma^{-2}\boldsymbol{g}$, a connection on $\mathcal{E}[w]$ is given by

$$\nabla^g \tau := \sigma^w \mathrm{d}(\sigma^{-w}\tau), \tag{2.9}$$

where d is the exterior derivative. A connection on weighted vector bundles is given by coupling (2.9) with the Levi-Civita connection associated with g. Because any section of a weighted bundle $\mathcal{B}[w]$ locally decomposes as $V \otimes \tau$, the connection is defined by

$$\nabla^g(V \otimes \tau) = V \otimes \nabla^g \tau + (\nabla^g V) \otimes \tau. \tag{2.10}$$

Proposition 2.4.6. (2.10) *is the Levi-Civita connection on weighted vector bundles.*

Proof. Choose a scale $g = \sigma^{-2}\boldsymbol{g}$, whereby

$$\nabla^g \boldsymbol{g} = \nabla^g \sigma^2 g = 2\sigma g \nabla^g \sigma + \sigma^2 \nabla^g g = 0.$$

Therefore, for any choice of scale g, ∇^g preserves the conformal metric. It follows that (2.10) is just the pushforward of the Levi-Civita connection on $\mathcal{E}^a$ by the isomorphism (2.8). □

The almost Einstein operator may now be viewed as a conformally invariant operator

$$A_{ab} : \Gamma(\mathcal{E}[1]) \to \Gamma(\mathring{\mathcal{E}}_{(ab)}[1]). \tag{2.11}$$

Proposition 2.4.7. *Let $(M, \boldsymbol{g})$ be a conformal manifold and choose $g, \widetilde{g} \in \boldsymbol{g}$. If $\widetilde{g} = e^{2\omega} g$, the Levi-Civita connection on $\mathcal{E}[w]$ transforms by*

$$\nabla_a^{\widetilde{g}} = \nabla_a^g + w\Upsilon_a. \tag{2.12}$$

Proof. Let $\tau \in \mathcal{E}[w]$. Then

$$\begin{aligned}
\nabla^{\widetilde{\sigma}}\tau &= \widetilde{\sigma}^w(\mathrm{d}(\widetilde{\sigma}^{-w}\tau)) \\
&= e^{-\omega w}\sigma^w(\mathrm{d}(e^{\omega w}\sigma^{-w}\tau)) \\
&= e^{-\omega w}\sigma^w(e^{\omega w}\mathrm{d}(\sigma^{-w}\tau) + \mathrm{d}(e^{\omega w})\sigma^{-w}\tau) \\
&= \nabla^g\tau + w\Upsilon\tau.
\end{aligned}$$

□

In principle, we now have the means to determine how conformal rescaling transforms the connection on any weighted tensor. All subsequent raising and lowering of tensor indices will use the conformal metric unless otherwise stated.

Chapter 3

Conformal Tractor Calculus

3.1 Almost-Einstein Equation

In this chapter we develop conformal tractor calculus, the natural invariant calculus on conformal manifolds, following [BEG94]. The following definitions and lemmas are applicable:

Definition 3.1.1. The trace of the Schouten tensor is denoted $J := \boldsymbol{g}^{ab}P_{ab}$. Evidently J has weight -2, and belongs to $\Gamma(\mathcal{E}[-2])$.

Lemma 3.1.2. *The Schouten tensor satisfies the following identity:*

$$\nabla_a P^a{}_b = \nabla_b J.$$

Proof. This follows from the contracted Bianchi Identity, 1.4.7. $\square$

Definition 3.1.3. Suppose (M, g) is a pseudo-Riemannian manifold of dimension at least 3. g is an *Einstein metric*, or simply *Einstein*, if $R_{ab} = \lambda g_{ab}$ for some $\lambda \in \Gamma(\mathcal{E})$.

The condition that λ is a smooth function can be replaced with $\lambda \in \mathbb{R}$ without loss of generality. Contracting indices gives $R = n\lambda$, so $\nabla_c R = n\nabla_c \lambda$. It follows from the contracted Bianchi identity that $n\nabla_c\lambda = 2\nabla_c R^c{}_b = 2\nabla_b\lambda$. By the assumption $n > 2$, $\nabla_c\lambda = 0$ is the only solution. Since manifolds are assumed to be connected, the result follows.

Let $(M, \boldsymbol{g})$ be a conformal manifold, and g a choice of scale. Working in this scale, the Schouten tensor can be expressed as $P_{ab} = \frac{1}{n-2}\left(R_{ab} - J\boldsymbol{g}_{ab}\right)$. Rearrangement gives

$$R_{ab} = (n-2)P_{ab} + J\boldsymbol{g}_{ab}. \tag{3.1}$$

It follows that g is Einstein if and only if $\overset{\circ}{P}_{ab}$ vanishes. If there exists a scale $g \in \boldsymbol{g}$ such that g is Einstein, the conformal manifold is said to be *conformally Einstein*. This motivates the following definition of [Gov10]:

Definition 3.1.4. Given a conformal manifold $(M, \boldsymbol{g})$, the *almost Einstein equation* (AE) is

$$A_{ab}\sigma = 0$$

for $\sigma \in \Gamma(\mathcal{E}[1])$.

Lemma 3.1.5. *A conformal manifold $(M, \boldsymbol{g})$ is conformally Einstein if and only if it has a solution $\sigma \in \Gamma(\mathcal{E}_+[1])$ to AE.*

Proof. Suppose $\sigma \in \Gamma(\mathcal{E}_+[1])$ solves AE in a scale g. Recall from (2.11) that A_{ab} is conformally invariant. Therefore σ also solves AE in the scale $\sigma^{-2}\boldsymbol{g}$. In this scale, we have

$$0 = A_{ab}\sigma = \mathring{P}_{(ab)}\sigma = \mathring{P}_{ab}\sigma,$$

since $\nabla^\sigma \sigma = 0$. Moreover, because σ is non-vanishing, $\mathring{P}_{ab}$ vanishes, so $(M, \boldsymbol{g})$ is conformally Einstein. Reversing this argument proves the converse. □

Proposition 3.1.6. *AE is equivalent to the following system of first order differential equations*

$$\begin{aligned} \nabla_a\sigma - \mu_a &= 0 \\ \nabla_a\mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho &= 0 \\ \nabla_a\rho - P^b{}_a\mu_b &= 0 \end{aligned}$$

where $\sigma \in \Gamma(\mathcal{E}[1])$, $\mu_a \in \Gamma(\mathcal{E}_a[1])$, and $\rho \in \Gamma(\mathcal{E}[-1])$.

Proof. See appendix A.2. □

Since the first order system is linear, it defines a connection on $\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. This will be shown to be conformally invariant, giving rise to a calculus that is effective for studying conformal geometry.

3.2 Tractor Connection and the D Operator

Definition 3.2.1. Let $(M, \boldsymbol{g})$ be a conformal manifold and g a choice of scale. Then $[\mathcal{T}]_g$ is the pair

$$[\mathcal{T}]_g := (\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1], g).$$

Definition 3.2.2. The connection $\nabla^{\mathcal{T}}$ on $[\mathcal{T}]_g$ given by

$$\nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho \\ \nabla_a\rho - P^b{}_a\mu_b \end{pmatrix} \tag{3.2}$$

is called the *tractor connection*.

This is a well defined connection [CG18] (chapter 3). By construction, parallel sections of the tractor connection are in one-to-one correspondence with solutions to AE.

Definition 3.2.3. Let $(M, \boldsymbol{g})$ be a conformal manifold, and g a choice of scale. The *D operator* is a second order differential operator $D_g : \mathcal{E}[1] \to [\mathcal{T}]_g$, defined by

$$D_g\sigma = \begin{pmatrix} n\sigma \\ n\nabla_a\sigma \\ -\Delta\sigma - J\sigma \end{pmatrix}. \tag{3.3}$$

This operator is canonically associated to the tractor connection, any parallel section (σ, μ_a, ρ) of $[\mathcal{T}]_g$ of the form $\frac{1}{n}D_g\sigma$: Suppose (σ, μ_a, ρ) is parallel, necessarily leading to $\mu_a = \nabla_a\sigma$ and $\boldsymbol{g}_{ab}\rho = -\nabla_a\mu_b - P_{ab}\sigma$. Contracting the second expression gives $n\rho = -\Delta\sigma - J\sigma$ as required.

Definition 3.2.4. Let $(M, \boldsymbol{g})$ be a conformal manifold, and $g, \widetilde{g} \in \boldsymbol{g}$. Suppose $\widetilde{g} = e^{2\omega}g$. The *rescaling operator* S^ω is defined as

$$S^\omega := \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a^b & 0 \\ -\frac{1}{2}\Upsilon^2 & -\Upsilon^b & 1 \end{pmatrix}.$$

Lemma 3.2.5. *The relation $\sim$ given by*

$$\Gamma([\mathcal{T}]_g) \ni \begin{pmatrix} \sigma \\ \mu_a \\ \rho \end{pmatrix} \sim \begin{pmatrix} \widetilde{\sigma} \\ \widetilde{\mu}_a \\ \widetilde{\rho} \end{pmatrix} \in \Gamma([\mathcal{T}]_{\widetilde{g}}) \textit{ if there exists } \omega \in \Gamma(\mathcal{E}) \textit{ such that } \begin{pmatrix} \widetilde{\sigma} \\ \widetilde{\mu}_a \\ \widetilde{\rho} \end{pmatrix} = S^\omega \begin{pmatrix} \sigma \\ \mu_a \\ \rho \end{pmatrix}$$

is an equivalence relation.

Proof. If $\omega = 0$, S^ω is the identity matrix, showing reflexivity. By computation,

$$S^{\omega_1}S^{\omega_2} = S^{\omega_1+\omega_2},$$

which guarantees both transitivity and symmetry. □

Definition 3.2.6. Given a conformal manifold $(M, \boldsymbol{g})$, the *(standard conformal) cotractor bundle* is defined as

$$\mathcal{T} = \coprod_{g\in[g]} [\mathcal{T}]_g \Big/ \sim$$

where $\sim$ is the equivalence relation of 3.2.5. The cotractor bundle is later shown to be canonically isomorphic to its dual, termed the *tractor bundle* (see (3.4)). A section of either of these is called a *tractor field* (or simply *tractor*).

Theorem 3.2.7. *The formula (3.2) for the tractor connection determines a conformally invariant connection*

$$\nabla^{\mathcal{T}} : \Gamma(\mathcal{T}) \to \Gamma(T^*M \otimes \mathcal{T})$$

which is termed the conformal tractor connection. The formula (3.3) for the differential operator D_g determines a conformally invariant differential operator

$$D : \Gamma(\mathcal{E}[1]) \to \Gamma(\mathcal{T}).$$

Proof. The D operator is tautologically conformally invariant by 3.2.4. See A.3 for the proof that the tractor connection is conformally invariant. □

3.3 Tractor Metric

Starting from a conformal n-manifold $(M, \boldsymbol{g})$, a rank $n+2$ bundle equipped with a conformally invariant connection has been constructed. Finally, carrying out conformal geometry on M using this bundle requires a conformally invariant metric.

Theorem 3.3.1. *Let $(M, \boldsymbol{g})$ be a conformal manifold of signature (p, q). The map $h : \Gamma(\mathcal{T}) \times \Gamma(\mathcal{T}) \to \Gamma(\mathcal{E})$ defined in a scale by*

$$h(V, V') := \sigma\rho' + \sigma'\rho + \boldsymbol{g}^{ab}\mu_a\mu'_b, \quad [V]_g = (\sigma, \mu_a, \rho), \quad [V']_g = (\sigma', \mu'_b, \rho')$$

is a signature $(p+1, q+1)$ metric on $\mathcal{T}$. Moreover, h is compatible with the tractor connection. The metric is termed the tractor metric.

Proof. Choose a scale g. The formula can be written in block matrix form as

$$h(V, V') = \begin{pmatrix} \sigma & \mu_a & \rho \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{g}^{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma' \\ \mu'_b \\ \rho' \end{pmatrix}.$$

The signature is clearly $(p + 1, q + 1)$ by consideration of the matrix form when it is diagonalised. To show that it is well defined, i.e., conformally invariant, let $\widetilde{g} = e^{2\omega}g$. Then

$$\begin{aligned}
\widetilde{\sigma}\widetilde{\rho}' + \widetilde{\sigma}'\widetilde{\rho} + \widetilde{\mu}^a\widetilde{\mu}'_a &= \sigma\left(\rho' - \Upsilon^a\mu'_a - \frac{1}{2}\Upsilon^2\sigma'\right) + \sigma'\left(\rho - \Upsilon^a\mu_a - \frac{1}{2}\Upsilon^2\sigma\right) \\
&\quad + \left(\mu^a + \Upsilon^a\sigma\right)\left(\mu'_a + \Upsilon_a\sigma'\right) \\
&= \sigma\rho' + \sigma'\rho + \mu^a\mu'_a - \sigma\Upsilon^a\mu'_a - \frac{1}{2}\Upsilon^2\sigma\sigma' - \sigma'\Upsilon^a\mu_a - \frac{1}{2}\Upsilon^2\sigma\sigma' \\
&\quad + \sigma\Upsilon^a\mu'_a + \sigma'\Upsilon^a\mu_a + \Upsilon^2\sigma\sigma' \\
&= \sigma\rho' + \sigma'\rho + \mu^a\mu'_a.
\end{aligned}$$

We now show that h compatible with the tractor connection. Calculation in a scale, (as the tractor connection and tractor metric have been shown to be conformally invariant), gives

$$\begin{aligned}
\nabla_c h(V,V') &= \nabla_c(\sigma\rho' + \sigma'\rho + \boldsymbol{g}^{ab}\mu_a\mu'_b) \\
&= ((\nabla_c\sigma)\rho' + \sigma'\nabla_c\rho + \boldsymbol{g}^{ab}(\nabla_c\mu_a)\mu'_b) + ((\nabla_c\sigma')\rho + \sigma\nabla_c\rho' + \boldsymbol{g}^{ab}\mu_a\nabla_c\mu'_b) \\
&= ((\nabla_c\sigma - \mu_c)\rho' + (\nabla_c\rho - P_{cd}\mu^d)\sigma' + (\nabla_c\mu_a + \boldsymbol{g}_{ac}\rho + P_{ac}\sigma)\mu^{a\prime}) \\
&\quad + ((\nabla_c\sigma' - \mu'_c)\rho + (\nabla_c\rho' - P_{cd}\mu'^d)\sigma + (\nabla_c\mu'_a + \boldsymbol{g}_{ac}\rho' + P_{ac}\sigma')\mu^a) \\
&= h(\nabla^{\mathcal{T}}_c V, V') + h(V, \nabla^{\mathcal{T}}_c V').
\end{aligned}$$

□

Abstract index notation $\mathcal{E}_A$ is used hereafter to denote $\mathcal{T}$. All tractor indices are denoted by upper-case font, from the start of the Latin alphabet. Conventions from tensor calculus are imported, for example, $\mathcal{E}^{(AB)}[w]$ denotes $S^2\mathcal{T}^* \otimes \mathcal{E}[w]$. The tractor metric provides an isomorphism

$$\begin{aligned}
\mathcal{E}_A &\xrightarrow{\simeq} \mathcal{E}^A \\
V &\longmapsto h(V,\cdot\,).
\end{aligned} \tag{3.4}$$

The isomorphism enables the raising and lowering of tractor indices. Since the tractor metric lives in $\Gamma(\mathcal{E}^{(AB)})$,

$$h(V,W) = h^{AB}V_AW_B = V\cdot W.$$

Hereafter the tractor bundle and its dual are distinguished solely by the positions of their indices. They are referred to together as "the tractor bundle".

