The slice Bennequin inequality

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These are expository notes on the slice Bennequin inequality, which we will prove using Khovanov homology. In the first section, the contact-geometric background is introduced. In particular, we introduce Legendrian and transverse knots, and their classical invariants. We then state the classical Bennequin inequality. The remainder of the notes does not rely on any definitions introduced in the first section, which is a bit like an historical background section.

In the second section, the relevant parts of Khovanov homology are introduced. In particular, we define the s invariant and describe some of its important properties. This section largely follows Rasmussen's pioneering paper on the s-invariant.

Finally in the third section, we use the s invariant to prove the *slice Bennequin inequality* which an improvement of the Bennequin inequality to slice genus. We look at at least one application.

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1 The classical Bennequin inequality

Traditionally knots are studied as purely topological objects - smoothly embedded circles in $\mathbb{S}^3$. However, in *contact geometry*, we can consider knots that are tangent or transverse to the contact structure to better understand the space.

In this section we introduce the basic concepts required in the study of knots in contact manifolds. This culminates in the statement of the classical Bennequin inequality.

We always assume our manifolds are oriented, unless otherwise stated.

Definition 1.1. Let (M, ξ) be a contact 3-manifold. That is, ξ is a non-degenerate plane field in M, or equivalently ξ is the kernel of a 1-form α with $\alpha \wedge d\alpha$ non-vanishing. A knot $K \subset M$ is said to be *Legendrian* (respectively, transverse) if it is tangent (respectively, transverse) to ξ.

In usual topology we consider knots up to isotopy, where isotopy really means "a homotopy whose image is a knot for all time". In contact geometry, we consider Legendrian (transverse) knots up to Legendrian (transverse) isotopy, which means we consider them up to a homotopy whose image is a Legendrian (transverse) knot for all time.

Legendrian knots are more rigid than transverse knots, since given any point in M, there is only a plane of possible directions for a Legendrian knot passing through the point, but there are many more possibilities for a transverse knot. This is reflected by the fact that Legendrian knots have particularly nice knot diagrams.

Proposition 1.2. Let $(\mathbb{R}^3, \xi)$ be the standard contact structure on $\mathbb{R}^3$. That is, $\xi = \ker(dz + xdy)$. Let L be a Legendrian knot in $(\mathbb{R}^3, \xi)$. Then the projection $\pi(L)$ onto the (y, z)-plane is a knot diagram satisfying the following properties:

- At any crossing, the arc with the greater gradient is the underpass.
- There are cusps instead of vertical tangencies.

This is called the *front projection.*

In fact, if L is Legendrian, then the x-coordinate of any point (x, y, z) on K satisfies $x = -dz/dy$. Therefore the embedding of K can be recovered from the front projection. We also frequently consider the *Lagrangian projection*, which is the projection onto the (x, y)-plane.

Remark. Whenever the projection onto a knot diagram isn't clear, we use π to denote the front projection and Π to denote the Lagrangian projection.

How common are Legendrian knots? Pretty common! Any topological knot in $\mathbb{R}^3$ can be C^0 approximated by a Legendrian knot, by coiling around the knot arbitrarily closely with the appropriate gradient. There is also a canonical way to obtain transverse knots from Legendrian knots:

Definition 1.3. Let L be a Legendrian knot. Let $\iota : \mathbb{S}^1 \times [-\varepsilon, \varepsilon] \hookrightarrow M$ be a transverse embedding of the annulus so that $\iota(\mathbb{S}^1 \times \{0\}) = L$. Then $\iota(\mathbb{S}^1 \times \{\varepsilon\})$ and $\iota(\mathbb{S}^1 \times \{-\varepsilon\})$ are both transverse knots, which we denote by $T_\pm(L)$. (It will later become clear how we differentiate between $T_+(L)$ and $T_-(L)$. Note that there is no convention in the literature, although we will choose one.)

We now define the *classical invariants* of Legendrian and transverse knots. For Legendrian knots, these are the *knot type* (simply the isotopy class of the knot in the topological sense), the *rotation number* and *Thurston-Bennequin invariant.* For transverse knots, they are the *knot type* and *self linking number.* It turns out that in some cases, the classical invariants determine the knot:

Theorem 1.4 (Eliashberg-Fraser)**.** *Two Legendrian unknots in* $\mathbb{R}^3$ *are Legendrian isotopic if and only if their classical invariants agree.*

On the other hand, it is not true in general that classical invariants suffice.

Theorem 1.5 (Chekanov)**.** *There exist Legendrian knots with the same classical invariants that are not Legendrian isotopic.*

The proof of Chekanov's theorem was to construct a new Floer theoretic invariant for Legendrian knots using the Lagrangian projection. Chekanov then exhibits two Legendrian knots of knot-type 5_2 which differ in this invariant. The construction is clearly explained in Chekanov's original paper.

So what are the classical invariants? In $\mathbb{R}^3$, they are easily defined using front projections.

Definition 1.6. Let $L \subset \mathbb{R}^3$ be Legendrian, and $T \subset \mathbb{R}^3$ transverse. The classical invariants are defined as follows:

- The *rotation number* $r(L)$ of L is given by

$$r(L) = \frac{1}{2}(c_d(\pi(L)) - c_u(\pi(L)))$$

  where c_d, c_u are the number of downward oriented cusps and upward oriented cusps respectively, of the front projection $\pi(L)$.

