Conformal Geometry with a View to Understanding the Cosmos

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Abstract

In general relativity, the universe is often formalized as a four-dimensional "space-time", i.e. a smooth manifold equipped with a signature (3,1) pseudo-Riemannian metric. I give an introduction to special and general relativity, motivating this formalism. We then discuss conformal compactifications of space-times, and observe that conformal geometry can be a more convenient approach to studying space-times than pseudo-Riemannian geometry. Finally we introduce conformal tractor calculus and outline its use in studying the asymptotic curvatures of certain hypersurfaces in space-times.

1 The rise of relativity

1.1 Newtonian mechanics and Electromagnetism

Quote (Kelvin, circa 1900)**.** *There is nothing new to be discovered in physics now. All that remains is more and more precise measurement.*

In the early 1900s, it was widely believed that physics was "almost completed". Many felt that the overarching physical theories founded on Newtonian mechanics and electromagnetism were correct modulo small improvements. Two notable necessary improvements that had to be made were:

- Resolving the ultraviolet catastrophe.
- Understanding the aether. "What does light look like if the observer is travelling at the speed of light?"

The former is something I won't go into, but it lead to the development of quantum mechanics. The latter grew into Einstein's theory of relativity.

At the turn of the 20th century, Newtonian mechanics was believed to be the correct theory of mechanics. The main axioms are as follows:

Postulates (Newton)**.**

- There exists an *absolute space*. This is a space which, without regard to anything physical, always remains similar and immovable. (This is an ambient space in which physical phenomena occur.)
- An *inertial frame* is a reference frame moving at a constant velocity with respect to absolute space. All inertial frames obey the laws of Newtonian mechanics.
- There exists an *absolute time*. All inertial frames experience this same notion of time.

Remark. *One can derive "Galilean transformations" from these postulates:*

- $t' = t$
- $x' = x - vt$
- $y' = y$
- $z' = z$

These axioms seem reasonable, because until the mid 1900s the most popular model of the universe was that it was a steady state. However, just before the turn of the 20th century, Maxwell's theory of electromagnetism was established. In this case, the axioms are the famous Maxwell's equations:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot B = 0$$
$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \nabla \times B = \mu_0\Big(J + \varepsilon_0 \frac{\partial E}{\partial t}\Big).$$

1.2 The aether

The idea was that these equations all hold in Newton's "absolute space". As a direct consequence of these equations, one can derive that all "electromagnetic waves" (including light) travel at a constant speed,

$$c = \frac{1}{\sqrt{\rho\varepsilon_0}}.$$

This raises some questions:

1. What does light look like to someone in an inertial frame which is moving at the speed of light with respect to absolute space?
2. If light is a wave that propagates at a constant speed in absolute space, then there must be a "medium (aether) that vibrates to give rise to light" with zero net motion with respect to absolute space. Therefore we should be able to detect absolute space by measuring the frame in which the aether is stationary.

Many physicists turned their attention to trying to detect the Aether. The most famous experiment being the *Michelson-Morley experiment*. Physicists considered the necessary properties of the aether:

- It must be a fluid, in order to fill space.
- It must be rigid (at least a million times more rigid than steel), to support the high frequencies of light.
- It had to be incompressible, massless, completely transparent, and non-viscous.

1.3 Special relativity

These issues lead to the development of special relativity, which is founded on different axioms to Newtonian mechanics.

Postulates (Einstein, SR)**.**

- The speed of light in a vacuum is constant for all observers.
- An *inertial frame* is a reference frame in which a body with zero net force acting upon it does not accelerate. All laws of physics (including Maxwell's equations) are true in every inertial frame.

Remark. *Accelerate with respect to what? Discussion.*

Remark. *Apparently Einstein wanted to call this the* theory of invariance *rather than* relativity*, but the word was recently used in a different theory so he opted for the un-used terminology. Pity, regarding the way the phrase "everything is relative!" is thrown around in popular culture.*

This means that if two observers are moving with respect to each other, they both see light as moving at the same speed. The only way this is possible is if the observed lengths of space or time change depending on your relative speeds.