3.4 Tractor Splitting Operators

From 3.2.6, the tractor bundle has the composition structure

$$\mathcal{T} = (\mathcal{E}[1] \oplus \mathcal{E}_a[1]) \oplus \mathcal{E}[-1]. \tag{3.5}$$

This is equivalent to two short exact VB-sequences

$$\begin{aligned}
\emptyset \longrightarrow \mathcal{E}_a[1] \cong \mathcal{T}_3 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{E}[1] \longrightarrow \emptyset \\
\emptyset \longrightarrow \mathcal{E}[-1] \cong \mathcal{T}_1 \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_2 \longrightarrow \emptyset
\end{aligned}$$

which capture that $\mathcal{E}[-1]$ is isomorphic to a subbundle $\mathcal{T}_1$ of $\mathcal{T}$, $\mathcal{E}_a[1]$ is isomorphic to a subbundle $\mathcal{T}_3$ of $\mathcal{T}_2 \cong {}^{\mathcal{T}}\!/_{\mathcal{T}_1}$, and $\mathcal{E}[1]$ is isomorphic to ${}^{\mathcal{T}_2}\!/_{\mathcal{T}_3}$. The *splitting operators*, which provide a canonical splitting of the tractor bundle for each choice of scale, can now be defined.

Definition 3.4.1. Let $(M, \boldsymbol{g})$ be a conformal manifold, and g a choice of scale. A natural isomorphism
$$\varphi_g : \mathcal{E}_A \xrightarrow{\simeq} \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$$
exists. The X, Y and Z splitting operators are defined by $[X^A]_g := \mathrm{pr}_1 \circ \varphi_g$, $[Y^A]_g := \mathrm{pr}_3 \circ \varphi_g$, and $[Z^A{}_a]_g := \mathrm{pr}_2 \circ \varphi_g$.

Using the metric h to raise and lower indices gives

$$\Gamma(\mathcal{E}^A[-1]) \cong \Gamma(\mathcal{E}^A \otimes \mathcal{E}[-1]) \cong \mathrm{Hom}_{\mathrm{VB}}(\mathcal{E}_A, \mathcal{E}[-1])$$

where $\mathrm{Hom}_{\mathrm{VB}}(E, F)$ is the space of VB-morphisms from E to F. It follows that the Y splitting operator can be identified with a section of $\mathcal{E}^A[-1]$. In summary the splitting operators are weighted tractors:

$$X^A \in \Gamma(\mathcal{E}^A[1]),\ Y^A \in \Gamma(\mathcal{E}^A[-1]),\ Z^A{}_a \in \Gamma(\mathcal{E}^A{}_a[1]).$$

Given $\widetilde{g} = e^{2\omega} g$, by 3.2.4,

$$\begin{aligned}
[X^A]_{\widetilde{g}} &= [X^A]_g \\
[Z^A{}_a]_{\widetilde{g}} &= [Z^A{}_a]_g + \Upsilon_a [X^A]_g \\
[Y^A]_{\widetilde{g}} &= [Y^A]_g - \Upsilon^a [Z^A{}_a]_g - \frac{1}{2}\Upsilon^2 [X^A]_g.
\end{aligned}$$

Remark. *Rather than constructing X^A in the above manner, it can be defined to be the canonical projection that evidently exists given the composition series 3.5. For this reason, X^A is termed the* <u>canonical tractor</u>.

Since the splitting operators are identified with tractors, their indices may be manipulated with the tractor metric. A splitting operator with lowered indices corresponds to an inclusion map, for example,
$$Y_A : \mathcal{E}[1] \to \mathcal{E}_A.$$

The following theorem shows how splitting operators can be useful.

Theorem 3.4.2. *On a conformal manifold $(M, \boldsymbol{g})$, there is a one-to-one correspondence between solutions $\sigma \in \Gamma(\mathcal{E}[1])$ of AE and parallel tractors I.*

Proof. Let σ be a solution to AE. Then $I_A = \frac{1}{n} D_A \sigma$ is parallel according to 3.1.6. If I_A is a parallel tractor, then $X^A I_A$ is a solution to AE. In fact, $(\frac{1}{n} D_B X^A) I_A = I_B$, and $(X^A \frac{1}{n} D_A)\sigma = \sigma$. □

Definition 3.4.3. A tractor I is called a *scale tractor* if there exists non-vanishing $\sigma \in \Gamma(\mathcal{E}[1])$ such that $I = \frac{1}{n} D\sigma$.

Theorem 3.4.2 demonstrates that a parallel tractor is necessarily a scale tractor (although the converse is not true in general).

Given a conformal manifold, contractions between splitting operators are as follows.

	X^A	Z^{Aa}	Y^A
X_A	0	0	1
Z_{Ab}	0	δ^a_b	0
Y_A	1	0	0

This is immediate from the matrix representation of the tractor metric. Moreover, it follows that the tractor metric can be expressed as

$$h_{AB} = X_A Y_B + Z_{Aa} Z_B{}^a + Y_A X_B.$$

Finally, the action of the tractor connection on the splitting operators is determined.

Proposition 3.4.4. *The following hold:*

$$\begin{aligned} \nabla^{\mathcal{T}}_b X^A &= Z^A{}_b, \\ \nabla^{\mathcal{T}}_b Z^{Aa} &= -Y^A \delta^a_b - X^A P^a{}_b, \\ \nabla^{\mathcal{T}}_b Y^A &= Z^{Aa} P_{ab}. \end{aligned}$$

Proof. Suppose $V_A \in \Gamma(\mathcal{E}_A)$, and g is a choice of scale. Choose σ, μ_a, and ρ such that

$$V_A = Y_A \sigma + Z_A{}^a \mu_a + X_A \rho.$$

Therefore the result follows by applying $\nabla^{\mathcal{T}}$ to V_A and equating terms. $\square$

Chapter 4

Conformally Compact Manifolds

4.1 Conformal Compactification

Definition 4.1.1. Let $(\underline{M}, \underline{g})$ be a pseudo-Riemannian manifold. Suppose $\underline{M}$ is the interior of a smooth compact manifold M with boundary. Let Σ denote ∂M so that $M = \Sigma \sqcup \underline{M}$. Then r is a *defining function for* Σ if

$$r : M \to \mathbb{R} \text{ is smooth, } \mathcal{Z}(r) = \Sigma, \text{ and } \mathrm{d}r_u \neq 0 \text{ for any } u \in \Sigma.$$

($\mathcal{Z}(r)$ denotes the zero locus of r.) $(\underline{M}, \underline{g})$ is *conformally compact* (in (M, g)) if there is a defining function r for Σ such that

$$g|_{\underline{M}} = r^2 \underline{g},$$

where g is a metric on M.

Remark. *It is not standard to refer to a manifold as being conformally compact "in (M, g)", but in this dissertation this terminology is used in the precise sense described above.*

Suppose $(\underline{M}, \underline{g})$ is conformally compact, and M, g, Σ, and r are given as above. Accordingly g induces a metric on Σ. Consider $(M, \widetilde{g})$ where $\widetilde{g} = \omega^2 g$ for a non-vanishing smooth function ω. Therefore $\widetilde{g}|_{\underline{M}} = (\omega r)^2 \underline{g}$, and ωr is a defining function for Σ. It follows that $\widetilde{g}$ also induces a metric on Σ, which is conformally related to g. This shows that there is no canonical metric on Σ, but the latter is equipped with a canonical conformal metric $\boldsymbol{g}_\Sigma$ determined by g. $(\Sigma, \boldsymbol{g}_\Sigma)$ is termed the *conformal infinity* of $\underline{M}$.

Definition 4.1.2. Let $(M, \boldsymbol{g})$ be a conformal manifold, and Σ a submanifold of M. Then $\sigma \in \Gamma(\mathcal{E}[1])$ is called a *defining density* for Σ if

$$\mathcal{Z}(\sigma) = \Sigma, \text{ and } \nabla^g_a \sigma_u \neq 0 \text{ for any } u \in \Sigma \text{ and any } g \in \boldsymbol{g}.$$

Proposition 4.1.3. *Let (M, g) be a compact pseudo-Riemannian manifold and $\boldsymbol{g}$ the conformal metric of g. A pseudo-Riemannian manifold $(\underline{M}, \underline{g})$ is conformally compact in (M, g) if and only if there is a defining density $\sigma \in \Gamma(\mathcal{E}[1])$ for ∂M such that $\boldsymbol{g}|_{\underline{M}} = \sigma^2 \underline{g}$.*

Proof. Suppose $(\underline{M}, \underline{g})$ is conformally compact in (M, g). Let $\boldsymbol{g}$ denote the conformal metric of g. Then there exists a non-vanishing $\tau \in \Gamma(\mathcal{E}[1])$ such that $g = \tau^{-2}\boldsymbol{g}$. If r is a defining function for ∂M, the density $\sigma := r\tau$ satisfies the required properties. The converse is similar. □

Since conformal infinity of a manifold is canonically equipped with a conformal metric, defining densities are a convenient approach to conformal compactification.

4.2 Asymptotic Curvature

Definition 4.2.1. Suppose $(\underline{M}, \underline{g})$ is conformally compact in (M, g). Then $(\underline{M}, \underline{g})$, or $\underline{g}$, is called *asymptotically flat* if on $\partial\underline{M}$ the defining function for $\partial\underline{M}$ satisfies

$$|\mathrm{d}r|_g^2 := g^{ab}\Upsilon_a\Upsilon_b = 0, \text{where } \Upsilon_a = \nabla_a r.$$

Asymptotically flat metrics are discussed in [GH78]. Suppose in addition that $\underline{g}$ is Riemannian. If instead the defining function for $\partial\underline{M}$ satisfies

$$|\mathrm{d}r|_g^2 = 1$$

on $\partial\underline{M}$, then $(\underline{M}, \underline{g})$, or $\underline{g}$, is called *asymptotically hyperbolic*. Such metrics are discussed in [MP11] (chapter 2).

In this dissertation $\underline{g}$ is said to be asymptotically hyperbolic if $|\mathrm{d}r|_g^2 = c > 0$ on $\partial\underline{M}$, since $\underline{g}$ can then be rescaled by a constant to ensure $|\mathrm{d}r|_g^2 = 1$.

Remark. *The notation $|V|^2$ is used hereafter to denote $V_{A_1\cdots A_n}V^{A_1\cdots A_n}$ for any tractor V.*

Theorem 4.2.2. *Suppose $(\underline{M}, \underline{g})$ is conformally compact in $(M, \boldsymbol{g})$ with defining function σ for ∂M. Let I be the scale tractor of σ. Then $|I|^2 = |\mathrm{d}r|_g^2$ for any $g \in \boldsymbol{g}$ and defining function r for ∂M satisfying $\underline{g} = r^{-2}g|_{\underline{M}}$.*

Proof. Let r be a defining function for ∂M satisfying $\underline{g} = r^{-2}g|_{\underline{M}}$ for some $g \in \boldsymbol{g}$. Then $\underline{g} = r^2\sigma^{-2}\boldsymbol{g}$. In the scale g, we have

$$I = \frac{1}{n}D\sigma = \begin{pmatrix} \sigma \\ \nabla_a^g \sigma \\ -\frac{1}{n}(\Delta^g\sigma + J\sigma) \end{pmatrix}.$$

Since σ vanishes on the boundary,

$$I_{\partial \underline{M}} = \begin{pmatrix} 0 \\ \nabla^g_a \sigma \\ -\frac{1}{n}\Delta^g \sigma \end{pmatrix}.$$

By defining $\Upsilon_a := \nabla_a r$ and $\tau := \sigma r^{-1}$, we have $\nabla^g_a \sigma = \tau \Upsilon_a$. Thus if h is the tractor metric on M, then

$$|I|^2 = \boldsymbol{g}^{ab}\tau^2 \Upsilon_a \Upsilon_b = g^{ab}\Upsilon_a\Upsilon_b = |\mathrm{d}r|^2_g. \tag{4.1}$$

□

Theorem 4.2.3. *Suppose $(\underline{M}, \underline{g})$ is conformally compact in (M, g) with defining function r for ∂M. Then*

$$\underline{R}_{abcd} = |\mathrm{d}r|^2_g \underline{g}_{ab} \owedge \underline{g}_{cd} + \mathcal{O}(r).$$

That is, $(\underline{M}, \underline{g})$ is asymptotically hyperbolic or asymptotically flat if and only if $\underline{g}$ has asymptotically constant sectional curvature -1 or 0, respectively.