- The *Thurston-Bennequin number* $tb(L)$ of L is given by

$$tb(L) = w(\pi(L)) - \frac{1}{2}c(\pi(L)) = w(\Pi(L)),$$

where w is the *writhe*, and c is the total number of cusps.

- The *self linking number* $sl(T)$ is given by

$$sl(T) = w(\pi(T)).$$

These definitions only make sense because of the canonical projections that come with the standard structure on $\mathbb{R}^3$. We can alternatively give definitions that work in any contact 3-manifold.

Definition 1.7. Let $L, T \subset M$ be null-homologous Legendrian and transverse knots respectively. The classical invariants are defined as follows:

- Let v be a vector field along L, tangent to L and inducing the orientation of L. Next let Σ be a Seifert surface of L. (This requires an implicit assumption that L is null-homologous.) The restriction $\xi|_\Sigma$ is a trivial 2-plane bundle, which induces a trivialisation $\xi|_L \cong L \times \mathbb{R}^2$. But now the vector field v determines a vector in $\mathbb{R}^2$ for each point in L, and has a winding number as we traverse $\xi|_L$. This winding number is the rotation number of L.

- Next we define the Thurston-Bennequin invariant. Let L have Seifert surface Σ. Let u be a vector field along L, but this time transverse to ξ. Let L' be a push-off of L in the direction of u. Then $tb(L)$ is defined to be $[\Sigma] \cdot [L'] = lk(L, L')$.

- Finally we define the self linking number. Let Σ be a Seifert surface of T, and w a non-vanishing vector field along T which lies in $\xi|_T$. Let T' be a push-off of T in the direction of w. Then $sl(T)$ is defined to be $[\Sigma] \cdot [T'] = lk(T, T')$.

Earlier we described how one could obtain two canonical transverse knots from a given Legendrian knot. These turn out to behave very nicely with respect to classical invariants: specifically,

$$sl(T_\pm(L)) = tb(L) \pm r(L).$$

(We mentioned that there is no convention for $T_\pm$ - but one of the annulus boundaries satisfies $sl = tb + r$ while the other $sl = tb - r$, so we use the convention which makes these signs agree.)

It can be difficult to transfer results back and forth between "topology without geometry" and "contact geometry", but this is addressed by a correspondence between transverse links and topological braids.

Theorem 1.8 (Bennequin)**.** *Any braid can be closed in a natural way to produce a transverse link in* $\mathbb{R}^3$*. Conversely, any transverse link is transversely isotopic to a closed braid.*

Proof idea. We use the contact structure $\ker(dz + xdy - ydx)$ in $\mathbb{R}^3$, which in cylindrical coordinates is given by $\ker(dz + r^2 d\theta)$. Any braid can be embedded away from the z-axis, and then closed (by adding arcs) to form a "ring" around the z-axis. By moving the strands arbitrarily close together, sufficiently C^1-close to the circle in the (x, y)-plane of $\mathbb{R}^3$, it is clearly transverse. Note that our contact structure is contactomorphic to the standard structure introduced earlier.

For the converse, we need to specify exactly what is meant by a closed braid. We mean a link that does not intersect the z-axis, and such that θ increases monotonically as we follow the orientation of the link. Therefore the theorem asserts that we can transversely isotope transverse links to be of this form. □

In the original theorem, there's a correspondence but it isn't entirely clear what we need to mod out by if we wish for the correspondence to be bijective. This was resolved in 2002.

Theorem 1.9 (Orevkov-Shevchishin, Wrinkle)**.** *Two braids represent transeversely isotopic links if and only if the braids are related by conjugation and positive stabilisation/destabilisation.*

By expressing braids algebraically, it is clear what conjugation means. Stabilisation on the other hand, corresponds to adding an extra strand together with a braid group generator involving that strand. Choosing the positively signed strand is a positive stabilisation. Similarly, destabilisation is the removal of a strand by removing a single positive crossing. In summary, we have a bijective correspondence

$$\{\text{braids modulo positive braid moves}\} \leftrightarrow \{\text{transverse links modulo transverse isotopy}\}.$$

In fact, our invariants behave very well here!

Definition 1.10. Let β be a braid. We define $n(\beta)$ to be the number of strands in β, and $w(\beta)$ to be the writhe of β. (These are clearly braid invariants.) However, they are not invariants of braids modulo positive braid moves. On the other hand, it is easy to see that $w(\beta) - n(\beta)$ is an invariant under positive braid moves.

Proposition 1.11. Let T be a transverse knot, transversely isotopic to the closure of a braid β. Then

$$sl(T) = w(\beta) - n(\beta).$$

We've now established a strong relationship between the theory of braids and that of transverse knots and links. Bennequin used this in his pioneering paper to prove the *Bennequin inequality.*

Notice that the write of a braid β is also the writhe of the link obtained as its closure. We can also define the number of strands in a closed braid (for example, half the number of intersection points of the closed braid with the (y, z)-plane). Therefore for any braid β, we can consider $w - n$ of its closure. Since the closure of a braid is an oriented link in $\mathbb{R}^3$, there exist Seifert surfaces of the closure. The *genus* $g(\beta)$ of a braid β is the minimum genus of a Seifert surface of the closure of β. Similarly, $\chi(\beta)$ is the maximum Euler characteristic of a Seifert surface of the closure of β.

