Morse theory

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This document contains Morse theory notes, largely following Audin and Damian [AD14]. The focus is on developing Morse homology and exploring some applications (such as the Morse inequalities). Some solutions to exercises are also given here. At the end of these notes we give a proof outline of the h-cobordism theorem (and prove the generalised Poincaré conjecture) following Milnor's lecture notes [Mil65]. Finally we explore the status of the generalised Poincaré conjecture and h-cobordism theorem (for each dimension) in several categories of manifolds ($\mathbf{Man}, \mathbf{Man}^\infty$, etc).

Contents

Chapter 1

Introduction to Morse theory

1.1 Introduction to the introduction

The fundamental idea in Morse theory is the following:

A well chosen map $f : M \to \mathbb{R}$ encodes a lot of information about M.

For example, consider the "height function" $h : \mathbb{T}^2 \to \mathbb{R}$ of a torus as depicted in figure 1.1:

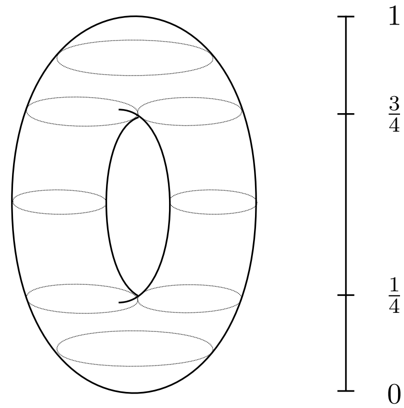

Figure 1.1: A torus with its height function next to it.

This height function has exactly four critical points, i.e. points where dh vanishes. We find that these critical points correspond to changes in the topology of the level sets of the function. Precisely, the level starts empty for negative values of a. Then at $a = 0$, there is a bifurcation and the level set is a point. As a continues to increase, the level set is a circle, until we reach $a = \frac{1}{4}$. Again there is a bifurcation, and the level set at this point is a figure 8. As we continue to increase a, the level set is now two disjoint circles and so on. We find that the changes in topology of the level sets occurs precisely at the critical points of h. On the other hand, when a is not a critical point, the submanifold theorem ensures that the level set $h^{-1}(a)$ is a submanifold of the torus. This agrees with our observations above.

The goal of Morse theory is to find invariants of manifolds by counting critical points of well chosen functions. The notion of a "well chosen function" is formalised to mean a *Morse function*.

Definition 1.1.1. A map $f : M \to \mathbb{R}$ is a *Morse function* if its critical points are non-degenerate. That is, if the Hessian of f at each critical point is non-singular.

A motivation for the existence of useful invariants of manifolds arising from Morse functions is *Reeb's theorem*.

Theorem 1.1.2 (Reeb's theorem)**.** *Let M be a compact manifold. Suppose there exists a Morse function on M with exactly two critical points. Then M is homeomorphic to a sphere.*

This theorem shows that a "choice" of Morse function can give results about the underlying space that are independent of the choice of Morse function. Eventually we generalise this idea and develop *Morse homology*. This is a homology theory constructed by counting critical points of Morse functions, which we show depends only on the diffeomorphism class of the manifold. The first section of these notes will culminate in the famous *Morse inequalities*.

1.2 Morse functions: existence and genericness

Definition 1.2.1. Let M be a smooth manifold, and $f : M \to \mathbb{R}$ a smooth map. Then any $x \in M$ such that $df_x = 0$ is a *critical point*. If M is compact, smooth functions always have critical points since they must attain their maxima and minima.

While first derivatives exist, second derivatives (Hessians) do not exist on smooth manifolds in general. However, they are well defined on critical points.

Definition 1.2.2. Let $x \in M$, and $f : M \to \mathbb{R}$. Suppose $df_x = 0$. The *Hessian* at x is defined by

$$d^2 f_x(X, Y) = (X(\widetilde{Y} f))(x),$$

where X, Y are tangent vectors in $T_x M$, and $\widetilde{Y}$ is any local extension of Y.

We must verify that the Hessian is a well defined symmetric bilinear form. Suppose $\widehat{Y}$ is any other extension of Y, and let $\widehat{X}$ be an extension of X. Then

$$(X(\widetilde{Y}f))(x) - (Y(\widetilde{X}f))(x) = [\widetilde{X}, \widetilde{Y}]_x f = df_x([\widetilde{X}, \widetilde{Y}]_x).$$

But the last term vanishes since $df_x = 0$ by assumption. Moreover, this calculation shows that the map is well defined, since

$$(X(\widetilde{Y}f))(x) = (Y(\widetilde{X}f))(x) = (X(\widehat{Y}f))(x).$$

A second approach to defining the Hessian is to use local charts as in the exercise 1 of Audin-Damian:

Exercise 1.2.3. (A-D, exercise 1) Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be smooth. Let V be another open subset of $\mathbb{R}^n$, and $\varphi : V \to U$ a diffeomorphism. Compute $d^2(f \circ \varphi)_y$ for $y \in V$. Let M be a manifold and $g : M \to \mathbb{R}$ a function. Show that $(d^2 g)_x$ is well defined on $\ker(dg)_x \subset T_x M$.

Solution: Here $(d^2 f)_x$ denotes the usual Hessian of f at x, defined by

$$(Hf_x)(u, v) = ((d^2 f)_x)_{ij} u^i v^j = \frac{\partial^2 f}{\partial x_i \partial x_j} u^i v^j.$$

Since φ is a diffeomorphism, it can be expressed as a smooth change of coordinates $(y_1, \ldots, y_n) \mapsto (\varphi_1(y_1, \ldots, y_n), \ldots, \varphi_n(y_1, \ldots, y_n)) = (x_1, \ldots, x_n)$. Then the Hessian of $f \circ \varphi$ is given by

$$\begin{aligned}
(H(f \circ \varphi)_y)(u, v) &= \frac{\partial^2 (f \circ \varphi)}{\partial y_i \partial y_j} u^i v^j \\
&= \Big(\frac{\partial x_k}{\partial y_i} \frac{\partial}{\partial x_k} \Big(\frac{\partial x_l}{\partial y_j} \frac{\partial f}{\partial x_l} \Big) \Big) u^i v^j \\
&= \Big(\frac{\partial x_k}{\partial y_i} \frac{\partial}{\partial x_k} \Big(\frac{\partial x_l}{\partial y_j} \Big) \frac{\partial f}{\partial x_l} \Big) u^i v^j + \frac{\partial^2 f}{\partial x_k \partial x_l} \Big(\frac{\partial x_k}{\partial y_i} u^i \Big) \Big(\frac{\partial x_l}{\partial y_j} v^j \Big).
\end{aligned}$$

In fact, this calculation shows that the Hessian of $g : M \to \mathbb{R}$ at x is well defined on the kernel of dg_x, since that is where the first term in the above formula vanishes. (Observe that φ corresponds to the choice of local chart on a manifold, and the second term is chart-invariant.) $\triangle$

Some examples and non-examples of Morse functions are explored in the exercises at the end of this section. We next prove that morse functions exist, and in fact, there are many of them!

Proposition 1.2.4. Let $M \subset \mathbb{R}^n$ be a submanifold. For almost any $p \in \mathbb{R}^n$, the function

$$f_p : M \to \mathbb{R}, \quad x \mapsto \|x-p\|^2$$

is a Morse function.

Remark. By the Whitney embedding theorem, it follows that Morse functions exist on all smooth manifolds.

Proof. Let f_p be as above. The derivative of f_p is given by

$$df_{p,x}(v) = 2(x-p, v).$$

Therefore the critical points occur exactly when T_xM is normal to $x-p$. (Such a p can always be found if $n > \dim M$, so critical points exist.) Choose local coordinates $(u_1, \dots, u_d)$ for M, so that

$$\frac{\partial f_p}{\partial u_i} = 2(x-p) \cdot \frac{\partial x}{\partial u_i}, \quad \frac{\partial^2 f_p}{\partial u_i \partial u_j} = 2\Big(\frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + (x-p) \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\Big).$$

Therefore (by definition) x is a non-degenerate critical point if and only if $x-p$ is normal to T_xM, and the matrix on the right is non-degenerate, i.e. has rank d. Recall that Sard's theorem states that *the set of critical points of a map* $g : M \to N$ *has measure zero in* N, where a critical point is any $x \in M$ such that dg_x does not have maximal rank (i.e. rank equal to $\min\{\dim M, \dim N\}$). Therefore by Sard's theorem, it suffices to show that the $p \in \mathbb{R}^n$ such that $x-p$ is normal to T_xM and the matrix on the right is singular, are the critical points of a smooth map.

To this end, consider the normal bundle of M in $\mathbb{R}^n$,

$$NM = \{(x,v) \in T_x\mathbb{R}^n : v \in T_xM^{\perp}\}.$$

Define the map $E : NM \to \mathbb{R}^n$ by $E(x,v) = x+v$. It can then be verified that $p = x+v \in \mathbb{R}^n$ is a critical point of E if and only if

$$2\Big(\frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + v \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\Big)$$

is singular. Therefore the set of all f_p (with p varying in $\mathbb{R}^n$) which are not Morse functions corresponds to a subset of the critical points of E, which by Sard's theorem, has measure zero in $\mathbb{R}^n$. Thus for almost all p, f_p is Morse. $\square$

This shows that manifolds have many Morse functions. However, it is not immediate that the Morse functions are *generic* in the sense that any function is approximated by a Morse function. This turns out to be the case too!

Proposition 1.2.5. Let M be a manifold, and $f : M \to \mathbb{R}$ smooth. Let $k \in \mathbb{N}$. Then on any compact subset of M, f can be approximated by a Morse function in C^k-norm.

Proof. This follows from the previous proposition, with details given in A-D. The idea is to choose an embedding into $\mathbb{R}^n$, and then use the previous proposition explicitly (that is, the proof makes use of the functions f_p). $\square$

An alternative but similar result is the following, which relies on transversality results and no embedding.