Proof. Suppose $(\underline{M}, \underline{g})$ is conformally compact in (M, g) with the defining function r for ∂M. By 2.3.4,

$$\underline{R}_{abcd} = r^{-2}\left(R_{abcd} + \left(\Psi_{ab} - \Psi_a\Psi_b + \frac{1}{2}\Psi^2 g_{ab}\right) \owedge g_{cd}\right),$$

where $\Psi_a = \frac{1}{2}r^2\nabla_a r^{-2} = -r^{-1}\nabla_a r$. The following expressions are substituted:

$$\begin{aligned} \Psi^2 &= g^{ab}\Psi_a\Psi_b = r^{-2}g^{ab}\Upsilon_a\Upsilon_b = r^{-2}|\mathrm{d}r|^2_g. \\ \Psi_{ab} &= \nabla_a\Psi_b = -r^{-1}\nabla_a\Upsilon_b + r^{-2}\Upsilon_a\Upsilon_b \\ \Psi_a\Psi_b &= r^{-2}\Upsilon_a\Upsilon_b. \end{aligned}$$

(Note that ∇ is the Levi-Civita connection associated with g.) Thus the Riemann curvature is given by

$$\begin{aligned} \underline{R}_{abcd} &= r^{-2}\left(R_{abcd} + \left(-r^{-1}\nabla_a\Upsilon_b + r^{-2}\Upsilon_a\Upsilon_b - r^{-2}\Upsilon_a\Upsilon_b + \frac{1}{2}r^{-2}|\mathrm{d}r|^2_g g_{ab}\right) \owedge g_{cd}\right) \\ &= r^{-2}R_{abcd} - r^{-3}\left(\nabla_a\Upsilon_b\right) \owedge g_{cd} + r^{-4}\frac{1}{2}|\mathrm{d}r|^2_g g_{ab} \owedge g_{cd}. \end{aligned}$$

But R_{abcd} and $(\nabla_a\Upsilon_b) \owedge g_{cd}$ are both polynomials which are bounded by compactness of $\overline{M}$, giving

$$\underline{R}_{abcd} = |\mathrm{d}r|^2_g \underline{g}_{ab} \owedge \underline{g}_{cd} + \mathcal{O}(r).$$

Therefore the sectional curvatures tend to $-|\mathrm{d}r|^2_g$ as $r \to 0$. □

Proposition 4.2.4. *Suppose $(\underline{M}, \underline{g})$ is asymptotically flat. Then its scalar curvature asymptotically tends to zero.*

Proof. This is immediate from 4.2.3. □

4.3 A Conformal Compactification of Minkowski Space

Definition 4.3.1. *Minkowski space*, also denoted $\mathbb{M}^n$, is the pseudo-Riemannian manifold $(\mathbb{R}^n, \eta)$ where η is the *Minkowski metric*, given by $\mathrm{diag}(-1, 1, \cdots, 1)$ in standard coordinates. This is a flat, complete metric of signature $(n-1, 1)$.

Later chapters we will study asymptotic curvatures of hypersurfaces in Minkowski space by using the compactification developed here. Since the aim is to use conformal tractor calculus, the compactification is constructed in an ambient space of dimension $n + 2$.

Consider $\mathbb{V} = \mathbb{R}^{n,2}$, that is, $\mathbb{R}^{n+2}$ equipped with the signature $(n, 2)$ inner product

$$\mathcal{H} = \mathrm{diag}(-1, -1, 1, \cdots, 1). \tag{4.2}$$

Let $\mathcal{N}$ denote the set of non-zero null vectors in $\mathbb{V}$, that is,

$$\mathcal{N} := \{x \in \mathbb{V} : \mathcal{H}(x, x) = 0, x \neq 0\}.$$

Define the equivalence relation $\sim$ on $\mathbb{V} \setminus \{0\}$ by $x \sim x'$ if $x' = rx$ for some $in\mathbb{R}_+$. (Here $\mathbb{R}_+$ denotes $\{r \in \mathbb{R} : r > 0\}$.) The *ray projectivisation* of $\mathbb{V}$ is defined as

$$\mathbb{P}_+(\mathbb{V}) := \{[x] : x \in \mathbb{V} \setminus \{0\}\} \cong \mathbb{S}^{n+1}.$$

Let $\tau : \mathbb{V} \setminus \{0\} \to \mathbb{P}_+(\mathbb{V})$ denote the map $x \to [x]$. $\mathbb{P}_+(\mathcal{N})$ denotes the image $\tau(\mathcal{N})$, and $\pi : \mathcal{N} \to \mathbb{P}_+(\mathcal{N})$ is defined by restricting τ. Topologically, $\mathbb{P}_+(\mathcal{N})$ is the product $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. An can be identified more explicitly with the set

$$\mathbb{T}^{1,n-1} := \left\{(x_1, x_2) : x_1^2 + x_2^2 = \frac{1}{2}\right\} \times \left\{(y_1, \cdots, y_n) : y_1^2 + \cdots + y_n^2 = \frac{1}{2}\right\},$$

where $\{x_1, x_2, y_1, \cdots, y_n\}$ are standard coordinates in $\mathbb{R}^{n+2}$.

Lemma 4.3.2. *The space $\mathbb{T}^{1,n-1}$ is canonically equipped with a conformal metric $\boldsymbol{g}$ of signature $(n-1, 1)$ (but no canonical scale). $(\mathbb{T}^{1,n-1}, \boldsymbol{g})$ is referred to as the Einstein toroid.*

Proof. See appendix A.4. □

The construction of tractor calculus on the Einstein toroid parallels that of the *conformal sphere* model of Riemannian conformal geometry described in [Gov10] (section 5.1). The tractor bundle of $\mathbb{T}^{1,n-1}$ is given by

$$\mathcal{T} = {}^{T\mathbb{V}|_{\mathcal{N}}}\big/_{\sim}$$

where $U_x \sim U_y$ if U_x is parallel to U_y in $T\mathbb{V}$ with respect to the canonical connection $\nabla^{\mathbb{V}}$ determined by the affine structure on $\mathbb{V}$, and $\pi(x) = \pi(y)$.

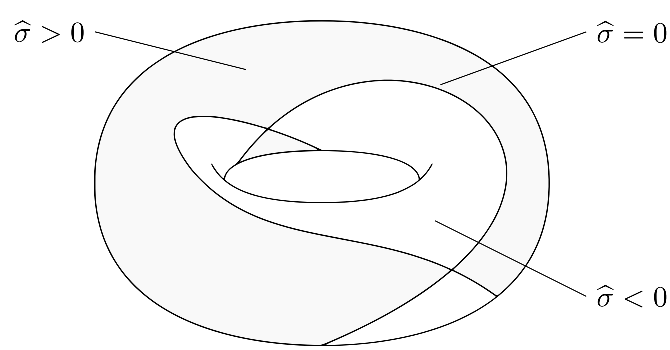
The connection $\nabla^{\mathbb{V}}$ determines a connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$ by declaring $\nabla^{\mathcal{T}}_W U \coloneqq \pi_* \left(\nabla^{\mathbb{V}}_{W'} U'\right)$, where W' and U' are lifts of W and U to a smooth section σ of π, which are then extended throughout $\mathcal{N}$ by parallel transport along the rays of $\mathcal{N}$.

Finally, the metric h on $\mathcal{T}$ is defined by $h(U,V) = \mathcal{H}(U',V')|_{\mathrm{Im}(\sigma)}$. This is well defined and compatible with $\nabla^{\mathcal{T}}$. The above constructions are summarised in the following proposition.

Proposition 4.3.3. *The Einstein toroid is canonically equipped with a tractor bundle $\mathcal{T}$, a signature $(n,2)$ tractor metric h, and a connection $\nabla^{\mathcal{T}}$ that is compatible with h. Each choice of scale determines an isomorphism $\mathcal{T} \xrightarrow{\simeq} T\mathbb{V}|_{\mathbb{T}^{1,n-1}}$, allowing the identification of tractors with vector fields in $\mathbb{V}$. A constant vector field in $\mathbb{V}$ corresponds to a parallel tractor.*

The Einstein toroid provides a conformal compactification of Minkowski space. Consider coordinates X^A, for $0 \le A \le n+1$, so that $X^0 = x_1, \cdots, X^{n+1} = y_n$. Given any vector V_A in $\mathbb{V}$, the polynomial $V_A X^A$ is homogeneous of degree one, thereby corresponding to a conformal density of weight one (see [Gov10], section 5.1). Every vector in $\mathbb{V}$ canonically determines a tractor field on $\mathbb{T}^{1,n-1}$. Thus X^A corresponds to the canonical tractor $X^A : \mathcal{E}_A \to \mathcal{E}[1]$.

Consider the null vector $(0,1,0,\cdots,0,1)$ in $\mathbb{V}$. This canonically determines the tractor field I on $\mathbb{T}^{1,n-1}$, given by $I_p = (0,1,0,\cdots,0,1)$ for each $p \in \mathcal{N}$. I is null and parallel, that is, $|I|^2 = 0$, and $\nabla^{\mathcal{T}} I = 0$. Define $\widehat{\sigma} \coloneqq h(I,X) = I_A X^A$, and define $\Sigma \coloneqq \mathcal{Z}(\widehat{\sigma})$. $\mathbb{T}^{1,n-1} \setminus \Sigma$ has two connected components, corresponding to the signs of $\widehat{\sigma}$, as shown below.



Let $\mathcal{M}$ denote the region where $\widehat{\sigma}$ is non-negative, $\underline{\mathcal{M}}$ the region where $\widehat{\sigma}$ is positive, and $\boldsymbol{\eta} = \boldsymbol{g}|_{\mathcal{M}}$. Since I is parallel, $\widehat{\sigma}$ corresponds to an almost-Einstein scale σ defined on $\mathcal{M}$ by 3.4.2. On $\underline{\mathcal{M}}$, this is also positive, so $\eta = \sigma^{-2}\boldsymbol{\eta}|_{\underline{\mathcal{M}}}$ is an Einstein metric on $\underline{\mathcal{M}}$.

In the scale σ,

$$I_A = \frac{1}{n} D_A \sigma = \begin{pmatrix} \sigma \\ \nabla_b \sigma \\ -\frac{1}{n}(\Delta\sigma + J\sigma) \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \\ -\frac{1}{n} J\sigma \end{pmatrix}.$$

It follows that $I^2 = h(I, I) = -\dfrac{2}{n} J\sigma^2 = -\dfrac{1}{n(n-1)} R\sigma^2$. However, σ is non-zero on $\underline{\mathcal{M}}$ while $I^2 = 0$ on $\mathcal{M}$. Accordingly R must vanish on $\underline{\mathcal{M}}$. Because η is Einstein, it follows that

$$R_{ab} = \frac{R}{n} \eta_{ab} = 0,$$

the condition for $(\underline{\mathcal{M}}, \eta)$ being Ricci flat. In addition, η is conformally flat by construction. The former indicates that the Schouten tensor vanishes on $\underline{\mathcal{M}}$, while the latter indicates that the Weyl tensor vanishes on $\underline{\mathcal{M}}$. Therefore, by 2.3.3, the Riemann curvature vanishes. In addition, η has signature $(n-1, 1)$ and is geodesically complete. Because $\underline{\mathcal{M}}$ can be shown to be diffeomorphic to $\mathbb{R}^n$, the conclusion can be drawn that $(\underline{\mathcal{M}}, \eta)$ is isometric to Minkowski space.

In fact, $(\underline{\mathcal{M}}, \eta)$ is Minkowski space. (See [Gov10] for the details in the Riemannian case.) From (4.2), the ambient metric $\mathcal{H}$ is given by $\mathrm{d}s^2 = -\mathrm{d}x_1^2 - \mathrm{d}x_2^2 + \mathrm{d}y_1^2 + \cdots + \mathrm{d}y_n^2$. Let $S := \pi^{-1}(\underline{\mathcal{M}}) \subset \mathcal{N}$. There exists a section $\widetilde{\sigma}$ of $\pi|_S : S \to \underline{\mathcal{M}}$ such that $\widetilde{\sigma}(\mathcal{M})$ is the hypersurface defined by $-x_2 + y_n = 1$. The section $\widetilde{\sigma}$ provides an identification of $\underline{\mathcal{M}}$ with $\widetilde{\sigma}(\mathcal{M})$, and η corresponds to the restriction of $\mathcal{H}$ to $\widetilde{\sigma}(\mathcal{M})$. Since $-x_2 + y_n = 1$, it follows that $\mathrm{d}x_2^2 = \mathrm{d}y_n^2$. Therefore η is $-\mathrm{d}x_1^2 + \mathrm{d}y_1^2 + \cdots + \mathrm{d}y_{n-1}^2$, which is precisely the Minkowski metric.

Definition 4.3.4. $(\mathcal{M}, \boldsymbol{\eta}, I)$ denotes the conformal compactification of Minkowski space, I being the scale tractor that determines the defining density to recover Minkowski space, $\mathbb{M}^n$, from $\mathcal{M}$. The *conformal infinity* of Minkowski space, $\partial\mathcal{M}$, is denoted $\mathscr{I}$.

Remark. *Although $(\mathcal{M}, \boldsymbol{\eta}, I)$ is compact, our construction is not technically a global conformal compactification, since $\nabla_a \sigma$ vanishes at $(0, 1, 0, \cdots, 0, 1)$ and $(0, -1, 0, \cdots, 0, -1)$, which are points in $\mathcal{Z}(\sigma)$. These points correspond to time-like and space-like infinity.*

Chapter 5

Conformal Hypersurfaces

5.1 Normal Tractor

Given a smooth manifold M, the term *hypersurface* will mean a submanifold of codimension 1 which is smoothly embedded in M.