Theorem 1.12 (Bennequin inequality)**.** *We state three forms of the (classical) Bennequin inequality.*

1. *Let β be a (closed) braid. Then*

$$w(\beta) - n(\beta) \leq -\chi(\beta).$$

2. *Let T be a transverse knot in $\mathbb{R}^3$. Then*

$$sl(T) \leq -\chi(T).$$

3. *Let L be a Legendrian knot in $\mathbb{R}^3$. Then*

$$tb(L) + |r(L)| \leq -\chi(L).$$

Proof. We do not give a proof of the first statement, but this is explained in the paper of Bennequin.

For the second statement, we noted above that

$$sl(T) = w(\beta) - n(\beta)$$

if the closure of β is transversely isotopic to T. We know that such a β exists, so we are done.

Finally for the third statement, recall that any Legendrian knot L has two canonical transverse knots $T_\pm(L)$ related to it. These satisfy

$$sl(T_\pm(L)) = tb(L) \pm r(L).$$

Since each of L and $T_\pm(L)$ have the same topological type, they have the same Euler characteristic, so the final inequality follows. □

The Bennequin inequalities can help us compute the various invariants appearing in the statements. A slightly silly application is the following:

Example. Consider the trefoil knot K embedded as a closed braid in $\mathbb{R}^3$. This has

$$w(K) = 3, n(K) = 2.$$

Therefore $1 = w(K) - n(K) \leq -\chi(K) = 2g(K) - 1$. It follows that the genus of the trefoil is at least 1, so the trefoil is not the unknot. More generally, knot which arises as a closed braid with higher writhe than number of strands cannot be the unknot.

Finally, I remark that the above inequalities hold in more general spaces. The first inequality in terms of braids cannot naively be interpreted in manifolds that aren't $\mathbb{R}^3$ (or $\mathbb{S}^3$), but the other two make sense in arbitrary contact 3-manifolds. It turns out that they hold in *tight* contact manifolds.

Theorem 1.13 (Bennequin inequality)**.** *Let* (M, ξ) *be a tight contact manifold. Then the above Bennequin inequalities for transverse and Legendrian knots holds in* M*.*

What is a *tight* contact structure? Contact structures can be classified as either *tight* or *overtwisted*. As the names suggest, tight contact structures are just twisty enough to be everywhere non-integrable, but overtwisted ones are somehow not efficient. In the standard contact structure in $\mathbb{R}^3$, we must go to infinity for our plane field to become "vertical". These notions are formalised by using characteristic foliations: an overtwisted contact structure is exactly one that admits an embedded disk on which the characteristic foliation has a closed orbit enclosing exactly one critical point. Otherwise the contact structure is called *tight*. For more details, see any introduction to contact geometry (such as Etnyre's notes).

In the remainder of these notes, the goal is to prove the *slice Bennequin inequality*. This is an improvement of the Bennequin inequality from Seifert genus to slice genus. Recall that slice genus is the minimal genus of a surface embedded in B^4 with boundary the given knot, rather than surfaces in $\mathbb{S}^3$. Therefore typically the slice genus is at most the Seifert genus, giving $-\chi_s(K) \leq -\chi(K)$. It turns out that the Bennequin inequality is true even when we replace $\chi(K)$ with $\chi_s(K)$!

2 Khovanov homology

In this section we introduce Khovanov homology by describing various properties, and giving a brief construction. Most proofs will be omitted. The process is essentially a more involved version of reading off a Jones polynomial from a knot diagram - the homology theory is intrinsically combinatorial and can be algorithmically computed from a diagram.

Khovanov homology idea. The outline of Khovanov homology is as follows.

1. For each link diagram D, there is a corresponding cochain complex

$$\mathrm{CKh}(D) = \bigoplus_{i,j \in \mathbb{Z}} \mathrm{CKh}^{i,j}(D).$$

This is equipped with boundary maps

$$d : \mathrm{CKh}^{i,j}(D) \to \mathrm{CKh}^{i+1,j}(D), \quad d^2 = 0.$$

2. The index i is the *homological grading*. The index j is the *quantum* or *Jones* grading.

3. The homology of this chain complex is defined to be the *Khovanov homology*:

$$\mathrm{Kh}(L) = H(\mathrm{CKh}(D)) = \bigoplus_{i,j} \mathrm{Kh}^{i,j}(L),$$

where L is the link with diagram D.

The j grading is called the Jones grading because the graded Euler characteristic of Khovanov homology is the Jones polynomial of the link:

$$\chi(\mathrm{Kh}(L)) = \sum_{i,j} (-1)^i q^j \,\mathrm{rk}\,\mathrm{Kh}^{i,j}(L) = (q + q^{-1}) J_L(q^2),$$

where $J_L(t)$ is the Jones polynomial of L. Recall that the Jones polynomial is characterised by skein relations, much like the Alexander polynomial.

Example. The Khovanov homology of the trefoil knot is shown in the following table.

j \ i	0	1	2	3	χ
9				$\mathbb{Z}$	-1
7				$\mathbb{Z}/2\mathbb{Z}$	0
5			$\mathbb{Z}$		1
3	$\mathbb{Z}$				1
1	$\mathbb{Z}$				1

Reading the Euler characteristic off the table, we recover

$$\chi(\mathrm{Kh}(3_1)) = q + q^3 + q^5 - q^9.$$

Indeed, the Jones polynomial of the trefoil is $t + t^3 - t^4$, and so

$$(q + q^{-1})(q^2 + q^6 - q^8) = q + q^3 + q^5 - q^9$$

as required.