One can derive the *Lorentz transformations*, literally via pythagoras' theorem (and no calculus):

- $t' = \gamma\Big(t - vx\Big)$
- $x' = \gamma(x - vt)$
- $y' = y$
- $z' = z$

In the above, γ is the *Lorentz factor* $\gamma = 1/\sqrt{1-v^2}$.

The mathematical cores of Newtonian mechanics and special relativity are the following invariants. Suppose we use coordinates (x, y, z, t) for parametrising space and time.

- In Newtonian mechanics, inertial frames are related by elements of the *Galilean group* $\mathrm{Gal}(3)$ (generated by rotations, translations, and uniform motions). In particular, $x^2 + y^2 + z^2$ and t are both invariant. This means that distances and times between objects doesn't depend on the reference frame.
- In Special relativity, inertial frames are related by elements of the *Lorentz group* $O(1, 3) = \{T \in GL(4, \mathbb{R}) : T^*\mu = \mu\}$ i.e. group of all isometries of Minkowski space. In particular, $x^2 + y^2 + z^2 - t^2$ is invariant when changing between inertial frames. When changing frames, time itself can dilate (and space can contract) provided this quantity remains invariant.

2 The formalism of general relativity

2.1 Space-times

A Riemannian manifold is a smooth manifold M equipped with a Riemannian metric g. A metric can be thought of as a smoothly varying inner product on the tangent spaces of M. Since on large scales we observe the universe to be a continuum, it's sensible to declare that the universe is a manifold.

Example. Consider the manifolds $\mathbb{R}^3$ and $\mathbb{R}$, each equipped with Euclidean metrics. Then $\mathbb{R}^3 \times \mathbb{R}$ (parametrised by (x, y, z, t)) is a possible model for a Newtonian universe. By construction, $x^2 + y^2 + z^2$ and t are both invariant between inertial frames.

We now want to have a similar model for the universe which matches the ideas of relativity. A pseudo-Riemannian manifold is a smooth n manifold M equipped with a smoothly varying non-degenerate symmetric bilinear form g. Given any orthogonal basis $\{e_1, \ldots, e_n\}$, if p is the number of positive values of $g(e_i, e_i)$, then $(p, n - q)$ is the signature of g.

Example. Consider the manifold $\mathbb{R}^4$ equipped with the metric

$$g((x_1, y_1, z_1, t_1), (x_2, y_2, z_2, t_2)) = x_1x_2 + y_1y_2 + z_1z_2 - t_1t_2.$$

This has signature $(3, 1)$, and has the correct invariant quantity for relativity. This is called *Minkowski space*, and is the flat model for a universe in relativity. Since the observable universe is flat on scales that humans can measure, Minkowski space describes the local geometry of an empty universe.

What are some interesting properties of Minkowski space? Given a point p in Minkowski space, we can partition its tangent space into regions where vectors have positive length, 0 length, and negative length, and we can choose a time orientation. The regions are then the *future light cone*, *past light cone*, and *elsewhere*. One can show that any object with mass must travel slower than the speed of light, which corresponds to the trajectory staying inside the lightcone. This gives rise to causality.

Definition 2.1. A *space-time* is a smooth manifold M equipped with a signature $(3, 1)$-metric. The collection of all feasible "manifold descriptions of the universe", given relativity, is some subset of the collection of space-times.

Remark. *By expanding our horizons from Minkowski space to arbitrary space-times, we're allowing for the shape of the universe to be different. Maybe it has interesting topological features, or other globally detectable features. But the most important change is that we can now have curvature in our universe, that is, we can have local variations. Time for a famous quote:*

2.2 Mass $\leftrightsquigarrow$ curvature

Quote (Wheeler, 1990)**.** *Space-time tells matter how to move; matter tells space-time how to curve.*

In a pseudo-Riemannian manifold, we no longer have the notion of "straight lines". It doesn't make sense to say things like "an object in motion will maintain a constant velocity unless acted on by an external force". Instead, the notion of a straight line is replaced with that of a geodesic of the manifold, and the statement that "an object in free fall will follow a geodesic". We now justify this.