Let $(M, \boldsymbol{g})$ be a conformal manifold. Let $\Sigma \subset M$ be a hypersurface in M. Σ is non-degenerate if the restriction of any $g \in \boldsymbol{g}$ to $T\Sigma$ is non-degenerate. In this case, $\boldsymbol{g}$ determines a conformal structure on Σ. This is well defined, since the restriction of any two metrics from the ambient conformal class are conformally related on the hypersurface.

Hereafter, any objects intrinsic to hypersurfaces are denoted with an overline to distinguish them from objects on the ambient space. For example, if $\Sigma \subset M$ is a hypersurface, a connection on M will be denoted ∇, whereas a connection on Σ will be denoted $\overline{\nabla}$.

Definition 5.1.1. Let $g \in \boldsymbol{g}$, and $\Sigma \subset (M, \boldsymbol{g})$ be a hypersurface. The *normal bundle* is defined as the quotient bundle

$$N\Sigma := {TM|_\Sigma}\big/{T\Sigma}.$$

The conormal bundle is denoted $N^*\Sigma$. A *conormal field* is a nowhere vanishing smooth section of $N^*\Sigma$.

Lemma 5.1.2. *Given any choice of scale $g \in \boldsymbol{g}$, $T\Sigma^\perp$ is defined by*

$$\coprod_{x\in\Sigma} T_x\Sigma^\perp = \coprod_{x\in\Sigma} \{u_x : g_x(u_x, v_x) = 0 \, \textit{for all } v_x \in T_x\Sigma\}.$$

Then Σ is non-degenerate if and only if $N\Sigma$ is isomorphic to $T\Sigma^\perp$.

Proof. If Σ is non-degenerate, $\varphi : T\Sigma^{\perp} \to N\Sigma$ defined by $V_x \mapsto V_x + T_x\Sigma$ is a VB-isomorphism. Conversely, if Σ is degenerate, $\dim T\Sigma^{\perp} \geq 2 \neq 1 = \dim N\Sigma$. $\square$

Lemma 5.1.3. *Let $\Sigma \subset M$ be a hypersurface. Σ is non-degenerate if and only if every non-vanishing $u \in \Gamma(T^*\Sigma^{\perp})$ is nowhere null.*

Proof. One direction is proved, the converse being similar. Choose a scale $g \in \boldsymbol{g}$ (as it is clear that the choice of g does not affect the proof). Suppose $u \in \Gamma(T^*\Sigma^{\perp})$ is non-vanishing and null at $x \in \Sigma$. Let V_x be the g-dual to u_x. Suppose with a view to contradiction that Σ is non-degenerate. By 5.1.2, $N\Sigma$ is then a line bundle. Therefore given any $W_x \in T_x\Sigma^{\perp}$, there exists $k \in \mathbb{R}$ such that $W_x = kV_x$. It follows that $g_x(W_x, V_x) = 0$. Because W_x was arbitrary, $V_x \in (T_x\Sigma^{\perp})^{\perp} = T_x\Sigma$. Therefore $\overline{g}$ is degenerate, because $\overline{g}_x(V_x, Z_x) = 0$ for any $Z_x \in T_x\Sigma$, a contradiction. $\square$

Hereafter $N\Sigma$ is identified with $T\Sigma^{\perp}$ whenever Σ is non-degenerate. The following are definitions of some conformally invariant analogues of pseudo-Riemannian concepts.

Firstly, suppose Σ is a non-degenerate hypersurface in $(M, \boldsymbol{g})$. Choose $n'_a \in \Gamma(N^*\Sigma[1])$ such that $|n'_a|^2 = \pm 1$. Such an n'_a exists by 5.1.2 and 5.1.3. This is arbitrarily extended to a smooth section of $\Gamma(\mathcal{E}_a[1])$ and denote as n_a. This is termed the *unit conormal field*.

Remark. *The appearance of $\pm$ or $\mp$ in a formula indicates that the sign depends on the sign of $|n_a|^2$. The upper part of $\pm$ or $\mp$ is used if n is space-like, the lower part if n is time-like. Σ is space-like if the induced metric is Riemannian. If the ambient space has Lorentzian signature, this is equivalent to the unit conormal being time-like.*

Secondly, the projection operator is defined by

$$\Pi_a^b := \delta_a^b \mp n_a n^b. \tag{5.1}$$

Given any point $p \in \Sigma$, $\Pi_a^b|_{T_pM}$ is the orthogonal projection from T_pM to $T_p\Sigma$.

Thirdly, we construct the second fundamental form. Choose a scale $g \in \boldsymbol{g}$, to make sense of ∇. The *second fundamental form* is the weight 1 tensor field defined along Σ by

$$L_{ab} := \Pi_a^c \nabla_c n_b. \tag{5.2}$$

This is independent of how n is extended off Σ and is therefore well defined. The transformation under conformal rescaling is determined as follows:

Lemma 5.1.4. *Given the conformal rescaling $g \to \widetilde{g} = e^{2\omega} g$, the second fundamental form transforms by $\widetilde{L}_{ab} = L_{ab} + \Upsilon^c n_c \overline{\boldsymbol{g}}_{ab}$.*

Proof. Let $\omega_a \in \Gamma(\mathcal{E}_a[w])$. By 2.2.1 and 2.4.7, $\nabla_b \omega_a$ transforms by

$$\widetilde{\nabla}_b \omega_a = \nabla_b \omega_a + (w-1)\Upsilon_b n_a - \Upsilon_a n_b + \Upsilon^c n_c \boldsymbol{g}_{ab}.$$

Applying this transformation to (5.2) gives $\widetilde{L}_{ab} = L_{ab} + \Upsilon^c n_c \overline{\boldsymbol{g}}_{ab}$. □

The averaged trace of the second fundamental form is a weight -1 mean curvature, denoted by H. Therefore

$$L_{ab} = \mathring{L}_{ab} + H\boldsymbol{g}_{ab}, \tag{5.3}$$

where evidently the trace free term is a conformal invariant, and H transforms by $\widetilde{H} = H + \Upsilon^a n_a$. From this transformation rule, the following definition is obtained:

Definition 5.1.5. Let $(\Sigma, \overline{\boldsymbol{g}})$ be a non-degenerate hypersurface in $(M, \boldsymbol{g})$, and let $n_a \in \Gamma(\mathcal{E}_a[1])$ be a unit conormal field. Choose a scale $g \in \boldsymbol{g}$ to split $\mathcal{E}_A$. Then the *normal tractor* to Σ is defined by

$$N_A = (0, n_a, -H) \in \Gamma(\mathcal{E}_A|_\Sigma).$$

Since $|N|^2 = \pm 1$, the normal tractor is viewed as the tractor bundle analogue to unit conormal fields in pseudo-Riemannian geometry. (N_A is confirmed to be a tractor since it transforms as a tractor under a conformal rescaling.)

5.2 Intrinsic Calculus on Hypersurfaces

Hereafter, $(M, \boldsymbol{g})$ denotes an arbitrary Lorentzian conformal manifold.

Recall from pseudo-Riemannian geometry that if a hypersurface is equipped with the restriction of the ambient metric, then the intrinsic Levi-Civita connection is just the restriction of the ambient Levi-Civita connection to tangential directions. That is, given any $\omega_b \in \Gamma(T^*\Sigma)$, the intrinsic Levi-Civita connection acts by

$$\overline{\nabla}_a \omega_b \coloneqq \Pi_a^c \Pi_b^d \nabla_c \omega_d. \tag{5.4}$$

Proposition 5.2.1 (Gauss Equation)**.** *The action of the intrinsic Levi-Civita connection on a hypersurface Σ decomposes into*

$$\overline{\nabla}_a V^b = \nabla_a^\top V^b \pm n^b L_{ac} V^c$$

for any $V^b \in \Gamma(T\Sigma)$, where $\nabla_a^\top$ is the tangential part of the ambient Levi-Civita connection*,* $\nabla_a^\top \coloneqq \Pi_a^b \nabla_b$.

Proof. This follows from a direct calculation. □

To study the conformal geometry of hypersurfaces, the relationship between the ambient tractor bundle and the tractor bundle intrinsic to hypersurfaces must be established.

Theorem 5.2.2. *The subbundle $N^{\perp} \subset \mathcal{E}_A$ is isomorphic to $\overline{\mathcal{E}}_A$, under the following map:*

$$[N^{\perp}]_g \ni S_A = \begin{pmatrix} \sigma \\ \mu_a \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu_a \mp H n_a \sigma \\ \rho \pm \frac{1}{2} H^2 \sigma \end{pmatrix} \in [\overline{\mathcal{E}}_A]_{\overline{g}}. \tag{5.5}$$

Proof. The map must be a conformally invariant VB-isomorphism. From the explicit formula above, it is a VB-isomorphism. Finally, to confirm that the map is conformally invariant, the metric is rescaled to $\widetilde{g} = e^{2\omega}\overline{g}$. Note that in the scale $\overline{g}$,

$$0 = N^A S_A = n^a \mu_a - \sigma H.$$

Thus $n_a \mu^a = \sigma H$. By rescaling the tractors on each side of the equation and using the above identity, conformal invariance can be verified. Details are given in [Sta06] (chapter 5). $\square$

Hereafter, $\overline{\mathcal{E}}_A$ is identified with $N^{\perp}$. By analogy to Π_a^b, $\Pi_A^B := \delta_A^B \mp N_A N^B$ is used to orthogonally project tractors from $\mathcal{E}_A|_{\Sigma}$ to the intrinsic bundle.

Lemma 5.2.3. *The intrinsic tractor projectors $\overline{X}_A, \overline{Z}_A{}^a$, and $\overline{Y}_A$ are related to the ambient tractor projectors $X_A, Z_A{}^b$, and Y_A along Σ in the following way:*

$$\begin{aligned} \overline{X}_A &= X_A \\ \overline{Z}_A{}^a &= \Pi_b^a Z_A{}^b \\ \overline{Y}_A &= Y_A \pm Z_A{}^a n_a H \mp \frac{1}{2} X_A H^2 \end{aligned}$$

Proof. These follow from 5.5. If $(\overline{\sigma}, \overline{\mu}_i, \overline{\rho})$ is an intrinsic tractor, then

$$\begin{aligned} \overline{Y}_A \overline{\sigma} + \overline{Z}_A{}^a \overline{\mu}_a + \overline{X}_A \overline{\rho} &= Y_A \overline{\sigma} + Z_A{}^a (\overline{\mu}_a \pm H n_a \overline{\sigma}) + X_A (\overline{\rho} \mp \frac{1}{2} H^2 \overline{\sigma}) \\ &= \Big(Y_A \pm Z_A{}^a n_a H \mp \frac{1}{2} X_A H^2 \Big) \overline{\sigma} + Z_A{}^a \overline{\mu}_a + X_A \overline{\rho}. \end{aligned}$$

Given any extension μ_a of $\overline{\mu}_a$ to $\mathcal{E}_a$, $\Pi_b^a Z_A{}^b \mu_a = Z_A{}^a \overline{\mu}_a$. $\square$

5.3 Existence of Defining Densities

A conormal field is a nowhere vanishing smooth section of $N^*\Sigma$, but it is not immediately clear that conormal fields exist. In this section, the question of existence of non-vanishing conormals for Σ is reformulated as the problem of existence of a defining density for Σ.

Lemma 5.3.1 (Local existence of defining densities)**.** *Let Σ be a non-degenerate hypersurface in $(M, \boldsymbol{g})$, with unit conormal n_a. Let $s \in \Sigma$. A neighbourhood U of s in M and $\tau \in \Gamma(\mathcal{E}[1]|_U)$ can be found, such that τ is a defining density for $\Sigma \cap U$. Moreover, the defining density must satisfy $\nabla_a \tau = h n_a$ along $\Sigma \cap U$ for some non-vanishing smooth function $h \in C^\infty(\Sigma \cap U)$.*

Proof. A proof is given in appendix A.5. In [Spi65] (chapter 5), it is interesting to note that manifolds are defined to be certain subsets of $\mathbb{R}^n$ with defining functions. □

In the proof of the above lemma, if τ is a defining density for Σ, then $\nabla_a \tau$ is normal to Σ. Therefore, if τ satisfies $|\nabla_a \tau|^2 = \pm 1$, $\nabla_a \tau$ is a unit conormal vector field to the hypersurface. This raises the question as to the setting in which local defining densities extend to global defining densities. A necessary condition is that $M \setminus \Sigma$ is disconnected: Suppose f is a defining function for Σ. Then $f^{-1}((0, \infty))$ and $f^{-1}((-\infty, 0))$ are disconnected. In fact, if M and Σ are both connected, then $M \setminus \Sigma$ being disconnected is also sufficient (given that Σ is non degenerate with unit conormal).