Proposition 1.2.6. Let M be a compact manifold. Then the set of Morse functions on M is a dense open subset of $C^\infty(M)$.

1.3 The Morse lemma

We know from Taylor's theorem that f near a critical point is approximated by its second derivative in the sense that

$$f(x) \approx f(c) + \frac{1}{2}(d^2 f)_c(x-c, x-c).$$

The Morse lemma states that in an appropriate chart, we have equality.

Theorem 1.3.1 (Morse lemma)**.** *Let $f : M \to \mathbb{R}$ be a Morse function. Suppose c is a critical point of f. Then there is a local chart $(x_1, \dots, x_n)$ (called a* Morse chart*) containing c such that, on this chart,*

$$f(x) = f(x_1, \dots, x_n) = f(c) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

The integer i depends only on the critical point, and is called the index *of the critical point.*

Remark. If i is the index of c, then $(n-i, i)$ is the signature of the bilinear form $(d^2 f)_c$.

Corollary 1.3.2. *Critical points of morse functions are isolated, by observing that on a Morse chart of c, df only vanishes at c. It follows that Morse functions on compact manifolds have finitely many critical points.*

In figure 1.1 we inspected the *height function* of the torus. This is a Morse function with four critical points. Starting from the bottom, we see that the critical points have index $0, 1, 1, 2$. In general a local maximum has full index n, while a local minimum has index 0. Saddle points have index strictly between 0 and n.

The above corollary does not in fact require the full power of the Morse lemma. Another proof is as follows.

Exercise 1.3.3. (A-D, exercise 2.) Characterise non-degenerate critical points of $f : M \to \mathbb{R}$ in terms of transversality of $df : M \to T^*M$. Deduce that non-degenerate critical points are isolated.

Solution: Let $f : M \to \mathbb{R}$. Suppose c is a critical point of f. Then $df_c = 0$. Interpreting df as a map $M \to T^*M$, this means that $df_c(V) = 0$ for all $V \in T_cM$, so df intersects the zero section $Z = \{\alpha \in T^*M : \alpha_x = 0 \text{ for all } x \in M\} \cong M \hookrightarrow T^*M$ at c. Recall that c is a non-degenerate critical point if and only if $d^2 f_c$ is a non-degenerate bilinear form, i.e. it has full rank. Thus c is non-degenerate if and only if $d(df)_c : T_cM \to T_{df(c)}(T^*_cM)$ defined by $d(df)_c : V_c \mapsto ((c, df_c), d(df)_c V_c)$ is an isomorphism. Equivalently, the image of $d(df)_c$ is $T_{df(c)}(T^*_cM)$. This happens if and only if $\operatorname{im} d(df)_c + T_{df_c}Z = T_{df(c)}(T^*M)$. Therefore non-degenerate critical points of f are precisely those $c \in M$ such that df intersects Z transversely.

We next prove that non-degenerate critical points are isolated. Recall that the intersection of two transverse submanifolds is itself a submanifold, with codimension given by the sum of the codimensions of the two submanifolds. Moreover, $\operatorname{im} df$ is an embedded submanifold of T^*M. (One can readily show, using the definition of a section, that df is an injective immersion. Using continuity of $\pi : T^*M \to M$, one can conclude that df is proper, so it is an embedding.) Since the zero section and $\operatorname{im} df$ each have codimension n in T^*M, the non-degenerate critical points must be an embedded 0-manifold. Therefore the non-degenerate critical points are isolated, as required. $\triangle$

1.4 Examples and exercises

Arguably the most important Morse functions are height functions and distance-to-a-point functions. The former was introduced in the introduction to the introduction, while the latter was introduced in the proof of the abundance of Morse functions. We now see more examples via some exercises.

Exercise 1.4.1. (A-D, exercise 3.) Monkey saddle: investigate $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $(x, y) \mapsto x^3 - 3xy^2$.

Solution: Observe that $\partial_x f = 3x^2 - 3y^2$, while $\partial_y f = -6xy$. Therefore there is a unique critical point, at $x = y = 0$. This is a degenerate critical point, since the Hessian also vanishes. This is in keeping with visual intuition, since f does not have a true saddle at $(0, 0)$: instead the level set $f^{-1}(0)$ is three intersecting lines, showing that the critical point is not "primitive" in a sense. Writing $f(x) = x(x^2 - 3y^2)$ and perturbing it to give $(x - \alpha)(x^2 - 3y^2)$ separates the critical point into a non-degenerate saddle at $(0, 0)$ and a degenerate critical point (with $f^{-1}(0)$ a line) at $x = \alpha$. $\triangle$

Exercise 1.4.2. (A-D, exercise 4.) Show that if $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ are Morse, then $f + g : M \times N \to \mathbb{R}$ is Morse, and the critical points are pairs of critical points of f and g.

Solution: Explicitly, $f+g$ is defined by $(f+g)(x, y) = f(x)+g(y)$. Suppose $(x, y) \in M \times N$ is a critical point. Then x is necessarily a critical point of f, and y is necessarily a critical point of g. To see this, observe that $d(f + g) = df + dg$, but $dg_M \equiv 0$, so whenever $d(f+g) = 0$, df_M must also vanish. Thus x is a critical point of f, and similarly for g. The converse also holds, so the critical points of $f + g$ are exactly the pairs of critical points of f and g. Similarly the Hessian of $f + g$ is the sum of the Hessians, which vanishes on the critical points. Therefore $f + g$ is Morse. $\triangle$

Chapter 2

Pseudo-gradients, topology, and the Smale condition

2.1 Existence of pseudo-gradients

If $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then its gradient is the vector field $\operatorname{grad} f$ defined by

$$\operatorname{grad}_x f = \Big(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\Big).$$

Equivalently, it is the vector field defined by

$$g(\operatorname{grad} f, Y) = df(Y)$$

for all vector fields Y on $\mathbb{R}^n$. Here g is the Euclidean metric on $\mathbb{R}^n$. This idea generalises to Riemannian manifolds.

Definition 2.1.1. Let $f : M \to \mathbb{R}$, (M, g) a Riemannian manifold. The *gradient* of f is the vector field $\operatorname{grad} f$ defined by

$$g(\operatorname{grad} f, Y) = df(Y)$$

for all vector fields Y.

The two key properties of gradients are the following:

1. (Since metrics are non-degenerate), the gradient vanishes if and only if $df = 0$, i.e. it vanishes precisely on critical points.

2. (Since metrics are positive-definite), f decreases along integral curves of f. More precisely, let φ be the flow of $-\operatorname{grad} f$. Then for any non-critical x,

$$\frac{d}{dt}(f(\varphi^t(x))) = (df)_{\varphi^t(x)}(-\operatorname{grad}_{\varphi^t(x)} f) = -g(\operatorname{grad}_{\varphi^t(x)} f, \operatorname{grad}_{\varphi^t(x)} f) < 0.$$

Using these properties we construct *pseudo-gradient fields* whose integral curves connect critical points of Morse functions. These allow the notions of stable and unstable manifolds of critical points, which later become significant. In general we do not have a Riemannian metric lying around, so with these two key properties in mind, we define *pseudo-gradients*.

Definition 2.1.2. Let $f : M \to \mathbb{R}$. A vector field X is a *pseudo-gradient adapted to f* if

1. $(df)_x(X) \leq 0$, and equality holds if and only if x is a critical point of f.
2. In a Morse chart around a critical point x, X agrees with $-\operatorname{grad} f$ (for the canonical metric of $\mathbb{R}^n$).

We now establish some notation that will be used hereafter. Let $f : M \to \mathbb{R}$ be a function and c a critical point of index i. Then there is a Morse chart in a neighbourhood of c in which f is of the form

$$f(x) = f(c) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2 = f(c) + Q(x).$$

Let V_- be the span of $x_1, \ldots, x_i$, and V_+ the span of $x_{i+1}, \ldots, x_n$. Then $V = V_- \oplus V_+$, and Q is negative definite on V_- while positive definite on V_+. For each $\varepsilon, \eta > 0$, the "standard balls" are defined by

$$U(\varepsilon, \eta) = \{x \in \mathbb{R}^n : -\varepsilon < Q(x) < \varepsilon, \|x_-\|^2 \|x_+\|^2 \leq \eta(\varepsilon + \eta)\}.$$

Since $Q : \mathbb{R}^n \to \mathbb{R}$, it has a gradient,

$$-\operatorname{grad}_{(x_-, x_+)} Q = 2(x_-, -x_+).$$

The boundary of $U(\varepsilon, \eta)$ consists of three pieces:

- $\partial_+ U = \{x \in \mathbb{R}^n : Q(x) = \varepsilon, \|x_-\|^2 \leq \eta\}$,
- $\partial_- U = \{x \in \mathbb{R}^n : Q(x) = -\varepsilon, \|x_+\|^2 \leq \eta\}$,
- $\partial_0 U = \{x \in \mathbb{R}^n : |Q(x)| \leq \varepsilon, \|x_-\|^2 \|x_+\|^2 = \eta(\varepsilon + \eta)\}$.

The first two pieces bound sublevel sets of Q, and the last piece is made up of segments of integral curves of the gradient of Q.

In summary, given a critical point c, there is a chart $U = U(\varepsilon, \eta) \subset \mathbb{R}^n$ so that its image $\Omega(c) \subset M$ under some diffeomorphism h is a neighbourhood of c. We also denote the boundaries of the neighbourhood by $\partial_\pm \Omega(c) = h(\partial_\pm U)$ and $\partial_0 \Omega(c) = h(\partial_0 U)$. We consistently try to denote neighbourhoods in M by Ω, and the model spaces (charts in $\mathbb{R}^n$) by U.