Fundamental observation: Inertial mass is indistinguishable from gravitational mass. Suppose an object is in free fall, and the only force F acting on it is due to a gravitational field g. The gravitational mass is defined by $F = m_G g$. Given this force, the object will experience an acceleration a by Newton's second law: $F = m_I a$, where m_I is the inertial mass of the object. Acceleration due to gravity is then given by

$$a = \frac{m_G}{m_a} g.$$

The ratio m_G/m_I is equal to a constant (which we can take to be 1 by using appropriate units) if and only if all objects fall with the same acceleration in a given gravitational field.

A priori there seems to be no reason to expect this to hold, but as of 2008 m_G/m_I has been measured to be a constant to within 10^{-12}. This motivated the important postulate that turns special relativity into general relativity:

Postulates (Einstein, GR. Strong equivalence principle)**.** In a sufficiently small region of space-time, uniform acceleration and uniform gravitational fields are indistinguishable.

A consequence is that if you're standing on a planet, locally this is indistinguishable from being forcefully accelerated through empty space. On the other hand, if you're free falling towards a planet, this is locally indistinguishable from having no acceleration.

2.3 General relativity

A geodesic on a manifold is defined to be a curve with no acceleration, so this postulate translates to the formulation that matter in free fall follows geodesics. Since geodesics are determined by curvature, this is exactly what gives rise to the correspondence between mass and curvature in general relativity. Formally the relationship is given by the *Einstein field equations*.

$$\mathbf{G} = \frac{8\pi G}{c^4}\mathbf{T}$$

Here $\mathbf{G}$ is the *Einstein tensor*, defined in terms of the Riemann curvature tensor, and $\mathbf{T}$ is the *stress-energy tensor*. The left hand side describes the curvature of the space-time, while the right hand side describes the matter content. More explicitly, $\mathbf{G} = G_{\mu\nu}$ is defined by

$$G_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathrm{Sc}$$

Where Ric and Sc are the Ricci and Scalar curvatures, respectively. These are defined in terms of the covariant derivative: given vector fields X, Y, Z on a space-time, the Riemann curvature is defined by

$$\mathrm{Riem}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

It turns out that Riem is a tensor field on our manifold, with $\mathrm{Riem}^{\mu}{}_{\nu\eta\sigma}$ in index notation being defined by

$$W_\mu \mathrm{Riem}^{\mu}{}_{\nu\eta\sigma}X^\nu Y^\eta Z^\sigma = g(\mathrm{Riem}(X,Y)Z, W).$$

The Ricci and Scalar curvatures are then defined by

$$\mathrm{Ric}_{\nu\eta} := \mathrm{Riem}^{\mu}{}_{\nu\eta\mu}, \quad \mathrm{Sc} := \mathrm{Ric}^{\nu}{}_{\nu}.$$

What follows from this is that the Einstein field equations are really a collection of partial differential equations.

Conclusion:

- A space-time (M, g) is only valid in general relativity if it is a solution to a collection of PDEs, called the Einstein field equations.
- This also motivates the study of space-times as solutions to IVPs, where an "initial value" might be a hypersurface in M representing a moment in time.

Are there solutions to the Einstein field equations? Recall that the stress-energy tensor represents the matter content of the space-time. Therefore a completely empty universe corresponds to $\mathbf{T} = 0$, in which case the Einstein field equations become $\mathbf{G} = 0$. In this case any space-time with vanishing curvature will solve the Einstein field equations - for example, Minkowski space.

3 Conformal × (compactification + geometry)

3.1 Penrose diagrams

An important tool in the study of space-times is conformal compactification. If we declare that the universe is a manifold, it's natural to be curious about the global structure of the universe. With this in mind, Penrose introduced what are now called *Penrose diagrams*.