Theorem 5.3.2 (Global existence of defining densities)**.** *Let Σ be a connected non-degenerate hypersurface in a connected conformal manifold $(M, \boldsymbol{g})$ with unit conormal n_a. If $M \setminus \Sigma$ is disconnected, there exists a defining density τ for Σ, and $\nabla_a \tau = n_a$ along Σ.*

Proof. See appendix A.6. □

If a global defining density τ for a hypersurface Σ exists, it can be normalised so that $\nabla_a \tau$ is the unit conormal to Σ. With reference to the definition of scale tractors 3.4.3, the following question arises:

$$\text{When is } D_A \tau|_\Sigma \text{ equal to } N_A?$$

The Riemannian case was answered in [Gov10], and is easily adapted to the Lorentzian case:

Theorem 5.3.3. *Let Σ be a non-degenerate hypersurface in M with defining density τ. Suppose $|T|^2 = \pm 1 + \mathcal{O}(\tau^2)$, where $T_A = \frac{1}{n} D_A \tau$. Then $T|_\Sigma$ is the normal tractor to Σ.*

Proof. In a scale g,

$$T_A = \frac{1}{n} D_A \tau \overset{g}{=} \begin{pmatrix} \tau \\ \nabla_a \tau \\ -\frac{1}{n}(\Delta + J)\tau \end{pmatrix}.$$

Since τ is a defining density for Σ, it vanishes along Σ. In addition, $\nabla_a \tau$ is a conormal to Σ, and from the above premise, $T_A T^A|_\Sigma = \nabla_a \tau \nabla^a \tau = \pm 1$. Therefore, it is the unit conormal field, as required. Finally it must be shown that

$$-\frac{1}{n}(\Delta + J)\tau = -H$$

along Σ, where H is the weighted mean curvature. From (5.3), $H(n-1) = \boldsymbol{g}^{ab} L_{ab} = \nabla^a n_a \mp n^a n^b \nabla_b n_a$. In addition, $\nabla_a \tau = n_a$, hence $\nabla^a n_a = \Delta \tau$. On the other hand,

$$|T|^2 = n_a n^a - \frac{2}{n}\tau(\Delta\tau + J\tau) = \pm 1 + \tau^2 h.$$

Therefore the second term in the mean curvature identity is

$$n^a n^b \nabla_b n_a = \frac{1}{2} n^b \nabla_b (n^a n_a) = \frac{1}{2} n^b \nabla_b (\pm 1 + \tau^2 h + \frac{2}{n}\tau(\Delta\tau + J\tau)).$$

Along Σ, this reduces to

$$n^a n^b \nabla_b n_a = \frac{1}{n} n^b (\nabla_b \tau)(\Delta\tau) = \pm\frac{1}{n}\Delta\tau.$$

Combining terms, this gives

$$-H = -\frac{1}{n}\Delta\tau = -\frac{1}{n}(\Delta\tau + J\tau)$$

along Σ as required. □

At this point, conditions have been established for the existence of a defining density, not yet for a defining density τ satisfying $T^2 = \pm 1 + \mathcal{O}(\tau^2)$. However, no additional assumptions are necessary to construct a defining density with this property. Using the existence of bump functions on smooth manifolds, if Σ admits a defining density, the existence of another defining density π' satisfying

$$|D\pi'|^2 = \pm 1 + \mathcal{O}(\pi')$$

can be shown. Now for any $k \in \mathbb{N}$ there exists a defining density π such that

$$|D\pi|^2 = \pm 1 + \mathcal{O}(\pi^k). \tag{5.6}$$

Details can be found in [GW15] (chapter 4). Since hypersurfaces are assumed to be non-degenerate, the existence of a defining density for the hypersurface Σ whose scale tractor restricts to the normal tractor along Σ can be assumed without loss of generality. Such a defining density is called a *unit defining density*. The corresponding scale tractor is called a *unit defining tractor*.

Definition 5.3.4. Let Σ be a hypersurface in $(M, \boldsymbol{g})$, and $\tau \in \Gamma(\mathcal{E}[1])$ a defining density for Σ. The scale tractor $T_A = \frac{1}{n} D_A \tau$ is called a *unit defining tractor* for Σ if T_A restricts to the normal tractor along Σ.

Chapter 6

Hypersurfaces in Minkowski Space

6.1 Totally Umbilic Hypersurfaces

Physicists and mathematicians wish to understand the various possible spacetimes satisfying Einstein's constraint equations. By compactifying spacetimes, their behaviour at infinity can be studied. More precisely, the behaviour of hypersurfaces at conformal infinity must be understood to better understand the spacetimes. This chapter shows that space-like totally umbilic hypersurfaces in $\mathbb{M}^n$, or more generally constant mean curvature hypersurfaces, are asymptotically hyperbolic.

Definition 6.1.1. A hypersurface in a conformal manifold is said to be *CMC* if a choice of scale is understood, and the hypersurface has constant unweighted mean curvature in this scale.

Definition 6.1.2. A hypersurface in a conformal manifold is said to be *totally umbilic* if the trace free part of the second fundamental form vanishes.

Lemma 6.1.3. *Let Σ be a hypersurface in $(M, \boldsymbol{g})$ with normal tractor N. Choose a scale σ, and let I be the corresponding scale tractor. Then $I_A N^A = \sigma H$, where H is the weighted mean curvature of Σ.*

Proof. In the scale σ,

$$I_A = \frac{1}{n} D_A \sigma \stackrel{\sigma}{=} \begin{pmatrix} \sigma \\ \nabla_b \sigma \\ -\frac{1}{n}(\nabla^2 \sigma + J\sigma) \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \\ -\frac{1}{n} J\sigma \end{pmatrix},$$

since $\nabla_b \sigma$ vanishes in the scale σ (see (2.9)). It follows that

$$h(I, N) = I^A N_A = -\sigma H. \tag{6.1}$$

□

Corollary 6.1.4. *Given Σ and σ as above, Σ is CMC if and only if $I_A N^A$ is constant.*

Proof. Given a scale $g = \sigma^{-2}\boldsymbol{g}$, $\sigma^{-1}L_{ab}$ is the usual unweighted second fundamental form. It follows that σH is the usual unweighted mean curvature. Therefore, Σ is CMC if and only if σH is constant. Corollary 6.1.4 now follows from 6.1.3. □

Lemma 6.1.5. *Let Σ be a hypersurface in $(M, \boldsymbol{g})$ with unit normal tractor N. Σ is totally umbilic if and only if N is parallel.*

Proof. Choose a scale, and let $N_A = (0, n_a, -H)$. If N is parallel along Σ, $\Pi_a^b \nabla_b N_A = 0$. In particular, $\Pi_a^b(\nabla_b n_c - \overline{\boldsymbol{g}}_{cb} H) = 0$. By rearrangement,

$$\mathring{L}_{ab} = \Pi_a^d \nabla_d n_b - \overline{\boldsymbol{g}}_{ab} H = \Pi_a^d(\nabla_d n_b - \overline{\boldsymbol{g}}_{db} H) = 0.$$

A proof of the converse is given in [CG18] (chapter 6). □

Lemma 6.1.6. *Let Σ be a totally umbilic hypersurface in $(M, \boldsymbol{g})$. In a scale with a parallel scale tractor, Σ is CMC, and the weighted mean curvature is parallel.*

Proof. Let σ be a choice of scale and suppose $I_A = \frac{1}{n} D_A \sigma$ is parallel. Then

$$-\sigma \nabla_a H = -\nabla_a(\sigma H) \stackrel{\sigma}{=} \nabla_a I_A N^A = N^A \nabla_a I_A + I_A \nabla_a N^A = 0.$$

□

These lemmas may now be used to study the geometry of certain hypersurfaces, initially by relating the intrinsic tractor connection to the ambient tractor connection. The intrinsic Levi-Civita connection was explicitly given by projecting the ambient Levi-Civita connection (see 5.4). By analogy, the *projected ambient tractor connection* is defined by

$$\check{\nabla}_a V^B := \Pi_D^B \Pi_a^c \nabla_c V^D \tag{6.2}$$

for any $V^B \in \Gamma(\overline{\mathcal{E}}^A)$.

Theorem 6.1.7. *Let Σ be a totally umbilic hypersurface in a conformally flat manifold. The intrinsic tractor connection $\overline{\nabla}$ agrees with the projected ambient tractor connection $\check{\nabla}$.*

Proof. From [CG18], given an intrinsic tractor V^A,

$$\check{\nabla}_a V^A = \overline{\nabla}_a V^A \mp S_a{}^A{}_B V^B$$

where S is the *tractor contorsion* tensor. The tractor contorsion is given by

$$S_c{}^A{}_B = X^A Z_B{}^d \mathcal{F}_{cd} - Z^A{}_c X_B \mathcal{F}_d{}^c$$

where $\mathcal{F}$ is the *Fialkow tensor*. Finally, given a choice of scale, the Fialkow tensor is given by

$$\mathcal{F}_{ab} = \frac{1}{n-2}\Big(W_{abcd}n^c n^d + \mathring{L}_{ab}{}^2 - \frac{|\mathring{L}|^2}{2(n-1)}\overline{\boldsymbol{g}}_{ab}\Big)$$

By assumption the ambient space is conformally flat. Therefore, $W_{abcd} = 0$. Moreover, as N is parallel, $\mathring{L}$ also vanishes. Thus $\mathcal{F}$ vanishes everywhere, so

$$\check{\nabla}_a V^A = \overline{\nabla}_a V^A$$

as required. □

Remark. *If either the assumption that the ambient space is conformally flat, or that the hypersurface has parallel normal tractor is dropped, generally the projected ambient tractor connection and intrinsic tractor connection will not agree.*

Theorem 6.1.8. *Let Σ be a totally umbilic hypersurface in a conformally flat manifold. If σ is a choice of scale in the ambient space with parallel scale tractor I, then $\Pi_A^B I_B$ is an intrinsic scale tractor for $(\Sigma, \boldsymbol{g}|_\Sigma)$, and $(\Pi_A^B I_B)\overline{X}^A = \sigma|_\Sigma$.*

Proof. Let σ be a choice of scale in the ambient space such that $I_A = \frac{1}{n} D_A \sigma$ is parallel. By 6.1.6, $N^B I_B = \sigma H$ is constant. It follows that

$$\begin{aligned}\check{\nabla}_a(\Pi_A^B I_B) &= \Pi_a^b \nabla_b I_A \mp \Pi_a^b \nabla_b (N_A N^B I_B)\\ &= \Pi_a^b \nabla_b I_A \mp N^B I_B \Pi_a^b \nabla_b N_A.\end{aligned}$$

Since I and N are parallel, both terms in the last line vanish. It follows from 6.1.7 that

$$\overline{\nabla}_a(\Pi_A^B I_B) = 0$$

where the connection and tractor are both intrinsic. By 3.4.2, $\Pi_A^B I_B$ is necessarily an intrinsic scale tractor. The corresponding scale is retrieved by $(\Pi_K^A I_A)\overline{X}^K$. Using 5.2.3, it follows from computation that $(\Pi_A^B I_B)\overline{X}^K = \sigma$ as required. □

Theorem 6.1.9. *Suppose Σ is a space-like totally umbilic hypersurface in $(\mathcal{M}, \boldsymbol{\eta}, I)$ (see 4.3.4). Then Σ is a CMC hypersurface. If the unweighted mean curvature of Σ is non-zero, then $\underline{\Sigma} = \Sigma \cap \mathbb{M}^n$ is asymptotically hyperbolic.*

Proof. In this setting, *asymptotically hyperbolic* means that:

1. Let $\underline{\overline{\eta}}$ denote the restriction of the ambient Minkowski metric to $\underline{\Sigma}$, and $\overline{\boldsymbol{\eta}}$ denote the restriction of $\boldsymbol{\eta}$ to Σ. If Σ intersects conformal infinity $\mathscr{I}$ of $\mathbb{M}^n$, there is a defining density $\overline{\sigma}$ for $\Sigma \cap \mathscr{I}$ such that $\underline{\overline{\eta}} = \overline{\sigma}^{-2}\overline{\boldsymbol{\eta}}|_{\underline{\Sigma}}$.

2. The intrinsic scale tractor $\overline{I}$ of $(\Sigma, \overline{\boldsymbol{\eta}})$ satisfies $|\overline{I}|^2 = c > 0$ on $\Sigma \cap \mathscr{I}$.

It will first be shown that $|\overline{I}|^2$ is strictly positive and constant on $\Sigma \cap \mathscr{I}$. From 6.1.8, we know

$$\overline{I}_A = \Pi_A^B I_B = I_A + N_A N^B I_B.$$

Consider the scale $\sigma = I_A X^A$. We denote $\sigma|_\Sigma$ by $\overline{\sigma}$. By 6.1.3, $N^A I_A = -\overline{\sigma} H$, where $\overline{\sigma} H$ is the unweighted mean curvature of Σ. In particular, $N^A I_A|_{\underline{\Sigma}}$ is the Minkowski mean curvature of $\underline{\Sigma}$. From 6.1.6, since I and N are parallel, $\overline{\sigma} H$ is constant. Since $|N|^2 = -1$, it follows that

$$|\overline{I}|^2 = I^2 + 2(-\overline{\sigma} H)^2 - (-\overline{\sigma} H)^2 = (\overline{\sigma} H)^2 > 0. \tag{6.3}$$

This is used to observe the following.

- $\mathcal{Z}(\overline{\sigma}) = \mathcal{Z}(\sigma) \cap \Sigma = \Sigma \cap \mathscr{I}$.
- For any p in $\Sigma \cap \mathscr{I}$, $[\overline{I}]_{\overline{\sigma}} = (0, \overline{\nabla}_a \overline{\sigma}(p), -\frac{1}{n}(\Delta + J)\overline{\sigma}(p))$. It follows that $|\overline{I}|^2 = |\overline{\nabla}_a \overline{\sigma}(p)|^2$. Therefore $\overline{\nabla}_a \overline{\sigma}(p) \neq 0$, since $|\overline{I}|^2$ is non-vanishing on $\Sigma \cap \mathscr{I}$ by (6.3).
- By restricting $\boldsymbol{\eta}$ and the Minkowski metric η to $\underline{\Sigma}$, $\underline{\eta} = \overline{\sigma}^{-2} \overline{\boldsymbol{\eta}}|_{\underline{\Sigma}}$.

These three properties ensure 1, while (6.3) ensures 2. Therefore $\underline{\Sigma}$ is asymptotically hyperbolic. □

Remark. *The above procedure does not prove that if $\overline{\sigma} H$ vanishes everywhere, then $\underline{\Sigma}$ is asymptotically flat, since in general $\overline{\nabla}_a \overline{\sigma}$ will not be non-vanishing on conformal infinity.*

6.2 Hypersurfaces with Constant Mean Curvature

The restriction to totally umbilic hypersurfaces resulted in many useful properties, one of which was constant mean curvature. The assumption of total umbilicity will now be dropped, and simply assume that hypersurfaces are CMC. First is necessary to further develop hypersurface tractor calculus, and determine how it relates to ambient tractor calculus.