Theorem 2.1.3. *Given any Morse function $f : M \to \mathbb{R}$, M compact, there is a pseudo-gradient adapted to f.*

Proof. We give a proof outline. One approach is to use the existence of Riemannian metrics on manifolds. A more elementary approach is to use partitions of unity, which we describe here.

1. f has finitely many critical points, $c_1, \ldots, c_n$. These have disjoint Morse charts $(U_1, h_1), \ldots, (U_n, h_n)$. This extends to a finite atlas $\{U_i : i \in I\}$, so that each c_j is contained in exactly one U_i.

2. On each Ω_i, define X_i to be the pullback of the vector field $-\operatorname{grad} f \circ h_i$ on U_i. Let φ_j be a partition of unity subordinate to $\{\Omega_i : i \in I\}$. Define
$$X = \sum_{i \in I} \varphi_i(x) X_i(x).$$

3. One can verify that X is a pseudo-gradient adapted to f. The key observation is that if X vanishes at x, x must be a critical point: otherwise every $\varphi_i(x)$ vanishes, which is absurd.

$\square$

2.2 Trajectories and stable/unstable manifolds

Let $f : M \to \mathbb{R}$, and let X be a pseudo-gradient field. The vector flows of X are called *trajectories* of X, denoted φ^t. The most important property of trajectories is that they are guaranteed to connect critical points. We begin this section by defining defining stable and unstable manifolds of critical points, which are collections of trajectories that tend to (or from) the critical point.

Definition 2.2.1. Let c be a critical point of $f : M \to \mathbb{R}$. The *stable manifold* of c is
$$W^s(c) = \{x \in M : \lim_{t \to \infty} \varphi^t(x) = c\}.$$
The *unstable manifold* of c is
$$W^u(c) = \{x \in M : \lim_{t \to -\infty} \varphi^t(x) = c\}.$$

With notation as established in the previous section, if our manifold if $U = U(\varepsilon, \eta)$, we have $W^s(0) = U \cap V_+$, $W^u(0) = U \cap V_-$.

Proposition 2.2.2. The stable and unstable manifolds of a critical point c are submanifolds. Moreover, they are diffeomorphic to open disks, and
$$\dim W^u(c) = \operatorname{codim} W^s(c) = \operatorname{ind}(c)$$

This loosely says that the trajectories belonging to stable and unstable manifolds describe the critical points. But it turns out that all trajectories belong to a stable and unstable manifold.

Proposition 2.2.3. Let M be compact, and $\varphi^t(x)$ a trajectory of a pseudo-gradient field X of f. Then there are critical points c, d of f such that

$$\lim_{t\to\infty} \varphi^t(x) = c, \quad \lim_{t\to-\infty} \varphi^t(x) = d.$$

Proof. We give a proof outline for the case $\lim_{t\to\infty} \varphi^t(x)$. First suppose that $\lim_{t\to\infty} \varphi^t(x)$ exists. Then the limit is necessarily a critical point, since X vanishes exactly on critical points. Therefore it suffices to show that the limit exists.

Suppose for a contradiction that the limit does not exist. By the definition of pseudo-gradients, f is then strictly decreasing along φ^t, so if it enters any Morse chart it must leave in finite time. Let t_0 be the time at which it leaves all Morse charts. By compactness of M, there exists ε_0 such that for all $t > t_0$, $(df)_x(X) \leq \varepsilon_0$. Therefore the limit of $f(\varphi^t(x))$ as $t \to \infty$ is $-\infty$, which is impossible by the compactness of M. $\square$

2.3 Critical values $\rightsquigarrow$ topology

The two important theorems of this section establish connections between critical values and topology. First, in the case where no critical points are crossed, the topology is unchanged. Second, in the case where a critical point of index k is crossed, the topology changes by the attachment of a k-cell.

Definition 2.3.1. Let M be a manifold, and $f : M \to \mathbb{R}$. It is well known that if a is a regular value of f, then the level set $f^{-1}(a)$ is an embedded submanifold. The same holds for sublevel sets: define $M^a = f^{-1}((-\infty, a])$. This is a submanifold with boundary. Similarly superlevel sets are denoted $\overline{M}^a$.

Theorem 2.3.2. *Let $f : M \to \mathbb{R}$. Suppose $a, b \in \mathbb{R}$, $f^{-1}([a, b])$ is compact, and f has no critical points in $f^{-1}([a, b])$. Then M^a is diffeomorphic to M^b.*

Proof. We give a proof outline. The idea is to flow along a pseudo-gradient to retract M^b to M^a. Consider a function $\rho : M \to \mathbb{R}$ satisfying

$$\rho(x) = \begin{cases} -\frac{1}{(df)_x(X)} & x \in f^{-1}([a, b]) \\ 0 & \text{outside of a compact neighbourhood of } f^{-1}([a, b]). \end{cases}$$

Let Y be the vector field ρX, and ψ^t the flow of Y. Then for each t, $\psi^t : M \to M$ is a diffeomorphism, and in particular ψ^{b-a} maps M^b onto M^a. $\square$

Corollary 2.3.3 (Reeb's theorem)**.** *Suppose a closed manifold M admits a Morse function with exactly two critical points. Then M is homeomorphic to a sphere.*

Proof. Let $f : M \to \mathbb{R}$ be a Morse function with two critical points, with M an n-manifold. Since M is compact, $\operatorname{im} f = [a, b]$ for some $a < b$. Then $f^{-1}(a)$ is necessarily a maximum and $f^{-1}(b)$ is a minimum. By the Morse lemma, for ε sufficiently small, $M^{a+\varepsilon} = f^{-1}([a, a+\varepsilon])$ and $\overline{M}^{b-\varepsilon} = f^{-1}([b-\varepsilon, b])$ are diffeomorphic to n-disks. By the previous theorem, $M^{a+\varepsilon}$ is diffeomorphic to $M^{b-\varepsilon}$. But now M is equal to $M^{b-\varepsilon} \cup \overline{M}^{b-\varepsilon}$, i.e. two n-disks glued along their boundary. This is homeomorphic to an n-sphere. □

Remark. The above result is still true if f is not Morse, that is, if the critical points are degenerate. Note also that the final conclusion is only true up to homeomorphism, since gluing two disks can result in exotic spheres.

In the above we discussed the special case of traversing to different sublevel sets without crossing any critical points. Next we investigate the case when we cross a critical point (of index k).

Theorem 2.3.4. *Let $f : M \to \mathbb{R}$. Suppose $a, b \in \mathbb{R}$, $f^{-1}([a, b])$ is compact, and f has exactly one critical point α in $f^{-1}([a, b])$, of index k. Then M^b is homotopy equivalent to M^a with a k-cell attached. (More explicitly, M^b is homotopic to $M^a \cup W^u(\alpha)$.)*

Rather than giving a proof, we give two examples (as in figure 2.1): In this figure,

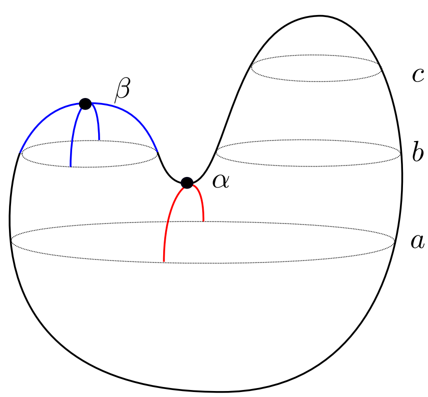

Figure 2.1: Examples of changing topologies of level sets.

M^a has the homotopy type of a point, and M^b has the homotopy type of a circle. We also observe that α is the unique critical point in $f^{-1}([a, b])$, and has an index of 1. The corresponding unstable manifold is shown in the figure, and are the two curves following a downward path from α shown in red. Thus $M^a \cup W^u(\alpha)$ is "M^a with a one-dimensional handle attached", and has the homotopy type of a circle as required.

Next observe that M^c once again has the homotopy type of a point. There is a unique critical point β lying between b and c. This has index 2, and we see that the corresponding unstable manifold is a disk lying below β, shown in blue. Thus $M^b \cup W^u(\beta)$ is a "cylinder with one end capped", and hence has the homotopy type of a point as required.

We now give closer attention to the stable and unstable manifolds, exploring the *Smale condition.*

2.4 Smale condition

Definition 2.4.1. Let $f : M \to \mathbb{R}$ be Morse. A pseudo-gradient adapted to f satisfies the *Smale condition* if all stable and unstable manifolds of f meet transversely, that is, for any a, b critical, $W^u(a) \pitchfork W^s(b)$.

We later find that the Smale condition ensures some combinatorial properties that tell us how to compare the index of distinct critical points. But first, some examples:

Example. The critical points of figure 2.1 consist of three extrema and one saddle. The unstable and stable manifolds of extrema are either n-dimensional submanifolds or points. Therefore whenever a stable or unstable manifold of an extremum intersects an unstable or stable manifold of another critical point, we find that the manifold corresponding to the extremum is n-dimensional, so the manifolds are transverse. But the remaining cases are intersections of stable and unstable manifolds of a fixed critical point. These always meet transversely, so this shows that the "wobbly sphere" (figure 2.1) satisfies the Smale condition.

Example. Two special cases (one of which was used explicitly above):

- By the Morse lemma, given any critical point c, $W^s(c) \pitchfork W^u(c)$.
- By the definition of pseudo-gradients, the unstable manifold always lies below a critical point, and the stable manifold above. Therefore whenever a, b are distinct critical points with $f(a) \leq f(b)$, then $W^s(b) \cap W^u(a)$ is empty. In particular, they are transverse.