We now give a construction of Khovanov homology from a link diagram (although we omit the proof of invariance under Reidemeister moves).

Definition 2.1. The Khovanov complex CKh of a link diagram D is defined as follows.

1. Let n be the number of crossings in D. Then $n = n_+ + n_-$, where $n_\pm$ are the number of positive/negative crossings in D.

2. Next, ignoring sign, each crossing admits exactly two resolutions:

   

   In total, D admits 2^n resolutions, each resolution corresponding to some $\alpha \in \{0,1\}^n$. This is the *cube of resolutions*, and the resolution of D corresponding to some α is denoted D_α.

3. Suppose two resolutions differ by one choice, e.g. $(0,0,1,0,1)$ and $(0,0,0,0,1)$. Then we join the resolutions by an *edge*. Formally we define edges to be $\xi \in \{0,1,*\}^n$ such that $\xi_j = *$ for a unique j. In our above example, the corresponding edge is

   $$\xi = (0,0,*,0,1).$$

4. Let $V = \mathbb{Z}[x]/(x^2)$. This is a graded module, with Jones grading

   $$J(1) = 1, J(x) = -1.$$

   At each $\alpha \in \{0,1\}^n$ we associate the graded module

   $$V_\alpha(D) = V^{\otimes k}\{|\alpha|\}, \quad |\alpha| = \sum \alpha_i, k = \#\text{circles in } D_\alpha.$$

   Note that for M graded (by the Jones grading), $M\{n\}$ denotes a grading shift up by n.

   The underlying module of CKh is the direct sum of all the $V_\alpha(D)$. It remains to define the homological grading and the boundary maps.

5. V is naturally a Frobenius algebra. The maps are

   - unit: $1 \in V$
   - counit: $\varepsilon : V \to \mathbb{Z}$, $\varepsilon(1) = 0, \varepsilon(x) = 1$
   - multiplication: $m : V \otimes V \to V$, the usual product on $\mathbb{Z}[x]/(x^2)$
   - comultiplication: $\Delta : V \to V \otimes V$, $\Delta(1) = 1 \otimes x + x \otimes 1$, $\Delta(x) = x \otimes x$.

   The multiplication and comultiplication maps induce boundary maps for every edge in the cube of resolutions. Any edge ξ joins two resolutions whose number of components differs by one. If the number of components decreases along ξ (as we increase $|\alpha|$ to $|\alpha|+1$) we declare $d_\xi : V_\alpha(D) \to V_{\alpha'}(D)$ to be defined by $m : V \otimes V \to V$ on the two components that fuse into one, and d_ξ is the identity on all other copies of V. On the other hand, if the number of components increases, then d_ξ is defined similarly but with Δ instead of m. Notice that the maps d_ξ do not affect the Jones grading.

6. We have already defined the Jones grading. The *homological grading* of a module $V_\alpha(D)$ is defined to be $|\alpha|$. Therefore each d_ξ has bidegree (1,0), where the first component is the homological grading, and the second the Jones grading.

   Define $(-1)^\xi$ to mean $(-1)^{\sum_{i<j}\xi_i}$, where j is the location of $*$ in $\xi = (\xi_1, \dots, \xi_n)$. For example, $(*, 0, 0) \rightsquigarrow 1$, $(1, *, 1) \rightsquigarrow -1$.

   The differential d^r of is defined by

$$d^r = \sum_{\xi \text{ starts at } \alpha, |\alpha|=r} (-1)^\xi d_\xi.$$

7. The preshifted complex is defined to be

$$([[D]]^r, d^r)$$

   where $[[D]]^r = \bigoplus_{\alpha,|\alpha|=r} V_\alpha(D)$. The homology of this complex is not invariant under Reidemeister moves, in the same way that the Kauffman bracket fails to be a knot invariant. In the same way that the Jones polynomial incorporates a grading shift, the complex is shifted (in both the homological and Jones gradings) to ensure that the homology is a knot invariant:

$$\mathrm{CKh}(D) = ((([[D]][-n_-]\{n_+ - 2n_-\}, d).$$

   We noted that the curly braces denote a grading shift in the Jones grading. Similarly the square braces denote a grading shift in the homological grading.

A key observation is that the definition of the Khovanov complex can incorporate any Frobenius algebra, rather than just $\mathbb{Z}[x]/(x^2)$. In any case, the resulting homology becomes a knot invariant!

Definition 2.2. The definition of the Khovanov complex can be modified by considering $V = \mathbb{Z}[x]/(x^2 - t)$ in the construction. This is a Frobenius algebra by modifying the comultiplication map to be

$$\Delta_t(1) = 1 \otimes x + x \otimes 1, \quad \Delta(x) = x \otimes x + t(1 \otimes 1).$$

With $t = 0$, we obtain Khovanov homology. Setting $t = 1$ gives the *Lee homology*, denoted by $\mathrm{Lee}(L)$ for a link L. Leaving t free, we obtain the *Khovanov-Lee homology* $\mathrm{Kh}'(L)$ over $\mathbb{Z}[t]$.