Theorem 6.2.1. *The D operator defined in 3.2.3 generalises to the conformally invariant* Thomas D operator

$$D_A : \Gamma(\mathcal{E}^\Phi[w]) \to \Gamma(\mathcal{E}_A \otimes \mathcal{E}^\Phi[w-1])$$

where $\mathcal{E}^\Phi$ is an arbitrary tractor bundle, defined in a scale g by

$$[D_A V]_g := \begin{pmatrix} (n+2w-2)wV \\ (n+2w-2)\nabla_a V \\ -(\Delta + wJ)V \end{pmatrix}.$$

Proof. This is discussed in depth in [ČG00]. □

Definition 6.2.2. Let Σ be a hypersurface in $(M, \boldsymbol{g})$ with unit defining tractor $T_A = \frac{1}{n} D_A \tau$ (see 5.3.4). The *tangential D operator* is defined by

$$D_A^T := D_A - \frac{T_A(T{\cdot}D)}{|T|^2} + \frac{X_A(T{\cdot}D)^2}{|T|^2(n-1)(n-2)}. \tag{6.4}$$

This is defined for Riemannian manifolds in [GWss].

Proposition 6.2.3. *Let Σ be a hypersurface in $(M, \boldsymbol{g})$. Then along Σ, on densities of weight 1,*

$$\frac{1}{n} D_A^T = \frac{1}{n-1} \overline{D}_A \pm \frac{|\mathring{L}|^2 X_A}{2(n-1)(n-2)}. \tag{6.5}$$

Proof. See appendix (A.7). This is a generalisation of lemma 4.9 of [GWss] to a Lorentzian ambient space, but restricted to weight 1 densities. □

In particular, by combining (6.4) and (6.5), it follows that if σ is a choice of scale in the ambient space, I is the corresponding scale tractor, and $\overline{I}$ is the intrinsic scale tractor of Σ, then

$$\Pi_A^B I_B \pm \frac{X_A(T{\cdot}D)^2\sigma}{n(n-1)(n-2)} \overset{\Sigma}{=} \overline{I}_A \pm \frac{|\mathring{L}|^2 X_A \overline{\sigma}}{2(n-1)(n-2)}. \tag{6.6}$$

Proposition 6.2.4. *Let Σ be a hypersurface in $(M, \boldsymbol{g})$. Let σ be a choice of scale for $\boldsymbol{g}$, and suppose Σ is CMC in this scale. Let $I_A = \frac{1}{n} D_A \sigma$, and $\overline{I}_A = \frac{1}{n-1} \overline{D}_A \overline{\sigma}$. Then*

$$\overline{I}_A = \Pi_A^B I_B \mp \frac{|\mathring{L}|^2 X_A \overline{\sigma}}{2(n-1)(n-2)} - \frac{J X_A \overline{\sigma}}{n(n-1)}. \tag{6.7}$$

Here J denotes the trace of Schouten in the scale σ.

Proof. From (6.6), we need only show that

$$(T{\cdot}D)^2 \sigma \overset{\Sigma}{=} \mp (n-2) J \overline{\sigma}.$$

Since Σ is CMC, the result follows immediately from equation (A.10). □

Lemma 6.2.5. *Let Σ be a hypersurface in $(M, \boldsymbol{g})$. Let σ be a choice of scale for $\boldsymbol{g}$, and suppose Σ is CMC in this scale. Let $I_A = \frac{1}{n} D_A \sigma$, $\overline{I}_A = \frac{1}{n-1} \overline{D}_A \overline{\sigma}$. Then*

$$|\overline{I}|^2 \overset{\Sigma}{=} |I|^2 \mp (N{\cdot}I)^2 - \frac{4}{n} Sc + \mathcal{O}(|\mathring{I\!I}|^2), \tag{6.8}$$

where II denotes the unweighted second fundamental form, $I\!I_{ab} = \sigma^{-1} L_{ab}$.

Proof. From (6.7), it follows that

$$|\overline{I}|^2 = |I|^2 \mp (N\cdot I)^2 \mp \frac{(I\cdot X)|\mathring{L}|^2\overline{\sigma}}{(n-1)(n-2)} - \frac{2(I\cdot X)J\overline{\sigma}}{n(n-1)} \mp (N\cdot I)^2 + (N\cdot N)(N\cdot I)^2.$$

Because $I\cdot X|_\Sigma = \overline{\sigma}$, and $J\overline{\sigma}^2 = 2(n-1)\mathrm{Sc}$, 6.2.5 follows. □

Suppose $\underline{\Sigma}$ is a hypersurface in $\mathbb{M}^n$. While the mean curvature of $\underline{\Sigma}$ is only defined on $\underline{\Sigma}$, conformal tractor calculus provides a method to extend this to conformal infinity: If Σ is a smooth extension of $\underline{\Sigma}$ to conformal infinity of a conformal compactification of $\mathbb{M}^n$, and N and I are the unit normal tractor of Σ and scale tractor of the compactification of $\mathbb{M}^n$ respectively, $N\cdot I|_{\underline{\Sigma}}$ is the minus the mean curvature of $\underline{\Sigma}$. Since $N\cdot I$ is defined on Σ, this extends the mean curvature of $\underline{\Sigma}$ beyond $\underline{\Sigma}$.

Theorem 6.2.6. *Let Σ be a space-like hypersurface in $(\mathcal{M}, \boldsymbol{\eta}, I)$. Suppose $\underline{\Sigma} = \Sigma \cap \mathbb{M}^n$ is CMC, with non-zero mean curvature. Then $\underline{\Sigma}$ is asymptotically hyperbolic (in the sense of 6.1.9).*

Proof. Let N be the unit normal tractor of Σ, $\sigma = I_A X^A$, and $\kappa \in \mathbb{R}$ the Minkowski mean curvature of $\underline{\Sigma}$. Then $N\cdot I|_{\underline{\Sigma}} = -\kappa \neq 0$ by 6.1.4. From the above explanation, $N\cdot I$ extends the Minkowski mean curvature to conformal infinity. Since $\underline{\Sigma}$ is dense in Σ, and $N\cdot I$ is continuous, it follows that $N\cdot I$ evaluates to $-\kappa$ everywhere.

By construction, $|I|^2 = 0$. Moreover, since Minkowski space is flat, the ambient scalar curvature vanishes. In the same way that $N\cdot I$ extends the mean curvature to conformal infinity, the Sc term in the formula (6.8) extends the scalar curvature to conformal infinity. This gives

$$|\overline{I}|^2 = \kappa^2 + \mathcal{O}(|\mathring{\mathrm{II}}|^2_\sigma).$$

Let $g \in \boldsymbol{\eta}$ be a choice of scale in the ambient space. Since Σ is a hypersurface in $(\mathcal{M}, \boldsymbol{\eta}, I)$, if it intersects conformal infinity $\mathscr{I}$ of Minkowski space, $|\mathring{\mathrm{II}}|^2$ must be finite on the intersection. There is a defining r for $\mathscr{I}$ such that

$$\underline{\eta} \coloneqq \sigma^{-2}\boldsymbol{\eta}|_{\underline{\mathcal{M}}} = r^{-2}g|_{\underline{\mathcal{M}}}.$$

Since $\mathring{L}_{ab}$ is conformally invariant, it can be shown that $|\mathring{\mathrm{II}}|^2$ transforms by $|\mathring{\mathrm{II}}|^2_\eta = r^2|\mathring{\mathrm{II}}|^2_g$. This forces $|\mathring{\mathrm{II}}|^2$ in the Minkowski scale to asymptotically approach zero. Since σ vanishes on $\Sigma \cap \mathscr{I}$, $|\overline{I}|^2 = \kappa^2 > 0$ on the boundary. Moreover, following the same procedure as theorem 6.1.9, if $\Sigma \cap \mathscr{I}$ is nonempty, then $\overline{\sigma} = \sigma|_\Sigma$ is a defining density for $\Sigma \cap \mathscr{I}$, which recovers the metric on $\underline{\Sigma}$. Therefore $\underline{\Sigma}$ is asymptotically hyperbolic. □

Remark. *It is necessary to assume that $\underline{\Sigma}$ has non-zero mean curvature. It is well known that zero-mean-curvature space-like hypersurfaces exist (see [CY76]). That is, CMC does not imply non-zero mean curvature.*

6.3 CMC Hypersurfaces of Asymptotically Flat Spacetimes

Definition 6.3.1. An *asymptotically flat spacetime* is an asymptotically flat Lorentzian manifold. Note that spacetimes are often required to satisfy the *shear-free condition*, as in [GH78].

Theorem 6.3.2. *Let $(\underline{M}, \underline{g})$ be an asymptotically flat spacetime, conformally compact in $(M, \boldsymbol{g})$. Let Σ be a topologically closed space-like hypersurface in $(M, \boldsymbol{g})$. Suppose $\underline{\Sigma} = \Sigma \cap \underline{M}$ is CMC with non-zero mean curvature. Then $\underline{\Sigma}$ is asymptotically hyperbolic.*

Proof. Let σ be the defining density for ∂M in the conformal compactification of $(\underline{M}, \underline{g})$. Let $I = \frac{1}{n} D\sigma$. By definition 4.2.1, $|\nabla_a \sigma|^2 = 0$ on ∂M, and $\sigma = 0$ on ∂M. Calculating in the scale σ, it follows that on ∂M,

$$|I|^2 = -\frac{2}{n}\sigma(\Delta + J)\sigma + |\nabla_a \sigma|^2 \stackrel{\partial M}{=} 0.$$

Let $\kappa \in \mathbb{R}$ denote the constant non-zero mean curvature of $\underline{\Sigma}$. As in 6.2.6, $N \cdot I = -\kappa$ everywhere on Σ, where N is the unit normal tractor to Σ. By 4.2.4, $\mathrm{Sc}|_{\underline{M}}$ asymptotically approaches zero since the ambient space is asymptotically flat. Thus $\mathrm{Sc} = 0$ on ∂M. Therefore (6.8) gives

$$|\bar{I}|^2 \stackrel{\Sigma}{=} |I|^2 + \kappa^2 + \mathcal{O}(|\mathring{\mathrm{II}}|^2).$$

From above, $|I|^2 = 0$ on ∂M. By the same reasoning 6.2.6, $|\mathring{\mathrm{II}}|^2$ in the scale σ must also asymptotically approach zero. Therefore

$$|\bar{I}|^2 = \kappa^2 > 0$$

on $\Sigma \cap \partial M$. Following the procedure in 6.1.9 shows that $\underline{\Sigma}$ is conformally compact in the 6.1.9 sense. It follows that $\underline{\Sigma}$ is asymptotically hyperbolic. $\square$

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Appendix A

Proofs and Calculations

A.1 Proof of theorem 2.3.4

The rescaling rule for the Riemann curvature tensor is first established. Let $V^a \in \Gamma(\mathcal{E}^a)$. Proposition 2.2.1 gives

$$\widetilde{\nabla}_a \widetilde{\nabla}_b V^c = \widetilde{\nabla}_a(\nabla_b V^c + \Upsilon_b V^c - \Upsilon^c V_b + \Upsilon_d V^d \delta_b^c).$$

Since $\widetilde{\nabla_a}$ is acting on a two-tensor, the product rule is used to determine its action. This gives us 20 terms, most of which cancel when the derivatives are commuted. This leaves:

$$\begin{aligned}
\widetilde{R}(\partial_a, \partial_b)V^c &= \widetilde{\nabla}_a \widetilde{\nabla}_b V^c - \widetilde{\nabla}_b \widetilde{\nabla}_a V^c \\
&= \nabla_a \nabla_b V^c - \nabla_b \nabla_a V^c + \Upsilon_{ab} V^c - \Upsilon_{ba} V^c + \Upsilon_\alpha \Upsilon_b V^d \delta_a^c \\
&\quad - \Upsilon_d \Upsilon_a V^d \delta_b^c + \Upsilon^c \Upsilon_a V_b - \Upsilon^c \Upsilon_b V_a + \Upsilon_{ad} V^d \delta_b^c \\
&\quad - \Upsilon_{bd} V^d \delta_a^c + \Upsilon_b{}^c V_a - \Upsilon_a{}^c V_b + \Upsilon^2 V_a \delta_b^c - \Upsilon^2 V_b \delta_a^c \\
&= (R_{abd}{}^c + \Upsilon_{ab} \delta_d^c - \Upsilon_{ba} \delta_d^c + \Upsilon_d \Upsilon_b \delta_a^c - \Upsilon_d \Upsilon_a \delta_b^c + \Upsilon^c \Upsilon_a g_{bd} - \Upsilon^c \Upsilon_b g_{ad} \\
&\quad + \Upsilon_{ad} \delta_b^c - \Upsilon_{bd} \delta_a^c + \Upsilon_b{}^c g_{ad} - \Upsilon_a{}^c g_{bd} + \Upsilon^2 g_{ad} \delta_b^c - \Upsilon^2 g_{bd} \delta_a^c) V^d
\end{aligned}$$

This gives the first result: $\widetilde{R}_{abc}{}^d = R_{abc}{}^d - \Lambda_{ab} \owedge \delta_c^d$. (Note relabelling of indices.) Contracting this with the metric gives the rescaling rules for the Ricci and scalar curvatures:

1. $\widetilde{R}_{ab} = R_{ab} + (n-2)\Lambda_{ab} + \left(\frac{n-2}{2}\Upsilon^2 - \nabla^c \Upsilon_c\right) g_{ab}$.
2. $\widetilde{R} = e^{-2\omega}(R - (n-1)(n-2)\Upsilon^2 - 2(n-1)\nabla^c \Upsilon_c)$.