Recall that for any critical point c, $\dim W^u(c) = \operatorname{codim} W^s(c) = \operatorname{ind}(c)$. But now If a, b are any two critical points of a pseudo-gradient satisfying the Smale condition, then $W^s(a) \pitchfork W^u(b)$, so

$$\begin{aligned}\dim(W^s(a) \cap W^u(b)) &= n - \operatorname{codim}(W^s(a) \cap W^u(b)) \\ &= n - (\operatorname{codim} W^s(a) + \operatorname{codim} W^u(b)) \\ &= n - (\operatorname{ind}(a) + (n - \operatorname{ind}(b))) = \operatorname{ind}(b) - \operatorname{ind}(a).\end{aligned}$$

Therefore the differences in indices is exactly the dimension of some submanifold of M. But what is this submanifold? It consists exactly of the trajectories of the pseudo-gradient connecting b to a!

$$\mathcal{M}(b,a) \coloneqq W^s(a) \cap W^u(b) = \{x \in M : \lim_{t\to\infty} \varphi^t(x) = a, \lim_{t\to-\infty} \varphi^t(x) = b\}.$$

In particular, if $\mathcal{M}(b,a)$ is non-empty, it consists of at least one trajectory and has dimension at least one. Therefore *indices of critical points always decrease along trajectories.*

Theorem 2.4.2 (Kupka-Smale theorem)**.** *Let M be a manifold (possibly with boundary). Let f be a Morse function on M with distinct critical values. Fix Morse charts about each critical point, and denote their union by Ω. Let X be a pseudo-gradient adapted to f, transverse to ∂M. Then there is a pseudo-gradient X' satisfying the Smale condition, arbitrarily close to X (in C^1-norm), and equal to X on Ω.*

Remark. All approximations in this section use the C^1-norm. We hereafter say that f *is approximated by* g to mean there exist arbitrarily good C^1 approximations g of f.

Remark. 1. Any Morse function $f : M \to \mathbb{R}$ can be approximated by Morse functions $\widetilde{f}$ with distinct critical values. Explicitly, perturb f by an appropriate function h which is constant on Morse charts, and has sufficiently small $|dh|$.

2. It is not true in general that every Morse function on a manifold with boundary has a pseudo-gradient transverse to the boundary. However, one can start by defining a vector field transverse to the boundary, and then define a Morse function for which an extension of this vector field is a pseudo-gradient.

Therefore the Smale theorem proves that on compact manifolds pairs (f, X) such that f is a Morse function whose critical points take distinct values and X is a pseudo-gradient adapted to f satisfying the Smale-condition exist and are generic.

Proof. A proof of the Smale theorem can be found in Audin and Damian. $\square$

Chapter 3

Morse homology fundamentals

3.1 Morse homology modulo 2

In the first chapter we established that given a compact manifold, Morse functions exist and are generic. In the second chapter we established moreover that pairs (f, X) where f is Morse and X is a pseudo-gradient adapted to f satisfying the Smale condition exist and are generic. Such (f, X) are said to be *Morse-Smale*.

Let M be a compact manifold, and (f, X) Morse-Smale on M. In this chapter we define the *Morse complex* on M using (f, X). We then show that the Morse complex is independent of the choice of (f, X), so it is an invariant of M. Finally we show that Morse homology is isomorphic to Singular homology.

To this end, we start by defining an appropriate space of coefficients via the quotient manifold theorem.

Proposition 3.1.1. Let G be a Lie group acting smoothly, freely, and properly on a smooth manifold M. Then M/G is a topological manifold of dimension $\dim M - \dim G$, with a unique smooth structure such that $\pi : M \to M/G$ is a smooth submersion.

Given critical points a, b of a Morse function f, we defined $\mathcal{M}(b, a)$ to be the collection of points lying on trajectories from b to a. Recall that $\mathcal{M}(b, a)$ is an $\mathrm{ind}(b) - \mathrm{ind}(a)$ dimensional submanifold of M. The Lie group $\mathbb{R}$ acts on $\mathcal{M}(b, a)$ by translations in time:

$$t \cdot x = \varphi^t(x).$$

The action is smooth since φ^t is smooth. In fact, φ^t is a diffeomorphism for any fixed t, so the action is also proper. To apply the quotient-manifold theorem, it remains to verify that the translation action is free. This follows from the fact that $\mathcal{M}(b, a)$ contains no critical points, so $f(\varphi^t(x))$ is a strictly decreasing function of t. This shows that the quotient manifold theorem applies, giving the following definition:

Definition 3.1.2. Let a, b be critical points of f. Then $\mathcal{L}(b,a) := \mathcal{M}(b,a)/t$ is the space of trajectories from b to a. By the quotient manifold theorem, $\mathcal{L}(b,a)$ is a smooth manifold of dimension $\operatorname{ind}(b) - \operatorname{ind}(a) - 1$.

Let M be compact, equipped with a Morse-Smale pair (f, X). For any i, let c_i denote a critical point of index i. For integral Morse homology, we use the signed cardinalities $N_X(c_{i+1}, c_i) \in \mathbb{Z}$ of $\mathcal{L}(c_{i+1}, c_i)$ as coefficients. For easier calculation ignoring orientation, we consider coefficients in $\mathbb{Z}/2\mathbb{Z}$. In other words, we settle for the cardinalities computed modulo 2, denoted $n_X(c_{i+1}, c_i) \in \mathbb{Z}/2\mathbb{Z}$.

Remark. By the previous definition, $\mathcal{L}(c_{i+1}, c_i)$ is a 0-dimensional manifold for any i. For the following definitions to be well defined, we require that $\mathcal{L}(c_{i+1}, c_i)$ is always finite (equivalently, compact). This is indeed true, and will be shown in a subsequent section.

Definition 3.1.3. For each k, let $\operatorname{Crit}_k(f)$ denote the set of critical points of f of index k. For any ring R, $C_k(f, R)$ denotes the free R-module of formal sums

$$C_k(f, R) := \Big\{ \sum_{c \in \operatorname{Crit}_k(f)} a_c c : a_c \in R \Big\}.$$

The vector spaces $C_k(f, \mathbb{Z}/2\mathbb{Z})$ will be the terms appearing in the mod 2 Morse complex.

Definition 3.1.4. Given any k, the *boundary map* $\partial = \partial_{k+1} : C_{k+1}(f, \mathbb{Z}/2\mathbb{Z}) \to C_k(f, \mathbb{Z}/2\mathbb{Z})$ is defined on critical points c_{k+1} by

$$\partial(c_{k+1}) = \sum_{c_k \in \operatorname{Crit}_k(f)} n_X(c_{k+1}, c_k) c_k.$$

This uniquely extends to a linear map on $C_{k+1}(f, \mathbb{Z}/2\mathbb{Z})$.

Definition 3.1.5. The *Morse complex* is defined to be the chain complex

$$\cdots \to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \to \cdots.$$

Remark. Well-definedness of the Morse complex now rests on two results, both of which are shown in a following section. Namely,

1. The boundary maps are well defined, i.e. $\mathcal{L}(c_{k+1}, c_k)$ is finite for each c_{k+1}, c_k.
2. The complex is truly a complex, i.e. $\partial^2 = 0$.

We conclude this section by exploring some examples.

Example. Spheres: the usual height function, and the height function on the wobbly sphere (as seen in figure 2.1).

We start by computing the Morse complex and corresponding homologies for the height function h on the usual sphere, $\mathbb{S}^n$, with $n \geq 2$. This has exactly two critical points, one of index 0 and one of index n. The Morse complex is then

$$\cdots \to 0 \to C_n(h, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots \to 0 \to C_0(h, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots .$$

Each boundary map is necessarily the zero-map, forcing the Morse homologies to be

$$H_k(h, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k \in \{0, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Next we consider the 2-sphere equipped with the Morse function f corresponding to the height function of the wobbly sphere, figure 3.1. This has four critical points, one of index 0 (a), one of index 1 (b), and two of index 2 (c, d). Therefore the Morse complex is

$$C_\bullet(f, \mathbb{Z}/2\mathbb{Z}) = \ \cdots \to 0 \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots .$$

Inspecting the diagram, the boundary map $\partial_1 : C_1(f, \mathbb{Z}/2\mathbb{Z}) \to C_0(f, \mathbb{Z}/2\mathbb{Z})$ sends b to 0 (since there are two trajectories, so $n_X(b, a) = 0$.) It follows that ∂_1 is the zero map. Next we observe from the diagram that $n_X(c, b) = n_X(d, b) = 1$. It follows that ∂_2 is surjective. Thus

$$\operatorname{im} \partial_1 = 0, \ker \partial_1 = \mathbb{Z}/2\mathbb{Z}, \quad \operatorname{im} \partial_2 = \mathbb{Z}/2\mathbb{Z}, \ker \partial_2 = \mathbb{Z}/2\mathbb{Z}.$$

Computing the Morse homology, we find that

$$H_k(f, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k \in \{0, n\} \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the Morse homologies of $\mathbb{S}^2$ calculated using f and h agree. In fact, we later show that the Morse homology is independent of the choice of Morse-Smale pair (f, X).

Example. A tilted torus and a tilted Klein bottle. These examples are notable, since $(\mathbb{T}^2, h)$ (where h is the usual height function) does not canonically give a Morse-Smale pair (h, X); the two index 1 critical points are joined by two trajectories, which is forbidden by the Smale condition. Therefore the torus (and the Klein bottle) must be tilted slightly, as shown in figures 3.2, 3.3. Let h denoted the tilted height function of the torus. By inspecting the figure, h has one critical point of index 0 (a), two of index 1 (b, c), and one of index 2 (d). Therefore the Morse complex is

$$C_\bullet(h, \mathbb{Z}/2\mathbb{Z}) = \ \cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots .$$

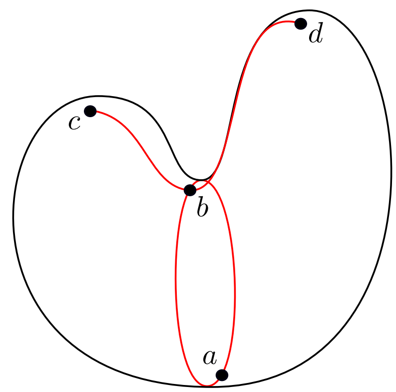

Figure 3.1: Wobbly sphere Morse complex.