We observe that the comultiplication map in Lee's complex (or the Khovanov-Lee complex) doesn't behave so well with respect to the Jones grading. Indeed, the map can be written as

$$d + t\Phi : \mathrm{CKh}'^{\,i}(D) \to \mathrm{CKh}'^{\,i+1}(D),$$

where d is the differential from the Khovanov complex (with bidegree $(1,0)$) and Φ is the additional term occuring in Δ_t, which has bidegree $(1,4)$. However, this implies that the Jones grading of any term in $(d+\Phi)(v)$ is always guaranteed to be at least the Jones grading of v. Thus $\mathrm{CKh}^{q\geq j}$ is closed under $(d+\Phi)$ for any j. This induces a filtration on the Lee complex,

$$\cdots \supset \mathrm{CLee}^{q\geq j} \supset \mathrm{CLee}^{q\geq j+1} \supset \cdots .$$

This in turn gives rise to a spectral sequence.

Proposition 2.3. The filtration of the Lee complex induces a spectral sequence such that

$$\begin{aligned} E^1 &= (\mathrm{CKh}(K), d), \\ E^2 &= (H(E^1), \Phi^*) = (\mathrm{Kh}(K), \Phi^*), \\ \Rightarrow E^\infty &= H(\mathrm{CKh}(K), d+\Phi) = \mathrm{Lee}(K). \end{aligned}$$

Proof. This is pretty much immediate: in both the Lee and Khovanov complexes, the underlying modules are isomorphic, only the maps differ. The Jones grading preserving part of the Lee complex is exactly the Khovanov complex, giving $E^1 = (\mathrm{CKh}(K), d)$. The second page follows. As for the infinity page, this is a general property of spectral sequences: they must converge to the homology of the complex from which the spectral sequence was induced. □

It turns out that the Lee homology is surprisingly trivial! Hereafter we work in rational coefficients.

Theorem 2.4. $\mathrm{Lee}(L;\mathbb{Q}) \cong \mathbb{Q}^{2^c}$*, where c is the number of components of L.*

Proof idea. Given a link L with c componenets, there are exactly 2^c choices of orientation of L. For each orientation $\mathcal{O}$ we define an element $\mathfrak{s}_{\mathcal{O}}$ of the Lee complex, and these turn out to be generators of the Lee complex which we call the *canonical generators.*

We start by choosing a new basis for $V = \mathbb{Q}[x]/(x^2-1)$, namely

$$a = x+1, \quad b = x-1.$$

Notice that in this basis the Lee complex boundary maps are actually a lot simpler!

Given any orientation $\mathcal{O}$ of a diagram D of L, there is a unique resolution $D_{\mathcal{O}}$ which is compatible with the orientation. Choose a component $C \in D_{\mathcal{O}}$. We define $\tau(D) \in \mathbb{Z}/2\mathbb{Z}$ to be the number of circles separating C from infinity, plus 1 if C is oriented clockwise. Finally we define

$$g_C = \begin{cases} a & \tau(C) = 0 \\ b & \tau(C) = 1. \end{cases}$$

This fixes the choice of "component-generator" corresponding to the C component for the module at $D_{\mathcal{O}}$. Overall the *canonical generator* of $\mathcal{O}$ is defined to be

$$\mathfrak{s}_{\mathcal{O}} = \bigotimes_{C \in D_{\mathcal{O}}} g_C.$$

One can show that these are indeed generators of $\mathrm{Lee}(L;\mathbb{Q})$. □

In particular, if L is a knot, then $\mathrm{Lee}(L;\mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$. It turns out that the gradings of the two components differ by exactly 2. Using this observation, Rasmussen defined the *s-invariant*.

Definition 2.5. Let K be a knot. Then

$$s_{min}(K) \coloneqq \min\{J(x) : x \in \mathrm{Lee}(K;\mathbb{Q}), x \neq 0\},$$
$$s_{max}(K) \coloneqq \max\{J(x) : x \in \mathrm{Lee}(K;\mathbb{Q}), x \neq 0\}.$$

In fact, Rasmussen showed that

$$s_{max}(K) = s_{min}(K) + 2,$$

and further defines $s(K) = s_{max}(K) - 1 = s_{min}(K) + 1$.

This completes the avalanche of definitions. We now study some properties! First we take a step back to the Khovanov homology. We noticed that the Jones grading was always odd for the trefoil knot, which is the one example we considered. More generally, one can show that *the parity of the Jones grading of any non-zero term is the parity of the number of components in the link*. As a corollary, we have

$$s(K) \text{ is even, for any knot } K.$$

The s invariant also behaves well under mirror images and connected sums.

Proposition 2.6. Let K be a knot. Then

$$s(\overline{K}) = -s(K).$$

Proof idea. The filtered complex $\mathrm{CLee}(\overline{K})$ is isomorphic to the dual of the complex $\mathrm{CLee}(K)$. (The isomorphism sends x to 1^* and 1 to x^*.) If two filtered complexes are dual, then so are their induced spectral sequences. □

Proposition 2.7. Let K_1, K_2 be knots. Then

$$s(K_1 \# K_2) = s(K_1) + s(K_2).$$

Proof idea. This follows from a short exact sequence

$$0 \to \text{Lee}(K_1 \# K_2) \xrightarrow{p_*} \text{Lee}(K_1) \otimes \text{Lee}(K_2) \xrightarrow{\partial} \text{Lee}(K_1 \# K_2) \to 0,$$

where p_* and ∂ are filtered of degree -1. One can show that canonical generators map to canonical generators under p_*, which gives

$$s_{min}(K_1 \# K_2) - 1 \leq s_{min}(K_1) + s_{min}(K_2).$$

But now adding 1 to each term to obtain s from s_{min}, we have

$$s(K_1 \# K_2) \leq s(K_1) + s(K_2).$$

Applying the same argument to $\overline{K}$, and using that $s(\overline{K}) = -s(K)$, we obtain the opposite inequality, and hence an equality. □