Since the Schouten tensor is explicitly defined by the Ricci and scalar curvatures, it can be shown that $\widetilde{P}_{ab} = P_{ab} + \Lambda_{ab}$. Finally, using the rescaling rule for the Riemann curvature tensor and Schouten tensor, the conformal invariance of the Weyl tensor can be shown. □

A.2 Proof of proposition 3.1.6

AE is equivalent to

$$\nabla_a \nabla_b \sigma + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho = 0,$$

an equation in two variables $\sigma \in \Gamma(\mathcal{E}[1])$ and $\rho \in \Gamma(\mathcal{E}[-1])$. The ρ term absorbs all trace terms, and the symmetrisation of indices can be dropped since $\nabla_a \nabla_b + P_{ab}$ act symmetrically on densities. By introducing $\mu_a = \nabla_a \sigma$, an equivalent system with two equations is formed:

$$\begin{aligned} \nabla_a \sigma - \mu_a &= 0 \\ \nabla_a \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho &= 0. \end{aligned}$$

Because this is not yet a closed system, AE is *prolonged*:

$$\nabla_a(\nabla_b \nabla_c \sigma + P_{bc}\sigma + \boldsymbol{g}_{bc}\rho) = \nabla_a \nabla_b \nabla_c \sigma + (\nabla_a P_{bc})\sigma + P_{bc}\nabla_a \sigma + \boldsymbol{g}_{bc}\nabla_a \rho = 0.$$

Contracting this by $\boldsymbol{g}^{ab}$ and $\boldsymbol{g}^{bc}$ respectively gives

$$\begin{aligned} \nabla^2 \nabla_c \sigma + (\nabla^a P_{ac})\sigma + P^a{}_c \nabla_a \sigma + \nabla_c \rho &= 0, \\ \nabla_a \nabla^2 \sigma + (\nabla_a J)\sigma + J\nabla_a \sigma + n\nabla_a \rho &= 0. \end{aligned}$$

Since $\nabla_a \nabla_b \sigma = \nabla_b \nabla_a \sigma$,

$$\nabla_a \nabla^2 \sigma - \nabla^2 \nabla_a \sigma = (\nabla_a \nabla_b \nabla^b - \nabla_b \nabla^b \nabla_a)\sigma = (\nabla_a \nabla_b - \nabla_b \nabla_a)\nabla^b \sigma = -R_{ab}\nabla^b \sigma.$$

Using the contracted Bianchi identity and the above result, taking the difference of the two contracted prolonged equations produces

$$(n-1)\nabla_a \rho + J\nabla_a \sigma - P^c{}_a \nabla_c \sigma - R_{ab}\nabla^b \sigma = 0.$$

Recalling the definition of P_{ab},

$$-R_{ab}\nabla^b \sigma = ((2-n)P_{ab} - J\boldsymbol{g}_{ab})\nabla^b \sigma = (2-n)P^c{}_a \nabla_c \sigma - J\nabla_a \sigma.$$

Substituting the above expression and dividing by $n-1$, gives

$$\nabla_a \rho - P^b{}_a \nabla_b \sigma = \nabla_a \rho - P^b{}_a \mu_b = 0.$$

$\square$

A.3 Proof of theorem 3.2.7

Let $(\sigma, \mu_b, \rho) \in \Gamma([\mathcal{T}]_g)$. Let ω be a smooth function, and $\widetilde{g} = e^{2\omega} g$. If $\nabla^{\mathcal{T}}$ and $\widetilde{\nabla}^{\mathcal{T}}$ denote the tractor connections on $[\mathcal{T}]_g$ and $[\mathcal{T}]_{\widetilde{g}}$ respectively, it will be shown that

$$\widetilde{\nabla}^{\mathcal{T}}_a(\widetilde{\sigma}, \widetilde{\mu}_b, \widetilde{\rho}) = S^{\omega} \nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho).$$

This is proven one component at a time. For the first component,

$$\widetilde{\nabla}_a \widetilde{\sigma} - \widetilde{\mu}_a = \nabla_a \sigma + \Upsilon_a \sigma - (\mu_a + \Upsilon_a \sigma) = \nabla_a \sigma + \mu_a$$

as required. For the second component, it will be shown that

$$\widetilde{\nabla}_a \widetilde{\mu}_c + \widetilde{P}_{ac} \widetilde{\sigma} + \boldsymbol{g}_{ac} \widetilde{\rho} = \Upsilon_c(\nabla_a \sigma - \mu_a) + \delta^b_c(\nabla_a \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho).$$

Considering the conformal transformation rules for $\mathcal{E}[w]$ and $\mathcal{E}_a$, any $\tau_a \in \Gamma(\mathcal{E}_a[w])$ conformally transforms by

$$\widetilde{\nabla}_a \tau_b = \nabla_a \tau_b + (w-1)\Upsilon_a \tau_b - \Upsilon_b \tau_a + \Upsilon^k \tau_k \boldsymbol{g}_{ab}.$$

From the previously derived the transformation rule for the Schouten tensor,

$$\begin{aligned}
\widetilde{\nabla}_a \widetilde{\mu}_c + \widetilde{P}_{ac} \widetilde{\sigma} + \boldsymbol{g}_{ac} \widetilde{\rho} &= \widetilde{\nabla}_a(\mu_c + \Upsilon_c \sigma) + \widetilde{P}_{ac}\sigma + \boldsymbol{g}_{ac}\left(\rho - \Upsilon^d \mu_d - \frac{1}{2}\Upsilon^2 \sigma\right) \\
&= (\nabla_a \mu_c - \Upsilon_c \mu_a + \Upsilon^d \mu_d \boldsymbol{g}_{ac}) + (\Upsilon_{ac} - 2\Upsilon_a \Upsilon_c + \Upsilon^2 \boldsymbol{g}_{ac})\sigma \\
&\quad + \Upsilon_c(\nabla_a \sigma + \Upsilon_a \sigma) + \left(P_{ac} + \Upsilon_a \Upsilon_c - \Upsilon_{ac} - \frac{1}{2}\Upsilon^2 \boldsymbol{g}_{ac}\right)\sigma \\
&\quad + \boldsymbol{g}_{ac}\left(\rho - \Upsilon^d \mu_d - \frac{1}{2}\Upsilon^2 \sigma\right) \\
&= \nabla_a \mu_c - \Upsilon_c \mu_a + \Upsilon_c \nabla_a \sigma + P_{ac}\sigma + \boldsymbol{g}_{ac}\rho
\end{aligned}$$

as required. For the third component, it will be shown that

$$\widetilde{\nabla}_a \widetilde{\rho} - \widetilde{P}^c{}_a \widetilde{\mu}_c = -\frac{1}{2}\Upsilon^2(\nabla_a \sigma - \mu_a) - \Upsilon^b(\nabla_a \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho) + \nabla_a \rho - P^b{}_a \mu_b.$$

From $\rho \in \Gamma(\mathcal{E}[-1])$, ρ transforms by $\widetilde{\nabla}_a \rho = \nabla_a \rho - \Upsilon_a \rho$. Therefore

$$\begin{aligned}
\widetilde{\nabla}_a \widetilde{\rho} &= \widetilde{\nabla}_a \left(\rho - \Upsilon^b \mu_b - \frac{1}{2}\Upsilon^2 \sigma \right) \\
&= \nabla_a \rho - \Upsilon_a \rho - \Upsilon^b (\nabla_a \mu_b - \Upsilon_b \mu_a + \Upsilon^c \mu_c \boldsymbol{g}_{ab}) \\
&\quad - ({\Upsilon_a}^b - 2\Upsilon_a \Upsilon^b + \Upsilon^2 \delta_a^b)\mu_b - \frac{1}{2}({\Upsilon_a}^c - 2\Upsilon_a \Upsilon^c + \Upsilon^2 \delta_a^c)\Upsilon_c \sigma \\
&\quad - \frac{1}{2}\Upsilon^c (\Upsilon_{ac} - 2\Upsilon_a \Upsilon_c + \Upsilon^2 \boldsymbol{g}_{ac})\sigma - \frac{1}{2}\Upsilon^2 (\nabla_a \sigma + \Upsilon_a \sigma) \\
&= \nabla_a \rho - \Upsilon_a \rho - \Upsilon^b \nabla_a \mu_b + \Upsilon_a \Upsilon^b \mu_b - {\Upsilon_a}^b \mu_b \\
&\quad - {\Upsilon_a}^b \Upsilon_b \sigma + \frac{1}{2}\Upsilon^2 \Upsilon_a \sigma - \frac{1}{2}\Upsilon^2 \nabla_a \sigma.
\end{aligned}$$

Moreover, the second term in $\widetilde{\nabla}_a \widetilde{\rho} - {\widetilde{P}^c}_a \widetilde{\mu}_c$ is given by

$$\begin{aligned}
{\widetilde{P}^c}_a \widetilde{\mu}_c &= \left({P^c}_a + \Upsilon^c \Upsilon_a - {\Upsilon^c}_a - \frac{1}{2}\Upsilon^2 \delta_a^c \right)(\mu_c + \Upsilon_c \sigma) \\
&= {P^c}_a \mu_c + {P^c}_a \Upsilon_c \sigma + \Upsilon^c \Upsilon_a \mu_c + \frac{1}{2}\Upsilon^2 \Upsilon_a \sigma - {\Upsilon^c}_a \mu_c - {\Upsilon^c}_a \Upsilon_c \sigma - \frac{1}{2}\Upsilon^2 \mu_a.
\end{aligned}$$

Combining the expressions gives

$$\widetilde{\nabla}_a \widetilde{\rho} - {\widetilde{P}^c}_a \widetilde{\mu}_c = \nabla_a \rho - \Upsilon_a \rho - \Upsilon^b \nabla_a \mu_b - \frac{1}{2}\Upsilon^2 \nabla_a \sigma + \frac{1}{2}\Upsilon^2 \mu_a - {P^b}_a \mu_b - {P^b}_a \Upsilon_b \sigma$$

Therefore $\widetilde{\nabla}_a^{\mathcal{T}}(\widetilde{\sigma}, \widetilde{\mu}_b, \widetilde{\rho}) = S^\omega \nabla_a^{\mathcal{T}}(\sigma, \mu_b, \rho)$ as desired. □

A.4 Proof of lemma 4.3.2

To induce a metric on $\mathbb{T}^{1,n-1} \cong \mathbb{P}_+(\mathcal{N})$: Let $x \in \mathcal{N}$. Consider g_x defined on $T_{\pi(x)}\mathbb{T}^{1,n-1}$ by $g_x(u, v) = \mathcal{H}(u', v')$, where u' and v' are *lifts* of u and v to $T_x\mathcal{N}$. (Lifts satisfy $\pi_*(u') = u$ and $\pi_*(v') = v$.)

g_x is well defined in the sense that it is independent of the choice of lifts. Suppose $v^1, v^2 \in T_x\mathcal{N}$, and $\pi_*(v^1) = \pi_*(v^2) = v$. Therefore $v^1 - v^2 \in \ker \pi_*$, and $v^1 - v^2 = kE(x)$ for some real number k (where E is the Euler vector field, $E(x) = x^i \partial_i$). Calculating the gradient of $F(x) = -x_1^2 - x_2^2 + y_1^2 + \cdots + y_n^2$ shows that $kE(x)$ is normal to $T_x\mathcal{N}$ (with respect to $\mathcal{H}$.) Therefore, g_x is independent of the choice of lift as required, and defines an inner product of some signature on $T_{\pi(x)}\mathbb{T}^{1,n-1}$.

To determine the signature of the inner product: $\mathcal{H}$ is invariant under rotations of the first two or last n components. Because $\mathcal{N}$ has rotational symmetry in the first two and last n components, only the signature of g_x for $x = (0, x_2, 0, \cdots, 0, y_n)$ needs to be determined. In this case, $T_{\pi(x)}\mathbb{T}^{1,n-1}$ is comprised of vectors of the form $(u_1, 0, u_3, \cdots, u_{n+1}, 0)$, so $g_x(u, v) = k(-u_1v_1 + u_3v_3 + \cdots + u_{n+1}v_{n+1})$ for some constant k. This shows that the inner product has signature $(n-1, 1)$. By symmetry, this applies to each point on $\mathbb{T}^{1,n-1}$.

Let $\boldsymbol{g}$ denote the restriction of $\mathcal{H}$ to vector fields in $T\mathcal{N}$ which are lifts of vector fields on $\mathbb{T}^{1,n-1}$. Let $U, V \in T\mathbb{T}^{1,n-1}$, and U', V' be lifts of U, V respectively. Then $\boldsymbol{g}_{sx}(U'_{sx}, V'_{sx}) = s^2\boldsymbol{g}_x(U'_x, V'_x)$ at each $x \in N$ and $s \in \mathbb{R}_+$. It follows that $g(U', V') : \mathcal{N} \to \mathbb{R}$ is a smooth function which is homogeneous of degree two, and $g_{sx} = s^2 g_x$ for each positive s. $\mathcal{N}$ is identified as the total space of the bundle of conformally related metrics on $\mathbb{T}^{1,n-1}$. Then $\boldsymbol{g}$ determines a smooth section of $S^2T^*\mathbb{T}^{1,n-1}[2]$ (which is also denoted by $\boldsymbol{g}$.) Given a section $\sigma \in \Gamma(\mathcal{E}_+[1])$, $g = \sigma^{-2}\boldsymbol{g}$ defines a metric as 2.4.5. This results in a canonical conformal structure on $\mathbb{T}^{1,n-1}$, but no distinguished metric. □

A.5 Proof of lemma 5.3.1

Choose $s \in \Sigma$, and an open set $V' \subset \Sigma$ such that $s \in V'$ and $V' \cong \mathbb{R}^{n-1}$. Since Σ is a hypersurface, it is smoothly embedded in M. Composing this inclusion map with the diffeomorphism from $\mathbb{R}^{n-1}$ to V', results in a smooth map $h : \mathbb{R}^{n-1} \to M$ such that $\mathrm{d}h(p)$ has full rank for each $p \in \mathbb{R}^{n-1}$. Let $\delta \in \Gamma(\mathcal{E}_+[1])$ be any choice of scale. Define $k : \mathbb{R}^{n-1} \times \mathbb{R} \to M$ by

$$k(p, t) = h(p) + t\delta^{-1}n(h(p))$$

for all $(p, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$. In the above display, n is the unit conormal to Σ. Since k is a composition of smooth maps, it is itself smooth. Moreover $\mathrm{d}k(p, 0)$ is invertible, since $\mathrm{d}h$ has full rank and is purely tangential, while $\delta^{-1}n(h(p))$ is purely normal. By the inverse function theorem, there is an open set $U \subset M$, such that $\widetilde{f} : U \to \mathbb{R}^{n-1} \times \mathbb{R}$ is the (smooth) inverse of k (given appropriate restrictions to the domains and codomains of k and $\widetilde{f}$). Consider the component $f_n : U \to \mathbb{R}$ of the inverse given by $f_n(k(p, t)) = t$ for each (p, t). $\mathcal{Z}(f_n) = \mathrm{im}h|_U = \Sigma \cap U$. Since $\widetilde{f}$ is bijective, $\mathrm{d}\widetilde{f}(p)$ is invertible on T_pM. Therefore $\mathrm{d}f_n$ is non-vanishing on $\Sigma \cap U$. By 5.1.1, $T_p\Sigma \oplus N_p\Sigma = T_pM$. The derivatives of the first $n-1$ components of $\widetilde{f}$ are all tangential, so $\mathrm{d}f_n(p)$ must belong to $N_p\Sigma$. Since $N_p\Sigma$ has rank 1, by defining $\tau = \widetilde{f}\delta$, the result follows. □

A.6 Proof of theorem 5.3.2

Let $\delta \in \Gamma(\mathcal{E}_+[1])$ be a choice of scale. From A.5, locally defining functions exist. Suppose existence of a global unit conormal, the following can be found:

- A collection of open sets $\{U_p : p \in \Sigma\}$ such that if $p \in \Sigma$, then $p \in U_p$, and $(M \setminus \Sigma) \cap U_p$ has two connected components;

- A collection of smooth functions $\{f_p : M \to \mathbb{R} : p \in \Sigma\}$, with $f_p|_{U_p}$ a defining function for $\Sigma \cap U_p$, and $\nabla_a f_p$ agreeing with $\delta^{-1} n_a$ on $U_p \cap \Sigma$.