From figure 3.2, we see that $n_X(d,c) = n_X(d,b) = 0$ since there are two trajectories in each case (shown in red). Therefore ∂_2 is the zero map. Similarly $n_X(c,a) = n_X(b,a) = 0$, so ∂_1 is also the zero map. It follows that the Morse homology is given by

$$H_k(h, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & k = 1 \\ \mathbb{Z}/2\mathbb{Z} & k \in \{0,2\} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that this agrees with the singular homology of the torus.

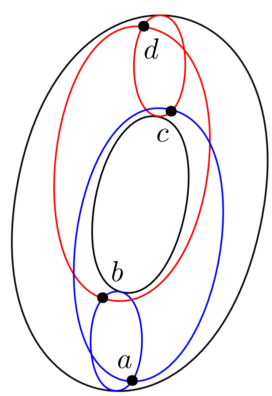

Figure 3.2: Tilted torus Morse complex.

Next we compute the Morse homology of the tilted Klein bottle, shown in figure 3.3. Let h' denote the tilted height function. We find that the Klein bottle has the same critical-point profile as the tilted torus: one critical point of index 0 (a'), two of index 1 (b', c'), and one of index 2 (d'). Therefore the Morse complex has the same objects as in the case of the torus;

$$C_\bullet(h', \mathbb{Z}/2\mathbb{Z}) = \ \cdots \to 0 \to \mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots .$$

In fact, counting trajectories modulo 2, we find that both ∂_1 and ∂_2 vanish in $C_\bullet(h', \mathbb{Z}/2\mathbb{Z})$. Therefore

$$H_k(h', \mathbb{Z}/2\mathbb{Z}) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & k = 1 \\ \mathbb{Z}/2\mathbb{Z} & k \in \{0, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

This shows that Morse homology (mod 2) cannot distinguish a Klein bottle from a torus. Of course, this is expected, since Singular homology mod 2 also does not distinguish the two.

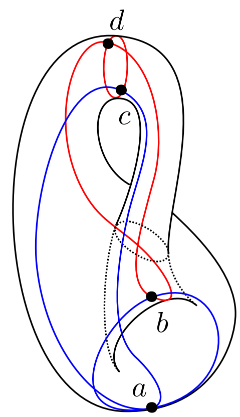

Figure 3.3: Tilted Klein bottle Morse complex.

As a corollary of the above examples, we have shown that a sphere is not diffeomorphic to a torus or a Klein bottle, but we still haven't shown that a torus is not diffeomorphic to a Klein bottle. For this we use Morse homology over the integers.

3.2 Integral Morse homology

Next we define Morse homology with coefficients in $\mathbb{Z}$. The main technicality is keeping track of orientations (signs). In fact, the mod 2 Morse homology in the previous section section follows exactly from the construction we carry out here, since signed counting is not detected modulo 2. Again, proofs of well-definedness are pushed into a subsequent section, here we just define the complex and look at some examples.

As defined in the previous section, the objects in the integral Morse complex are

$$C_k(f,\mathbb{Z}) = \Big\{ \sum_{c\in \mathrm{Crit}_k(f)} a_c c : a_c \in \mathbb{Z} \Big\}.$$

The corresponding boundary maps are defined to be $\partial_{k+1} : C_{k+1}(f,\mathbb{Z}) \to C_k(f,\mathbb{Z})$,

$$\partial(c_{k+1}) = \sum_{c_k\in \mathrm{Crit}_k(f)} N_X(c_{k+1},c_k)c_k,$$

where $N_X(c_{k+1},c_k)$ is the signed count of trajectories from c_{k+1} to c_k. The integral Morse complex is

$$\cdots \to C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \to \cdots .$$

This is a well defined chain complex for the same reason that the mod 2 Morse complex is a chain complex, which we soon show.

We now describe the process of signed counting, and compute some examples. The aim is to induce orientations on each $\mathcal{L}(c_{k+1},c_k)$. Since these are zero dimensional manifolds, orientations are exactly choices of sign for each point.

Start by choosing an orientation for each stable manifold $W^s(c)$. These are homeomorphic to disks (of some dimension), so they are orientable. Choose any $x \in \mathcal{M}(c_{k+1},c_k)$. There is a short exact sequence

$$0 \to T_x\mathcal{M}(c_{k+1},c_k) \to T_xW^s(c_k) \to N_xW^u(c_{k+1}) \to 0.$$

But $N_xW^u(c_{k+1})$ is canonically isomorphic to $T_xW^s(c_{k+1})$, giving a short exact sequence

$$0 \to T_x\mathcal{M}(c_{k+1},c_k) \to T_xW^s(c_k) \to T_xW^s(c_{k+1}) \to 0.$$

Since the middle and right term are oriented, there is an induced orientation on $T_x\mathcal{M}(c_{k+1},c_k)$. But we also have a short exact sequence

$$0 \to \mathbb{R} \to T_x\mathcal{M}(c_{k+1},c_k) \to T_x\mathcal{L}(c_{k+1},c_k) \to 0,$$

where $\mathbb{R}$ is oriented by time. This induces an orientation on $T_x\mathcal{L}(c_{k+1},c_k)$ as required.

Remark. Although choices are being made when orienting the stable manifolds, reversing the orientation of a given stable manifold $W^s(c)$ corresponds to multiplying $N_X(d,c)$ and $N_X(c,b)$ by -1, for any d and b. Therefore the isomorphism classes of the integral Morse homologies are independent of the choice of orientation.

Example. Integral Morse homology of a torus. Let h denoted the tilted height function of the torus. By inspecting figure 3.4, h has one critical point of index 0 (a), two of index 1 (b,c), and one of index 2 (d). Therefore the Morse complex is

$$C_\bullet(h,\mathbb{Z}) = \ \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \to \cdots .$$

Next we determine the boundary maps. The arrows on figure 3.4 denote the chosen orientations of stable manifolds. The stable manifold of d is a point, so it assigned the orientation $+$. Working through the exact sequences above, we find that the two trajectories in $\mathcal{L}(d, c)$ have orientations $+$ and $-$, so $N_X(d, c) = 0$. It turns out that all of the N_X vanish (with trajectory orientations as shown in the figure), so every boundary map is trivial. Therefore the integral Morse homology of the torus is equal to

$$H_k(h, \mathbb{Z}) = \begin{cases} \mathbb{Z}^2 & k = 1 \\ \mathbb{Z} & k \in \{0, 2\} \\ 0 & \text{otherwise.} \end{cases}$$

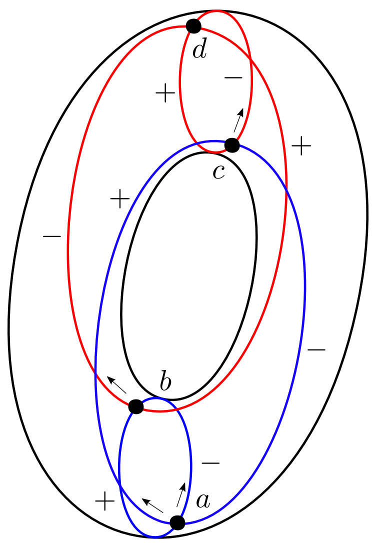

Figure 3.4: Integral (signed) torus Morse complex.

Example. Integral Morse homology of a Klein bottle. This is again similar to the torus. Let h denoted the height function of the Klein bottle. By inspecting figure 3.5, h has one critical point of index 0 (a), two of index 1 (b, c), and one of index 2 (d). (These are not labelled on the figure to prevent clutter; the labels $a, \ldots, d$ are in height-ascending order.) The Morse complex is

$$C_\bullet(h, \mathbb{Z}) = \ \cdots \to 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \to \cdots .$$

Next we determine the boundary maps. For the most part, the signed counting ends up being identical to that of the torus. The only difference is the trajectories from d to b, which both end up having negative sign. One can verify that a "tubular neighbourhood" of the two trajectories from d to b is an embedded Mobius strip. All signs are shown in figure 3.5, from which we conclude that ∂_1 is trivial, and ∂_2 is defined by

$$\partial_1(xd) = -2xb$$

for all $xd \in C_2(h, \mathbb{Z})$. It follows that $\operatorname{im} \partial_2 \cong 2\mathbb{Z}$, and $\ker \partial_2$ is trivial. Therefore the integral Morse homology of the Klein bottle is equal to

$$H_k(h, \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & k = 1 \\ \mathbb{Z} & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

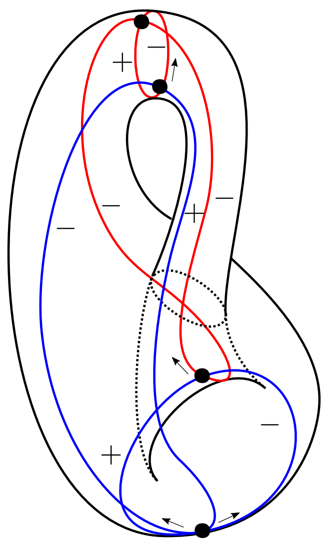

Figure 3.5: Integral (signed) Klein bottle Morse complex.

In summary, while mod 2 Morse homology could not detect orientation, the integral Morse homology can distinguish a Klein bottle from the torus. As a corollary, a torus and Klein bottle are not diffeomorphic. Observe that again the Morse and singular homologies agree.