The four main theorems from Rasmussen's paper introducing the s-invariant are the following:

Theorem 2.8. *Let K be a knot. Then*

1. $|s(K)| \leq 2g_s(K)$, *where $g_s(K)$ is the* slice genus *of K.*
2. *$s : \mathcal{C} \to \mathbb{Z}$ is a well defined map, and in fact, a homomorphism. (Here $\mathcal{C}$ is the knot concordance group.)*
3. *If K is alternating, then $s(K) = \sigma(K)$, where $\sigma(K)$ is the classical knot signature of K.*
4. *If K can be represented by a positive diagram D, then*

$$s(K) = 2g_s(K) = 2g(K) = n(D) - O(D) + 1,$$

where $g(K)$ is the Seifert genus, $n(D)$ is the number of crossings in the diagram D, and $O(D)$ is the number of Seifert circles of D.

Before proceeding with proofs or examples, we need to define a few things!

Definition 2.9. Let K be a knot in $\mathbb{S}^3$. $\mathbb{S}^3$ bounds a 4-ball B^4, so we can view K as being embedded in the boundary of B^4. A *slice disk* of K is a smoothly embedded $D^2 \subset B^4$ whose boundary is K. If such a disk exists, K is said to be a *slice knot.* In general, knots are not slice, and the *slice genus* of K is the minimum genus of an embedded surface in B^4 with boundary K.

In the 4th theorem, we mention *Seifert circles.* This is the number of componenets (circles) obtained in the positive resolution of a knot (or link). Recall that this is the firts step in the Seifert algorithm for finding a Seifert surface!

Proof idea for 1 and 2. The key idea is that Khovanov (and Lee) homology are functorial under link cobordisms: given links L_0, L_1, suppose there is a cobordism $\Sigma \subset \mathbb{S}^3 \times [0,1]$ between them. Then there are induced maps

$$F_\Sigma : \mathrm{Kh}(L_0) \to \mathrm{Kh}(L_1), \quad F_{\Sigma,\mathrm{Lee}} : \mathrm{Lee}(L_0) \to \mathrm{Lee}(L_1).$$

In fact, it turns out that if Σ is a *connected* cobordism between knots, then

$$F_{\Sigma,\mathrm{Lee}} : \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q}$$

is an isomorphism.

Now we prove 1. Let $\Sigma \subset B^4$ be a surface with boundary K, realising the slice genus of K. Removing a disk from Σ gives a connected cobordism Σ' from the unknot to K. Now $F_{\Sigma'}$ and $F_{\Sigma',\mathrm{Lee}}$ are maps from Khovanov and Lee homologies of K to those of the unknot. We study how they change the Jones gradings. Reidemeister moves leave the Jones gradings invariant, while Morse moves of index 0 and 1 increase the Jones grading by 1, and Morse moves of index 1 decrease the Jones grading by 1. It follows that $F_{\Sigma'}$ changes J by exactly $\chi(\Sigma')$, and $F_{\Sigma',\mathrm{Lee}}$ by at least $\chi(\Sigma')$. Now if $x \in \mathrm{Lee}(K)$ is a non-zero element attaining $s_{max}(K)$, then

$$1 \geq J(F_{\Sigma'}(x)) \geq J(x) + \chi(\Sigma') = s + 1 - 2g_s(K).$$

The first inequality comes from the fact that $F_{\Sigma'}(x)$ lives in $\mathrm{Lee}(0_1)$ via the isomorphism. therefore $s \leq 2g_s(K)$. Now considering $\overline{K}$ gives the full result.

Next we prove 2. This is surprisingly straightforward! From the mirror and connected sum properties of s, it only remains to verify that s is a well defined map on $\mathcal{C}$. But to see this, suppose K_1 and K_2 are concordant. Then $K_1 \# \overline{K_2}$ is slice. But now by the first result, $s(K_1) - s(K_2) = s(K_1 \# \overline{K_2}) = 0$. Therefore s is well defined on concordance classes as required! □

A fun corollary is the following:

Corollary 2.10. *Suppose K_+ and K_- differ by a single crossing change, the crossing being positive in K_+ and negative in K_-. Then*

$$s(K_-) \leq s(K_+) \leq s(K_-) + 2.$$

The above inequality was shown by Livingston to hold for any knot invariant satisfying theorems 1 and 2 above!

We skip the third theorem, as it will not appear in our discussion on the slice-Bennequin inequality. However, we'll now take a look at theorem 4. The crux of theorem 4 is the following.

Proposition 2.11. If K has a positive diagram D with $n(D)$ crossings and $O(D)$ Seifert circles, then $s(K) = n(D) - O(D) + 1$.