Penrose observed that a certain compact 4-manifold with boundary equipped with a certain metric, often drawn as a diamond, has an interior conformally equivalent to Minkowski space:

Draw Penrose diagram on a black-board...

- Describe names of different parts of a Penrose diagram. Explain how it corresponds to Minkowski space.
- Describe the asymptotic behaviour of particles, light, mention causality. Mention holography.
- There exist penrose diagrams for blackholes, and space-times other than Minkowski space.

Abstracting the idea of a Penrose diagram to mathematics, its essence is conformal compactification.

3.2 Conformal compactification

Definition 3.1. Let M be a smooth manifold, and g, g' two metrics on M. Then g and g' are said to be *conformally equivalent* if there is a non-vanishing smooth function r on M such that $g' = r^2 g$ on M.

Remark. *Conformal essentially means "angle preserving". One can verify that any two conformally equivalent metrics share the same angles but might measure different lengths.*

Definition 3.2. Let $\Sigma \subset M$ be a submanifold. A function $f : M \to \mathbb{R}$ is a *defining function* for Σ if $\mathcal{Z}(r) = \Sigma$, and ∇r is non-vanishing on Σ.

Definition 3.3. Suppose (M, g) is a pseudo-Riemannian manifold. A compact manifold $(\overline{M}, \overline{g})$ with boundary is said to be a *conformal compactification* of M if M is the interior of $\overline{M}$, and there is a defining function r for ∂M such that

$$\overline{g}|_M = r^2 g.$$

Definition 3.4. In the case where we are compactifying a space-time, (M, g) is called the *physical space-time* and g the *physical metric*. $(\overline{M}, \overline{g})$, g are the *unphysical* space-time and metric respectively. ∂M is called *conformal infinity* or *scri*, which is a disjoint union of time-like, light-like, and space-like infinity.

The reason this is such a powerful tool is because it gives physicists a guideline for solving problems concerning asymptotics of quantities in space-times:

1. Translate the physical problem concerning asymptotics into a pseudo-Riemannian geometry problem. (E.g. mass $\rightsquigarrow$ curvature.)
2. Rescale all of the relevant geometric machinery from the physical space-time to the unphysical space-time.
3. Solve the problem on conformal infinity.
4. Undo the rescaling to interpret the solution in the physical space-time.

In practice, rescaling is actually a big hassle. Given a pseudo-Riemannian manifold (M, g), the *fundamental theorem of Riemannian geometry* guarantees the existence of a unique torsion free connection ∇^g which is compatible with the metric. Suppose g, g' are conformally equivalent metrics on M, with $g' = r^2 g$. Then how are ∇^g and $\nabla^{g'}$ related? It turns out that if X, Y are vector fields on M, then

$$\nabla^{g'}_X Y = \nabla^g_X Y + X(\log r)Y + Y(\log r)X - g(X, Y)\operatorname{grad}\log r.$$

Recalling that the Riemann curvature tensor on (M, g) was defined by

$$\mathrm{Riem}^g(X, Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z,$$

it's pretty clear that simply rescaling something as fundamental as the curvature tensor is a lot of work. Staying within the realm of pseudo-Riemannian geometry, the calculations get very complicated.

3.3 Conformal geometry

Since conformal rescaling can quickly become intractable, one way around this issue is to do all of your work in a conformal manifold as opposed to a pseudo-Riemannian manifold.

Definition 3.5. Let M be a manifold equipped with an equivalence class $[g]$ of metrics, where $g \sim g'$ if they are conformally equivalent. Then $(M, [g])$ is called a *conformal manifold*.

In this formulation, a space-time corresponds to a specific choice of metric within the equivalence class. That is, a specific metric in the conformal class is designated as the "physical metric". A new outline for solving problems involving asymptotics would be:

1. Translate the physical problem concerning asymptotics into a conformal geometry problem. (E.g. mass $\rightsquigarrow$ curvature.)
2. Solve the problem on conformal infinity.
3. Interpret the solution in the physical space-time.