Let $p \in \Sigma$. Then $f_p^{-1}((0,\infty)) \cap U_p$ is a subset of a connected component of $M \setminus \Sigma$. Denote this component by U_+, and $(M \setminus \Sigma) \setminus U_+$ by U_-. For any other $q \in \Sigma$, $f_q^{-1}((0,\infty)) \cap U_q$ is also a subset of U_+, from the requirement that $\nabla_a f_p|_{U_q \cap \Sigma} = \delta^{-1} n_a|_{U_q \cap \Sigma}$.

Since M is a smooth manifold and $\mathcal{U} = \{U_p : p \in \Sigma\} \cup \{U_+, U_-\}$ is an open cover of M, there exists a partition of unity $\{\varphi_p : p \in \Sigma\} \cup \{\varphi_+, \varphi_-\}$ subordinate to $\mathcal{U}$. (See [Lee00].) Define $f : M \to \mathbb{R}$ by

$$f \coloneqq \varphi_+ + \sum_{p \in \Sigma} f_p \varphi_p - \varphi_-.$$

This is well defined, since the sum has only finitely many non-zero terms at any point in M by nature of being a partition of unity. In addition, the function is smooth.

The zero locus $\mathcal{Z}(f)$ is in fact Σ. If $x \in U_+$, then $\varphi_-(x) = 0$. Moreover, if $x \in U_p \cap U_+$ for some $p \in \Sigma$, then $\sum_{p \in \Sigma} f_p \varphi_p(x)$ is strictly positive, or otherwise $\varphi_+(x)$ is strictly positive. Thus $U_+ \cap Z(f) = \varnothing$. Similarly, $U_- \cap Z(f) = \varnothing$. On the other hand, if $x \in \Sigma$, then each term vanishes.

Finally, $\nabla_a f = \delta^{-1} n_a$. Since $\nabla_a \varphi_+$ vanishes on U_-, continuity ensures that it vanishes on Σ. Similarly, $\nabla_a \varphi_-$ vanishes on Σ. From the product rule, for any $x \in \Sigma$,

$$\begin{aligned} \nabla_a \Bigg(\sum_{p \in \Sigma} f_p \varphi_p \Bigg)(x) &= \sum_{p \in \Sigma} (\nabla_a f_p)(x) \varphi_p(x) + \sum_{p \in \Sigma} f_p(x) (\nabla_a \varphi_p)(x) \\ &= \delta^{-1} n_a \sum_{p \in \Sigma} \varphi_p(x) = \delta^{-1} n_a. \end{aligned}$$

It follows that $f\delta$ is a global defining density for Σ, with $\nabla_a (f\delta) = n_a$. □

A.7 Proof of proposition 6.2.3.

In the following lemmas and proposition, $T_A = \frac{1}{n} D_A \tau$ is a unit defining tractor for a hypersurface Σ of a Lorentzian conformal manifold $(M, \boldsymbol{g})$. Choose a scale g, and define $n_a =$

$\nabla_a \tau$, $\rho = -\frac{1}{n}(\Delta + J)\tau$, and $\nabla_n = n^a \nabla_a$. By 5.3.3,

$$[T_A]_g = \begin{pmatrix} \tau \\ n_a \\ \rho \end{pmatrix} \stackrel{\Sigma}{=} \begin{pmatrix} 0 \\ n_a \\ -H \end{pmatrix} = [N_A]_g.$$

As usual, $n_a n^a = \pm 1$. The following lemmas follow the proof outline for equation (4.7) in [GWss].

Lemma A.7.1. *Given a choice of scale, the intrinsic trace of Schouten is related to ambient curvatures by*

$$\overline{J} = J \mp P_{bc} n^b n^c \mp \frac{|\mathring{L}|^2}{2(n-2)} \pm \frac{(n-1)^2 H^2}{2(n-1)}. \tag{A.1}$$

Proof. Let $g \in \boldsymbol{g}$ be a choice of scale. By equation (3.3.1) of [Cur12], the following *Gauss Equation* holds when $n_a n^a = s$:

$$\overline{R}_{abcd} = R^{\top}_{abcd} - sL_{ac}L_{bd} + sL_{ad}L_{bc}.$$

Hence for $n_a n^a = \pm 1$,

$$\overline{R}_{abcd} = R^{\top}_{abcd} \mp L_{ac}L_{bd} \pm sL_{ad}L_{bc}. \tag{A.2}$$

Contracting A.2 by $\overline{\boldsymbol{g}}^{ad}\overline{\boldsymbol{g}}^{bc}$ gives the *Ricci relation*,

$$\overline{R} = R \mp 2R_{bc}n^b n^c \mp |\mathring{L}|^2 \pm (n-1)^2 H^2. \tag{A.3}$$

Substituing the expression in 3.1 gives the result. □

Lemma A.7.2. *Given a choice of scale,* $\nabla_n \rho = P_{ab} n^a n^b + \frac{|\mathring{L}|^2}{n-2}$.

Proof. By 5.6, suppose without loss of generality that $|T|^2 = \pm 1 + \tau^3 B$ for some $B \in \Gamma(\mathcal{E}[-3])$. In addition $|T|^2 = n_a n^a + 2\tau\rho$. Therefore

$$\frac{1}{2}\nabla_n |T|^2 = n_a n^a \rho + \tau(\nabla_n \rho) \mp \Pi^{ab}\nabla_b n_a \mp n\rho \mp J\tau.$$

Using $n_a n^a = \pm 1 - 2\tau\rho + \tau^3 B$,

$$\frac{1}{2}\nabla_n |T|^2 \pm (n-1)\rho - \tau(\nabla_n \rho) = \mp \Pi^{ab}\nabla_b n_a + \tau(\mp J - 2\rho^2 + \tau^2 B\rho). \tag{A.4}$$

Apply ∇_n to A.4. This gives

$$\frac{1}{2}\nabla_n^2 |T|^2 \pm (n-2)\nabla_n \rho = \mp \nabla_n(\Pi^{ab}\nabla_b n_a) - J \mp 2\rho^2. \tag{A.5}$$

Since $n^c(\nabla_c\nabla_a - \nabla_a\nabla_c)n_b = R_{cadb}n^c n^d$,

$$\nabla_n\nabla_a n_b = \nabla_a\nabla_n n_b + R_{cadb}n^c n^d - (\nabla_a n^c)(\nabla_c n_b).$$

It follows that

$$\begin{aligned}\nabla_n(\Pi^{ab}\nabla_b n_a) &= \nabla_n(\Pi^{ab})\nabla_b n_a + \Pi^{ab}(\nabla_a\nabla_n n_b + R_{cadb}n^c n^d - (\nabla_a n^c)(\nabla_c n_b))\\ &\overset{\Sigma}{=} -2H^2 + \frac{1}{2}\Pi^{ab}\nabla_a\nabla_b(n_c n^c) - R_{cd}n^c n^d - L_{ab}L^{ab}\\ &= (n-3)H^2 - R_{cd}n^c n^d - L_{ab}L^{ab}.\end{aligned}$$

Substitution into A.5 shows that along Σ,

$$\frac{1}{2}\nabla_n^2|T|^2 \pm (n-2)\nabla_n\rho = \pm R_{cd}n^c n^d \pm |\mathring{L}|^2 - J. \tag{A.6}$$

For $n \geq 3$, by 3.1, $R_{ab} = (n-2)P_{ab} + Jg_{ab}$. Therefore along Σ,

$$\nabla_n\rho = P_{ab}n^a n^b + \frac{|\mathring{L}|^2}{n-2}. \tag{A.7}$$

□

Lemma A.7.3. *Let $\Delta^\top = \overline{\boldsymbol{g}}^{ab}\nabla_a^\top\nabla_b^\top$. Along Σ,*

$$\Delta = \Delta^\top \pm \nabla_n^2 \pm (n-2)H\nabla_n. \tag{A.8}$$

Proof. Expanding the definition of $\nabla^\top$ gives

$$\begin{aligned}\Delta^\top &= (\nabla^a \mp n^a\nabla_n)(\nabla_a \mp n_a\nabla_n)\\ &= \Delta \mp \nabla_n(n^a\nabla_a) - (\nabla_n n_a)\nabla^a \mp (\nabla_a^\top n^a)\nabla_n \mp n_a(\nabla^a \mp n^a\nabla_n)\nabla_n.\end{aligned}$$

The result follows by substituting $\nabla_a^\top n^a = (n-1)H$ and $\nabla_n n_a = n_a H$. □

Lemma A.7.4. *Let g be a choice of scale in which the mean curvature vanishes. (Locally there exists such a scale: The proof in [Gov10], proposition 4.1. easy adapts to the case $|n_a|^2 = \pm 1$.) As an operator on densities of weight 1,*

$$(T{\cdot}D)^2 = (n-2)\Big(\mp\Delta^\top \mp \overline{J} + (n-1)(P_{ab}n^a n^b + \nabla_n^2) + \frac{n|\mathring{L}|^2}{n-2} - \frac{|\mathring{L}|^2}{2(n-2)}\Big) \tag{A.9}$$

along Σ.

Proof. $T{\cdot}D = \tau(-\Delta - J) + n\rho + n\nabla_n$ when acting on weight one densities. Since $T{\cdot}D$ has weight -1, along Σ this gives

$$\begin{aligned}(T{\cdot}D)(T{\cdot}D) &\overset{\Sigma}{=} (n-2)\nabla_n(\tau(-\Delta - J) + n\rho + n\nabla_n)\\ &= (n-2)(\mp\Delta \mp J + n\nabla_n\rho + n\nabla_n^2).\end{aligned} \tag{A.10}$$

Use (A.8) to replace ∇ with $\nabla^\top$, (A.1) to replace J with $\overline{J}$, and (A.7) to remove $\nabla_n\rho$. □

Lemma A.7.5. *Let g be a choice of scale in which the mean curvature vanishes. (Locally there exists such a scale, see [Gov10], proposition 4.1.) Then*

$$-\frac{1}{n}(\Delta + J) \pm \frac{(T{\cdot}D)^2}{n(n-1)(n-2)} = -\frac{1}{n-1}(\Delta^\top + \overline{J}) \pm \frac{|\mathring{L}|^2}{2(n-1)(n-2)} \tag{A.11}$$

Proof. By (A.8), (A.9) and (A.1), in the scale g, the Schouten and ∇_n^2 terms cancel, giving

$$\begin{aligned}-\Delta - J \pm \frac{(T{\cdot}D)^2}{(n-1)(n-2)} = -\frac{n}{n-1}\Delta^\top - \frac{n}{n-1}\overline{J} \mp \frac{|\mathring{L}|^2}{2(n-2)}\\ \pm \frac{n|\mathring{L}|^2}{(n-1)(n-2)} \mp \frac{|\mathring{L}|^2}{2(n-1)(n-2)}.\end{aligned}$$

This simplifies to the desired result. □

Proposition A.7.6. *Along Σ, (6.5) holds. That is,*

$$\frac{1}{n}D_A^T = \frac{1}{n-1}\overline{D}_A \pm \frac{|\mathring{L}|^2 X_A}{2(n-1)(n-2)}. \tag{A.12}$$

Proof. The left hand side of (A.11) is the bottom slot of $\frac{1}{n}D_A^T$. Because $(\nabla_a \mp n_a n^b\nabla_b)\sigma = \nabla_a^\top\sigma$, the middle slot of $\frac{1}{n}D_A^T$ is simply $\nabla_a^\top$. These prove that in the scale g,

$$\frac{1}{n}D_A^T = \begin{pmatrix} 1 \\ \nabla_a^\top \\ -\frac{1}{n-1}(\Delta^\top + \overline{J})\end{pmatrix} \pm \begin{pmatrix} 0 \\ 0 \\ \frac{|\mathring{L}|^2}{2(n-1)(n-2)}\end{pmatrix}$$

as an operator on densities of weight 1. Moreover, acting on densities, $\nabla^\top$ and $\Delta^\perp$ agree with $\overline{\nabla}$ and $\overline{\Delta}$. (This can be verified by computation. See (A.8) above, and lemma A.2 of [GWss].) Therefore the above expression becomes

$$\frac{1}{n}D_A^T = \frac{1}{n-1}\overline{D}_A \pm \frac{|\mathring{L}|^2 X_A}{2(n-1)(n-2)},$$

which is evidently conformally invariant. □