Proof idea. The exact formula comes from considering *canonical generators* of $\text{Lee}(K;\mathbb{Q})$. One can show that

$$s(K) = \frac{J([\mathfrak{s}_{\mathcal{O}}] + [\mathfrak{s}_{\overline{\mathcal{O}}}]) + J([\mathfrak{s}_{\mathcal{O}}] - [\mathfrak{s}_{\overline{\mathcal{O}}}])}{2}.$$

This is simple because one of terms in the numerator corresponds to the copy of $\mathbb{Q}$ in $s_{max} = s(K) + 1$, and the other lies in grading $s_{min} = s(K) - 1$. Moreover, we have

$$J([\mathfrak{s}_{\mathcal{O}}]) = J([\mathfrak{s}_{\overline{\mathcal{O}}}]) = s(K) - 1.$$

Explicitly, $J([\mathfrak{s}_{\mathcal{O}}])$ is defined to be the maximum Jones grading among terms homologous to $\mathfrak{s}_{\mathcal{O}}$. But D is a positive diagram, meaning our resolution is the 0-resolution and lives in the lowest homological grading. Nothing can map into this grading, so $\mathfrak{s}_{\mathcal{O}}$ must be the unique class homologous to itself! Writing the class out explicitly, we have

$$\mathfrak{s}_{\mathcal{O}} = (a \text{ or } b) \otimes (a \text{ or } b) \otimes \cdots = (x \pm 1) \otimes (x \pm 1) \cdots .$$

In summary we know that $\mathfrak{s}_{\mathcal{O}}$ has $O(D)$ factors, and lies in the same Jones grading as $\otimes^k x$. Now from the definition of Khovanov homology,

$$s(K) - 1 = J(\mathfrak{s}_{\mathcal{O}}) = -O(D) + (n_+(D) - 2n_-(D)) = n(D) - O(D).$$

The result follows. □

As a simple corollary, once can chain together inequalities as follows:

$$2g_s(K) \leq 2g(K) \leq 2g(\Sigma) = n(D) - O(D) + 1 = s(K) \leq 2g_s(K)$$

where Σ is the Seifert surface obtained from the Seifert algorithm on the positive diagram D, which can be shown to have genus half of $n(D) - O(D) + 1$.

3 The slice Bennequin inequality, from Khovanov homology

Finally we prove the slice Bennequin inequality using the s-invariant, as promised. Rather than using contact structures and so on, our statement of the slice Bennequin inequality will be in terms of braids, analogous to the first form of the Bennequin inequality: recall the following theorem from section 1.

Theorem 3.1 (Bennequin inequality)**.** *Let β be a closed braid. Then*

$$w(\beta) - n(\beta) \leq -\chi(\beta).$$

Here $\chi(\beta)$ is the Seifert Euler characteristic of β.

The main result is that we can improve this inequality by replacing $\chi(\beta)$ with $\chi_s(\beta)$, i.e. the maximum Euler characteristic of a slice surface of β (instead of a Seifert surface).

Theorem 3.2 (Slice Bennequin inequality)**.** *Let β be a closed braid. Then*

$$w(\beta) - n(\beta) \leq -\chi_s(\beta).$$

The first step in this proof is to generalise the formula

$$s(K) = n(D) - O(D) + 1 = 2g_s(K)$$

to diagrams that are not necessarily positive:

Theorem 3.3. *Let K be a knot with diagram D. Then*

$$w(D) - O(D) + 1 \leq s(K).$$

Proof. This follows by a simple inductive argument using the result for positive knots, and the corollary relating the s invariant for knots that differ at one crossing:

$$s(K_-) \leq s(K_+) \leq s(K_-) + 2.$$

If the knot is positive, we're done. Now suppose we change a positive crossing to a negative one. The left side of the inequality decreases by exactly 2. On the other hand, from the aforementioned corollary, the right side decreases by at most 2. □

We're almost ready to prove the slice Bennequin inequality. The main point of interest is that closed braids need not be knots, but we can still use the s-invariant to extract information. In preparation, we must prove one more lemma.

Lemma 3.4. Let $\beta_1 = w_1 w_2$ and $\beta_2 = w_1 \sigma_i^{\pm 1} w_2$ be braids, where w_i are arbitrary words and σ_i is a standard generator. Let $\widehat{\beta_i}$ denote the closures of the β_is. Then

$$|\chi_s(\widehat{\beta_1}) - \chi_s(\widehat{\beta_2})| \leq 1.$$

Proof. Let $\Sigma \subset B^4$ be a slice surface for $\widehat{\beta_1}$. Note that Σ is connected, and in particular has no closed components. $\widehat{\beta_2}$ differs by the addition of a twist, which can be realised on the level of surfaces by the addition of a twisted band. The orientation of the braid ensures that the twisted band respects the orientation of Σ, so we end up with a new oriented connected surface Σ', now with boundary $\widehat{\beta_2}$.