3.3 Well-definedness of the Morse complex

The goals of this section are to show that $\mathcal{L}(b, a)$ is finite (when $\operatorname{ind}(b) = \operatorname{ind}(a) + 1$), and that the Morse complex is truly a complex. To achieve this, we construct the notion of *broken trajectories*. Recall that for any critical points a, b of a Morse function, $\mathcal{L}(b, a)$ consists of the trajectories from b to a, and is a manifold of dimension $\mathrm{ind}(b) - \mathrm{ind}(a) - 1$. The space of *broken trajectories* from b to a is a certain compactification of $\mathcal{L}(b, a)$:

Definition 3.3.1. Let a, b be critical points of a Morse function f. Then

$$\overline{\mathcal{L}}(b, a) := \bigcup_{c_i \in \mathrm{Crit}(f)} \mathcal{L}(b, c_1) \times \cdots \times \mathcal{L}(c_q, a)$$

is the space of *broken trajectories* from b to a.

Each factor $\mathcal{L}(x, y)$ is endowed with the quotient of the subspace topology, and each term in the union is equipped with the product topology. However, to make sense of the union, we must define an appropriate topology on the whole space. The above definition can be motivated by visualising the trajectories on a torus, and observing that if a and b respectively denote the minimum and maximum (of the usual height function), then $\mathcal{L}(b, a)$ is a disjoint union of four open intervals. However, $\overline{\mathcal{L}}(b, a)$ should be a figure-eight.

A description of the topology of $\overline{\mathcal{L}}(b, a)$ is given at the start of section 3.2 in Audin and Damian. They describe a neighbourhood system as follows, to define the topology:

1. Let $\lambda = (\lambda_1, \ldots, \lambda_q) \in \overline{\mathcal{L}}(b, a)$ be a broken trajectory. We will define a neighbourhood $\mathcal{W}(\lambda, U^-, U^+)$ of λ.

2. Each λ_i is a trajectory which exists some Morse neighbourhood $\Omega(c_{i-1})$ and enters $\Omega(c_i)$. More specifically, the exit point x_i of λ_i in $\Omega(c_{i-1})$ has a neighbourhood U_i^- contained in $\Omega(c_{i-1}) \cap f^{-1}(x_i)$. Similarly, there are neighbourhoods U_i^+ of entry points. Let U^- denote the family of U_i^-, and similarly for U^+.

3. The collection of $\mathcal{W}(\lambda, U^-, U^+)$ now defines a neighbourhood system by declaring that $\eta = (\eta_1, \ldots, \eta_p) \in \mathcal{W}(\lambda, U^-, U^+)$ whenever
   (a) the η_j belong to $\mathcal{L}(c_{i_j}, c_{i_{j+1}})$, where c_{i_j} is a subsequence of the critical points occurring in λ, and
   (b) each η_j exists from within the corresponding neighbourhood in U^-, and enters the corresponding neighbourhood in U^+.

With this topology, $\mathcal{L}(b, a)$ is a subspace of $\overline{\mathcal{L}}(b, a)$. Moreover, the following key result holds:

Theorem 3.3.2. *The space $\overline{\mathcal{L}}(b, a)$ of broken trajectories is compact.*

Proof. We give a proof outline. Observe that the neighbourhood system given above contains a countable neighbourhood system, using compactness and second-countability of the ambient manifold M. Therefore to prove compactness it suffices to prove sequential compactness.

A further reduction can be made since a sequence in a finite product converges if and only if it converges pointwise. Therefore it suffices to prove that any sequence in $\mathcal{L}(b,a)$ has a convergence subsequence. Let (ℓ_n) be a sequence of trajectories in $\mathcal{L}(b,a)$. Each ℓ_n exits $\Omega(b)$ at some ℓ_n^-, and enters $\Omega(a)$ at some ℓ_n^+. These form sequences in compact subsets of M, so they have convergent subsequences. Recalling that a trajectory is a solution of a differential equation, and these are unique given initial conditions, we conclude that ℓ_n has a convergent subsequence. $\square$

This result completes one of the goals of this section:

Proposition 3.3.3. Suppose b and a are critical points with $\mathrm{ind}(b) = \mathrm{ind}(a) + 1$. Then $\mathcal{L}(b,a)$ is finite. In particular, the boundary maps defined for the mod 2 and integral Morse complex are well defined.

Proof. Given the premise, $\mathcal{L}(b,a)$ is a 0 dimensional manifold. Therefore it suffices to prove compactness of $\mathcal{L}(b,a)$. But this follows from the general fact that $\overline{\mathcal{L}}(b,a)$ is compact. $\square$

Our next goal is proving that the Morse complex is indeed a complex. Specifically, it remains to show that $\partial^2 = 0$. We give an outline, the details of which are in Audin and Damian. The key result that must be proven is the following:

Theorem 3.3.4. *Let a, b be critical points of f with $\mathrm{ind}(b) - \mathrm{ind}(a) = 2$. Then $\overline{\mathcal{L}}(b,a)$ is a compact one dimensional manifold with boundary.*

To see why this gives the desired result, fix a, b as above. We prove that $\partial^2 b = 0$ for both integral Morse homology and mod 2 Morse homology.

Let $i+2$ be the index of b. By definition, we have

$$\begin{aligned}
\partial b &= \sum_{c_{i+1} \in \mathrm{Crit}_{i+1}(f)} N_X(b, c_{i+1}) c_{i+1}, \\
\partial^2 b &= \sum_{c_i \in \mathrm{Crit}_i(f)} N_X(\partial b, c_i) c_i = \sum_{\substack{c_i \in \mathrm{Crit}_i(f) \\ c_{i+1} \in \mathrm{Crit}_{i+1}(f)}} N_X(b, c_{i+1}) N_X(c_{i+1}, c_i) c_i.
\end{aligned}$$

To show that $\partial^2 = 0$, it suffices to show that $\sum_{c_i} N_X(b, c_{i+1}) N_X(c_{i+1}, a)$ is zero. On the other hand, using the previous theorem we know that

$$\overline{\mathcal{L}}(b,a) = \mathcal{L}(b,a) \sqcup \partial\overline{\mathcal{L}}(b,a) = \mathcal{L}(b,a) \sqcup \bigcup_{c_i} \mathcal{L}(b, c_i)\mathcal{L}(c_i, a).$$

Therefore the expression $\sum_{c_i} N_X(b, c_{i+1}) N_X(c_{i+1}, a)$ is in fact the cardinality of $\partial\overline{\mathcal{L}}(b,a)$; i.e. the cardinality of the boundary of a compact one dimensional manifold with boundary. By the classification of one dimensional manifolds with boundary, this is cardinality is always 0 modulo 2. Similarly when the manifold is oriented, the signed count of boundary points is always 0. Therefore $\sum_{c_i} N_X(b, c_{i+1}) N_X(c_{i+1}, a) = 0$. Since a, b were arbitrary, it follows that $\partial^2 = 0$. Therefore the Morse complex is truly a complex, as required.

We now explore the key theorem:

Proof. We give a proof outline that $\overline{\mathcal{L}}(b, a)$ is a compact one dimensional manifold with boundary, provided $\operatorname{ind}(b) - \operatorname{ind}(a) = 2$. We already know that $\mathcal{L}(b, a)$ is a one dimensional manifold, and we know that $\overline{\mathcal{L}}(b, a)$ is a compact (metrisable) topological space. Therefore it is sufficient to prove the following result:

Let M be compact, and (f, X) a Morse-Smale pair on M. Fix k and let b, c, a be critical points of index $k+2, k+1$, and k respectively. Let λ_1, λ_2 be trajectories from b to c and c to a respectively. Then there exists a continuous embedding ψ from $[0, \delta)$ onto a neighbourhood of (λ_1, λ_2) in $\overline{\mathcal{L}}(b, a)$ that is differentiable on $(0, \delta)$, and satisfies $\psi(0) = (\lambda_1, \lambda_2), \psi(s) \in \mathcal{L}(b, a)$ for $s \neq 0$. Moreover, if (ℓ_n) is a sequence in $\mathcal{L}(b, a)$ that tends to (λ_1, λ_2), then ℓ_n is contained in the image of ψ for sufficiently large n.

Proving the above result turns out to be fairly technical, but all of the details are given in Audin and Damian. □

3.4 Morse-Smale pair invariance of the Morse homology

Earlier in the chapter we computed the Morse homology of a 2-sphere using the standard height function as well as the wobbly height function. In each case we observed that the Morse homologies were unchanged! This is in fact a general result: the Morse homology does not depend on the choice of Morse function or pseudo-gradient field. That is, the Morse homology depends only on the smooth manifold.

In an ideal world we could interpolate between two Morse functions with Morse functions: given $f_0, f_1 : M \to \mathbb{R}$, we can define a homotopy $F : M \times I \to \mathbb{R}$ from f_0 and f_1. However, F is not generally Morse, if the number of critical points changes. The idea is that we can actually bypass this issue, provided we can construct a morphism of complexes inducing isomorphisms on homology which do not refer to the degenerate points. We describe a proof outline, but full details are given in Audin and Damian.

Theorem 3.4.1. *Let M be compact, and $(f_0, X_0), (f_1, X_1)$ two Morse-Smale pairs on M. Then there exists a morphism of complexes $\Phi_* : C_\bullet(f_0, X_0) \to C_\bullet(f_1, X_1)$ which induces isomorphisms on homology.*

Proof. We somewhat categorify the proof. Let F be any interpolation between f_0 and f_1 that is constant on $[0, 1/3] \cup [2/3, 1]$. More explicitly, suppose $F : M \times [0, 1] \to \mathbb{R}$ is a smooth function such that

$$F_t|_{[0,1/3]} \equiv f_0, \quad F_t|_{[2/3,1]} \equiv f_1.$$

We call such an F an *end-constant interpolation.* Let $\mathbf{EndConstInt}(M)$ denote the category of Morse-Smale pairs on M, where morphisms between Morse-Smale pairs are equivalence classes of end-constant interpolations. (Two end-constant interpolations are equivalent if they take the same constant values, i.e. if they have the same domain and codomain.) Composition of morphisms is given by

$$G \circ F = \begin{cases} F_t & t \in [0, 1/3] \\ G_t & t \in [2/3, 1], \end{cases}$$

and given any f, the identity morphism is given by $F_t \equiv f$. One can readily verify that $\mathbf{EndConstInt}(M)$ is a category.