The reason this approach is uncommon is because conformal geometry is less rigid, so there is much less machinery to work with. The fact that the Levi-Civita connection had a complicated conformal rescaling formula shows that it isn't well defined on a conformal manifold. In conformal geometry we no-longer have access to the tensor calculus toolkit of pseudo-Riemannian geometry - we must instead develop a conformally invariant analogue.

4 Conformal tractor calculus

4.1 Tractor calculus

Since we have no chance of finding an adequate analogue of the Levi-Civita connection for differentiating tensor fields, the idea is to first define a new bundle (instead of the tangent bundle) which contains at least all of the information of the tangent bundle, but does admit a canonical connection which is compatible with a conformal metric on the bundle. On a very abstract level, these are called *tractor bundles.*

Definition 4.1 (Tractor bundle)**.** Let $P \to G$ be an inclusion of Lie groups, where P is parabolic. Suppose $E \to M$ is the P-frame bundle of a (G, P)-Cartan geometry on M. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Then the associated bundle

$$E \times_H V \to M$$

is called a *tractor bundle.* $E \times_H V$ admits a canonical G-invariant connection. (See Čap and Souček, *Curved Casimir Operators and the BGG Machinery* for details.)

The important result for us is that a conformal manifold admits a transparent tractor bundle. If $(M, [g])$ is a conformal manifold of dimension n, there is a rank $n+2$ bundle $\mathcal{T} \to M$ called the conformal tractor bundle, which splits as

$$L[1] \oplus TM[1] \oplus L[-1] \to M,$$

where L is a line bundle, given any choice of metric g in the conformal class. The $[k]$ denotes that the bundles have weight k. The conformal tractor bundle is canonically equipped with a conformally invariant connection, termed the tractor connection, which in a choice of metric $g \in [g]$, is given by

$$\nabla_a^{\mathcal{T}} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab}\sigma + \mathbf{g}_{ab}\rho \\ \nabla_a \rho - P^b{}_a \mu_b \end{pmatrix}.$$

Moreover, there is a conformally invariant metric h on the tractor bundle, compatible with $\nabla^{\mathcal{T}}$, which is given by

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & [g] & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In summary, even though a conformal n-manifold doesn't admit a "tensor calculus", it has a rank $n+2$ tractor bundle which is equipped with a conformally invariant metric and compatible conformally invariant connection. From here we can develop a calculus for solving problems in conformal geometry.

5 Asymptotic curvature of hypersurfaces

Rather than keeping things abstract, it's best to see tractor calculus in action. I mentioned earlier that one way that physicists study space-times is in an "initial value problem formulation". I will now elaborate on this.

The Einstein field equations are a collection of six independent partial differential equations, and a solution to these equations is a metric. These are too difficult to solve in a general setting, so almost all physicists use the *ADM formalism* of general relativity. This involves the inclusion of one significant assumption.

Postulates (ADM)**.** The universe is a space-time that is foliated into a family of space-like hyper-surfaces Σ_t, labelled by a time coordinate. (A hypersurface is *space-like* if at every point the unit normal vector is time-like.)

Under this assumption, the Einstein field equations divide into two *constraint equations* and two *evolution equations*. One can show that if initial data satisfies the constraint equations at any given time, then the time evolution of the data will also satisfy the constraint equations. Here "initial data" means a 3-manifold equipped with a Riemannian metric, which corresponds to Σ_0 in the ADM formalism. Finding solutions to the Einstein field equations reduces to the problem of finding initial data that satisfies the constraint equations.

Unfortunately the problem of understanding when initial data satisfies the constraint equations is also difficult - but it becomes significantly easier if we prescribe the data as having constant mean curvature, since this causes the two constraint equations to decouple. For this reason constant mean curvature initial data is of great interest. Understanding constant mean curvature initial data may shed light on the structure of the universe.

A example of a question regarding constant mean curvature initial data is the following:

Is a constant (non-zero) mean curvature space-like hypersurface in
an asymptotically flat space-time necessarily asymptotically hyperbolic?

This can be answered in the affirmative by using conformal tractor calculus to extend statements about curvature to conformal infinity.