If the twisted band joined two disjoint boundary components of Σ, then the genus of Σ' is the same as that of Σ, while the number of boundary components has decreased by one. On the other hand, if the two strands being twisted together were the same boundary component, then addition of the band separates them into two boundary components while increasing the genus by one. In either case, we have

$$\chi(\Sigma') = \chi(\Sigma) - 1.$$

Therefore

$$\chi_s(\widehat{\beta_2}) \geq \chi_s(\widehat{\beta_1}) - 1.$$

For the other direction, notice that $\beta_1 = w_1 \sigma_i^{\mp 1} \sigma_i^{\pm 1} w_2$, the same argument carries through with the roles of β_1 and β_2 reversed to give

$$\chi_2(\widehat{\beta}) \geq \chi_s(\widehat{\beta_2}) - 1.$$

Combining these we have

$$|\chi_s(\widehat{\beta_1}) - \chi_s(\widehat{\beta_2})| \leq 1$$

as required. □

Theorem 3.5 (Slice Bennequin inequality)**.** *Let β be a braid, and $\widehat{\beta}$ its closure. Then*

$$w(\beta) - n(\beta) = w(\widehat{\beta}) - n(\widehat{\beta}) \leq -\chi_s(\widehat{\beta}).$$

Proof. First we prove the result for knots. Since a knot has one component,

$$-\chi_s(\widehat{\beta}) = 2g_s(\widehat{\beta}) - 1.$$

Moreover, $O(\widehat{\beta}) = n(\widehat{\beta})$. Now from the previous result, we have

$$w(\widehat{\beta}) - n(\widehat{\beta}) = w(\widehat{\beta}) - O(\widehat{\beta}) + 1 - 1 \leq s(\widehat{\beta}) - 1 \leq 2g_s(\widehat{\beta}) - 1 = -\chi_s(\widehat{\beta}).$$

This proves the result in the case where $\widehat{\beta}$ has one component. For the more general case, we assume $\widehat{\beta}$ is an arbitrary link. We write β^+ to denote the braid obtained by deleting all generators in β which appear with a negative exponent. (In terms of standard generators, every braid is of the form $\sigma_{n_1}^{k_1} \sigma_{n_2}^{k_2} \cdots \sigma_{n_m}^{k_m}$, and we remove any $\sigma_{n_i}^{k_i}$ for which k_i is negative.)

We can assume without loss of generality that $\widehat{\beta^+}$ is a knot as follows: suppose the i and jth strands of $\widehat{\beta^+}$ are distinct components, with $i < j$. Then appending $\sigma_i \cdots \sigma_{j-1} \cdots \sigma_i$ to β^+ results in the two strands being twisted together into a single component in the closure.

On the other hand, if we append $\sigma_i \sigma_i^{-1}$ to β, the resulting braid is topologically unchanged. (In particular the writhe and number of strands is unchanged.) Therefore we can append as many $\sigma_i \sigma_i^{-1}$ as necessary onto the end of β so that $\widehat{\beta^+}$ is a knot. (Note that appending $\sigma_i \sigma_i^{-1}$ to β corresponds to appending σ_i to β^+.)

But now $\widehat{\beta^+}$ is a knot, so the slice Bennequin inequality holds for β^+. By construction, β is obtained by adding some number of negative crossings into β^+. The previous lemma showed that χ_s can change by *at most* 1 upon the addition of a single negative crossing. On the other hand, the writhe decreases by *exactly* 1 (while the number of strands is unchanged). Therefore inductively the slice Bennequin inequality holds for all braids. □

A famous application is the *Milnor conjecture*, which is a statement about the slice genus of torus knots. (Actually the Milnor conjecture is equivalent to the slice Bennequin inequality, but is the “easier to prove” version.)

Theorem 3.6 (Milnor conjecture). *The slice genus the torus knot $T_{p,q}$ is*

$$g_s(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

Proof. The torus knot $T_{p,q}$ is canonically the closure of a braid with p strands and $q(p-1)$ crossings. In fact, each of these crossings is positive, so that

$$w(T_{p,q}) = q(p-1), \quad n(T_{p,q}) = p.$$

This gives $w(T_{p,q}) - n(T_{p,q}) = (q-1)(p-1) - 1$. By the slice-Bennequin inequality, this is bounded above by $-\chi_s(T_{p,q}) = 2g_s(T_{p,q}) - 1$. Therefore we obtain

$$(p-1)(q-1)/2 \leq g_s(T_{p,q}).$$

For the other direction, note that the naive Seifert surface for $T_{p,q}$ has genus $(p-1)(q-1)/2$, so that

$$g_s(T_{p,q}) \leq g(T_{p,q}) \leq (p-1)(q-1)/2.$$

□

Theorem 3.7. *The slice Bennequin inequality holds for transverse and Legendrian knots in the standard structures of $\mathbb{R}^3$ (or $\mathbb{S}^3$). That is, if T is transverse and L is Legendrian, then*

$$sl(T) \leq -\chi_s(T), \quad tb(L) + |r(L)| \leq -\chi_s(L).$$

Proof. These follow from the braid-form of the slice Bennequin inequality, in the same way that the classical Bennequin inequality did. □

Finally we add that the slice Bennequin inequality doesn't even make sense if we want to interpret it in a general (tight) contact manifold: there is no reason these should bound some canonical 4-manifold - what do we consider to be slice surfaces?

That being said, in some cases we can indeed generalise the slice Bennequin inequality to tight contact structures. We now introduce the relevant definitions to state the generalisation.

Definition 3.8. A *Stein manifold* is a complex manifold X admitting a proper strictly plurisubharmonic function $\varphi : X \to \mathbb{R}$. This means that $\omega = d(d\varphi \circ J)$ is a symplectic structure on X, compatible with J.

A contact manifold (M, ξ) is *Stein fillable* if M is the boundary of a Stein manifold X such that ξ is the kernel of $d\varphi \circ J$.

Lisca and Matić showed that the slice Bennequin inequality generalises to knots in Stein fillable contact manifolds. Note that their proof is inherently non-combinatorial/geometric, as it relies on Seiberg-Witten theory.