On the other hand, there is also a category of Morse complexes, which we denote $\mathbf{MoCplx}(M)$. The objects are $C_\bullet(f, X)$ for (f, X) a Morse-Smale pair on M, and morphisms are chain maps.

Suppose there is a functor $\Phi : \mathbf{EndConstInt}(M) \to \mathbf{MoCplx}(M)$, and let F be an end-constant interpolation between (f_0, X_0) and (f_1, X_1), and G an end-constant interpolation between (f_1, X_1) and (f_0, X_0). Then $\Phi^F \circ \Phi^G$ and $\Phi^G \circ \Phi^F$ must both be the identity maps on their respective complexes, so Φ^F induces an isomorphism on homologies. This shows that to prove the theorem, it suffices to find a functor $\Phi : \mathbf{EndConstInt}(M) \to \mathbf{MoCplx}(M)$.

This process is broken into two parts:

1. For a given end-constant interpolation F, define $\Phi^F : C_\bullet(f_0) \to C_\bullet(f_1)$. Show that Φ^F depends only on the equivalence class of F.
2. Verify that the induced morphism of complexes is functorial:
   (a) Show that if I is an identity morphism in $\mathbf{EndConstInt}$, then $\Phi^I = \mathrm{id}$.
   (b) Show that for morphisms F, G, H in $\mathbf{EndConstInt}$ (with compatible domains and codomains), $\Phi^G \circ \Phi^F = \Phi^H$.

We describe the construction of Φ, but skip the proof of 2 which verifies that our choice of Φ is indeed a functor.

1. Let $F : M \times I \to \mathbb{R}$ be an end constant interpolation from f_0 to f_1. Extend F to $[-1/3, 4/3]$ by keeping the ends constant. Although F is not in general Morse, we define a new function using F which is guaranteed to be Morse. Specifically, choose $g : \mathbb{R} \to \mathbb{R}$ to be Morse, with two critical points 0 and 1 (the maximum and minimum respectively), such that g is decreasing sufficiently rapidly between 0 and 1. Precisely, we require

$$\frac{\partial F}{\partial t}(x, t) + g'(t) < 0$$

for all $x \in M$ and $t \in (0, 1)$. This can be achieved by compactness of M, by allowing the critical value of 0 to be very large. Then $F + g : M \times [-1/3, 4/3] \to \mathbb{R}$ is Morse, with critical points exactly being

$$\mathrm{Crit}(f_0) \times \{0\} \cup \mathrm{Crit}(f_1) \times \{1\}.$$

Moreover, the critical points $(b, 0)$ have index $\mathrm{ind}(b)+1$, and the critical points $(a, 1)$ have index $\mathrm{ind}(a)$. Using a partition of unity, there exists a pseudo-gradient field X adapted to $F + g$ which coincides with $X_0 - \mathrm{grad}\, g$ on $M \times [-1/3, 1/3]$, and with $X_1 - \mathrm{grad}\, g$ on $M \times [2/3, 4/3]$. $(F + g, X)$ is not necessarily Morse-Smale, but by genericness of Morse-Smale pairs, there is an approximation $\widetilde{X}$ of X which is Morse-Smale. Since $(F + g, X)$ restricted to $M \times [-1/3, 1/3]$ or $M \times [2/3, 4/3]$ is indeed Morse-Smale, in these instances the small-perturbation-invariance of the Morse complex gives the following:

$$C_\bullet(F + g|_{M\times[-1/3,1/3]}, \widetilde{X}) = C_{\bullet+1}(f_0, X_0)$$
$$C_\bullet(F + g|_{M\times[2/3,4/3]}, \widetilde{X}) = C_\bullet(f_1, X_1).$$

This gives a decomposition $C_{k+1}(F + g, \widetilde{X}) = C_k(f_0, X_0) \oplus C_{k+1}(f_1, X_1)$. But now the boundary map $\partial_{\widetilde{X}}$ decomposes as

$$\partial_{\widetilde{X}} = \begin{pmatrix} \partial_{X_0} & 0 \\ \Phi^F & \partial_{X_1} \end{pmatrix}.$$

One can show that Φ^F is the desired chain map, and moreover $F \mapsto \Phi^F$ is a functor. $\square$

3.5 Morse homology is singular homology

The most standard proof of the Morse-Smale pair invariance of Morse homology is to simply show that the Morse homology is canonically isomorphic to the cellular homology (and hence singular homology). This proves not only that Morse homology is independent of the Morse-Smale pair used to define it, but further that Morse homology depends only on the topological structure of the manifold, and not its smooth structure.

Theorem 3.5.1. *Let M be a manifold, and (f, X) a Morse-Smale pair on M. Let $C_\bullet(M)$ denote the associated Morse complex. There is a cellular decomposition of M (with associated cellular complex $K_\bullet(M)$), and an isomorphism*

$$F : K_\bullet(M) \to C_\bullet(M).$$

That is, a map which is an isomorphism in each degree, with $F \circ \partial = \partial_X \circ F$.

It follows that the Morse and singular homologies of a manifold are isomorphic. In the above it was not clarified whether the homology was integral, mod 2, or something else - this does now matter as the theorem holds for any coefficient ring.

An amazingly brief overview of the proof is as follows:

1. Show that the Morse-Smale pair (f, X) induces a cellular decomposition of M; the cells are the unstable manifolds of each critical point.

2. Show that the corresponding complexes are isomorphic.

A full proof is given in Audin and Damian. Here we compute two examples: the usual 2-sphere and the wobbly 2-sphere. These examples should show that the isomorphism of complexes is not difficult to prove, and the real difficulty lies in proving that a Morse-Smale pair truly induces a cellular decomposition.

Example. The usual 2-sphere equipped with the height function. The height function has two critical points, a of index 0 and b of index 2. The corresponding unstable manifolds are a 0-cell $W^u(a)$, and a 2-cell $W^u(b)$. The cellular decomposition has no 1-cells, so the cellular complex is

$$\cdots \to 0 \to G \to 0 \to G \to 0 \to \cdots$$

where G is the coefficient ring. The boundary maps are all automatically trivial, so the cellular homology is

$$H_k^{\text{Cell}}(\mathbb{S}^2, G) = \left\{ \begin{matrix} G & \text{if } k \in \{0, 2\}, \\ 0 & \text{otherwise.} \end{matrix} \right\} = H_k^{\text{Morse}}(\mathbb{S}^2, G)$$

as required.

Example. The wobbly 2-sphere equipped with the height function. The height function has four critical points, a of index 0, b of index 1, and c and d of index 2. The corresponding unstable manifolds are a 0-cell $W^u(a)$ (blue), a 1-cell $W^u(b)$ (red), and two 2-cells $W^u(c), W^u(d)$ (white) as shown in figure 3.6.

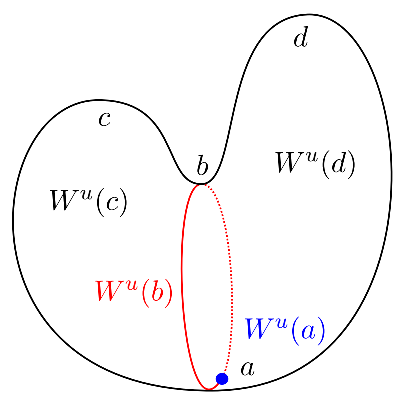

Figure 3.6: Induced cellular decomposition of the wobbly sphere.

Therefore the cellular complex is

$$\cdots \to 0 \to G^2 \to G \to G \to 0 \to \cdots$$

where G is the coefficient ring.

The two potentially-nontrivial boundary maps ∂_2 and ∂_1, which we now compute. On each 2-cell, ∂_2 is defined by $\partial_2 e^2 = N(e^2, W^u(b))W^u(b)$ where $N(e^2, W^u(b))$ is the degree of the induced map

$$\mathbb{S}^1 \to M^1 \to M^1/M^0 \to \mathbb{S}^1$$

where M^i is the i-skeleton of M. (M^{-1} is taken to be the empty set.) In this case the map is a composition of identity maps, so

$$N(W^u(c), W^u(b)) = N(W^u(d), W^u(b)) = g$$

where g is generator of G. Next for the case of ∂_1, we see that $N(W^u(b), W^u(a)) = 0$. Intuitively this is because the one-cell $W^u(b)$ forms a cycle. In terms of degrees, consider the induced map $f : \mathbb{S}^0 \to M^0/\varnothing = *$ with the two points in the domain signed by -1 and 1. The degree of f is 0 since the preimage of $*$ contains both points.

These calculations are remarkable in that the boundary maps of the cellular complex are identical to the boundary maps in the Morse complex, and even the computations to determine the boundary maps are similar.

It follows that the cellular homology is

$$H_k^{\text{Cell}}(\mathbb{S}^2, G) = \left\{ \begin{matrix} G & \text{if } k \in \{0, 2\}, \\ 0 & \text{otherwise.} \end{matrix} \right\} = H_k^{\text{Morse}}(\mathbb{S}^2, G)$$

as required.

Chapter 4

Morse homology applications

4.1 The Morse inequalities

As we have established that Morse homology is isomorphic to singular homology, we can define the *Betti numbers* of a manifold using Morse homology and obtain an equivalent definition as in the singular (or de Rham) cases.

Definition 4.1.1. The *Betti numbers* $b_k(M)$ of a manifold M are the ranks of the k-th homology groups;

$$b_k(M) := \operatorname{rank} H_k(M, \mathbb{Z}).$$

The rank of a $\mathbb{Z}$-module A is the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes A$. By flatness of $\mathbb{Q}$, the rank-nullity theorem from linear algebra passes over to $\mathbb{Z}$-modules (more generally modules over a PID) in the following sense: given any short exact sequence $0 \to A \to B \to C \to 0$ of $\mathbb{Z}$-modules, $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$. Using this fact we can easily derive the *Morse inequalities.* (Note that Audin and Damian only derive the *weak Morse inequalities.* Here we derive the *strong Morse inequalities*, which are what are usually referred to as the Morse inequalities.)

Theorem 4.1.2 (Strong Morse inequalities)**.** *Let M be a manifold, and f a Morse function on M. Let N_i denote the number of index i critical points of f. Then for any $k \geq 0$,*

$$\sum_{i=0}^{k} (-1)^{k-i} N_i \geq \sum_{i=0}^{k} (-1)^{k-i} b_i(M).$$

Proof. Recall that any Morse function f can be perturbed so that it has the same critical points, but admits a pseudo-gradient satisfying the Smale property. Therefore without loss of generality, f belongs to a Morse-Smale pair (f, X), induing the Morse complex $C_\bullet(M, \mathbb{Z})$. Recall that $C_i(M, \mathbb{Z})$ is the free $\mathbb{Z}$-module generated by critical points of index

i, so $N_i = \operatorname{rank} C_i(M,\mathbb{Z})$. Moreover, by the first isomorphism theorem, $\operatorname{rank} C_i(M,\mathbb{Z}) = \operatorname{rank}\ker\partial_i + \operatorname{rank}\operatorname{im}\partial_i$. Therefore the left hand side of the inequality we wish to derive is

$$\begin{aligned}\sum_{i=0}^{k}(-1)^{k-i}N_i &= N_k - N_{k-1} + \cdots + (-1)^k N_0 \\ &= \operatorname{rank}\ker\partial_k + \operatorname{rank}\operatorname{im}\partial_k \\ &\quad - \operatorname{rank}\ker\partial_{k-1} - \operatorname{rank}\operatorname{im}\partial_{k-1} \\ &\quad + \cdots \\ &\quad + (-1)^k\operatorname{rank}\ker\partial_0 + (-1)^k\operatorname{rank}\operatorname{im}\partial_0.\end{aligned}$$

Since $C_{-1}(M,\mathbb{Z}) = 0$, $\operatorname{rank}\operatorname{im}\partial_0 = 0$. On the other hand, $\operatorname{rank}\operatorname{im}\partial_{k+1} \geq 0$. By regrouping terms, this gives an inequality

$$\begin{aligned}\sum_{i=0}^{k}(-1)^{k-i}N_i &\geq -\operatorname{rank}\operatorname{im}\partial_{k+1} + \operatorname{rank}\ker\partial_k \\ &\quad + \operatorname{rank}\operatorname{im}\partial_k - \operatorname{rank}\ker\partial_{k-1} \\ &\quad + \cdots \\ &\quad - (-1)^k\operatorname{rank}\operatorname{im}\partial_1 + (-1)^k\operatorname{rank}\ker\partial_0.\end{aligned}$$

By by definition, $b_i(M) = \operatorname{rank} H_i(M,\mathbb{Z}) = -\dim\operatorname{im}\partial_{i+1} + \dim\ker\partial_i$. Therefore the above inequality is exactly the inequality we set out to prove. $\square$

In the above proof, $\operatorname{rank}\operatorname{im}\partial_{k+1} \geq 0$ was the only inequality contributing to the inequality in the final result. In the case where $k = n$, $\partial_{k+1} = 0$. Therefore in the top dimensional case the Morse inequalities are an *equality*.

Corollary 4.1.3. *Let f be a Morse function on an n dimensional manifold M. Then*

$$\sum_{i=0}^{n}(-1)^i N_i = \sum_{i=0}^{n}(-1)^i b_i(M) = \chi(M).$$

Theorem 4.1.4 (Weak Morse inequalities)**.** *Let M be a manifold, and f a Morse function on M. Let N_k denote the number of index k critical points of f. Then*

$$N_k \geq b_k(M).$$

Proof. This is immediate from the strong Morse inequalities by fixing k, and adding the strong Morse inequality with $\sum_{i=0}^{k-1}(-1)^{k-i-1}N_i$ to the strong Morse inequality with $\sum_{i=0}^{k}(-1)^{k-i}N_i$. $\square$

Example. The wobbly sphere is equipped with a height function with one critical point of index 0, one of index 1, and two of index 2. The alternating sum of the N_is is therefore $1-1+2=2=\chi(\mathbb{S}^2)$. The Klein bottle is equipped with a height function with one critical point of index 0, two of index 1, and one of index 2, giving $1-2+1=0=\chi(\mathrm{K}^2)$.

Example. Suppose M is a closed n-manifold admitting a Morse function with two critical points. These are necessarily a maximum and minimum, i.e. index 0 and index n. It then follows from the Morse inequalities that for each i, $\operatorname{rank} H_i(M) = \operatorname{rank} H_i(\mathbb{S}^n)$. Since $H_0(M)$ and $H_n(M)$ are always free, it follows that M has isomorphic homology to the n-sphere. By Whitehead's theorem, M is a homotopy n-sphere. By the Poincaré conjecture, M is homeomorphic to the n-sphere. This gives a very clean but completely unnecessary (and probably circular) proof of the Reeb sphere theorem.

An immediate corollary of the weak Morse inequalities is the following:

Corollary 4.1.5. *Let f be a Morse function on M. Then f has at least as many critical points as the sum of the ranks of the homology groups of M.*

In summary the Morse inequalities allow us to gain a lot of topological insight from finding good functions on a manifold, following the theme of Reeb's theorem and the section concerning changes in topology passing from one sublevel set to another. One more interesting theorem which will not be proven here is a result from *discrete Morse theory.*

Theorem 4.1.6. *Let M be a manifold equipped with a cellular decomposition. Let m_k denote the number of k-cells in the decomposition. Then for each k,*

$$m_k \geq b_k(M).$$

This is the weak version of the discrete Morse inequality. This doesn't quite follow from our proof that the Morse and cellular homologies of a manifold are equal, as we have not shown that every cellular decomposition induces a suitable Morse-Smale pair. As a corollary, it follows that there are no cellular decompositions of a torus into fewer than 4 cells. From our (standard) version of the Morse inequalities, we conclude that any Morse function on a torus has at least four critical points.

4.2 Morse functions and simple connectedness

Unfortunately Homology does not in general detect simple connectedness. A well known aspect of Hurewicz's theorem is that there is an isomorphism

$$\pi_1^{\mathrm{ab}}(M) := \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]} \cong H_1(M).$$

Since non-trivial *perfect groups* exist, i.e. non-trivial groups that are equal to their derived subgroups (such as A_5), there are manifolds with vanishing first homology but non-trivial fundamental group. Without further assumptions about the manifold, one cannot conclude from a vanishing first homology that a manifold is simply connected.

Theorem 4.2.1. *If a closed n-manifold M admits a Morse function with no critical points of index 1, M is simply connected.*

Remark. By the Morse inequalities, $H_1(M)$ has rank zero. This doesn't generally imply that $H_1(M) = 0$, but even if it did, by the above observation we still cannot conclude that M is simply connected.

Proof. The proof does not use Homology. Fix any minimum $a \in M$ of f to be the base point of M. Let γ be a loop in M based at a. We can assume without loss of generality that γ is smooth.

Now let b be any critical point of index greater than 0. By assumption, b must have index at least 2, so its stable manifold has dimension at most $n-2$. On the other hand, γ is a smooth map, so by Sard's theorem it is homotopic to a function which is transverse to $W^s(b)$. This argument can be repeated for the finitely many critical points of index at least 2, with all homotopies fixing the basepoint. Since $d\gamma$ has dimension 1, transversality is equivalent to the statement that γ does not meet any of the stable manifolds of critical points of index at least 2.

Let X be a pseudo-gradient adapted to f, and let x be a point in the image of γ. The limit $\lim_{t\to\infty} \varphi_X^t(x)$ is a critical point, so M is a union of the stable manifolds of critical points of f. Since γ has empty intersection with stable manifolds of critical points of index at least 2, and there are no critical points of index 1, γ lies in the union of stable manifolds of index 0. These are all disjoint, so γ is contained in one stable manifold. Stable manifolds are diffeomorphic to disks, so γ is contractible. $\square$

4.3 Poincaré duality realised in Morse homology

Since we have established that Morse homology is isomorphic to Singular homology, we suddenly have a lot of results such as the following:

- Poincaré duality
- Excision theorem
- Homology long exact sequence, Mayer–Vietoris sequence
- Künneth formula
- Hurewicz theorem
- Universal coefficient theorem

These can be established purely using Morse theory, rather than transporting the result from singular or cellular homology. The first four bulleted results are established in this manner in Audin and Damian. Here we describe a version of Poincaré duality, as I found the idea to be particularly clean. In Audin and Damian, Poincaré duality is proven in the original form as stated by Poincaré:

Theorem 4.3.1. *Let M be a closed n-manifold. Then there is an isomorphism $H_k(M, \mathbb{Z}/2\mathbb{Z}) \cong H_{n-k}(M, \mathbb{Z}/2\mathbb{Z})$. Moreover if M is oriented, then $b_k(M) = b_{n-k}(M)$.*

In these notes we obtain a stronger result which looks more similar to the usual general statement of Poincaré duality for singular homology.

Theorem 4.3.2. *Let M be a closed n-manifold. Then there is an isomorphism*

$$t : C_\bullet(M, \mathbb{Z}/2\mathbb{Z}) \to C^\bullet(M, \mathbb{Z}/2\mathbb{Z}), \quad t_k : C_k(M, \mathbb{Z}/2\mathbb{Z}) \to C^{n-k}(M, \mathbb{Z}/2\mathbb{Z})$$

of chain complexes. In particular, for each k there is an isomorphism $H_k(M, \mathbb{Z}/2\mathbb{Z}) \cong H^{n-k}(M, \mathbb{Z}/2\mathbb{Z})$. Moreover if M is oriented, then there is an isomorphism

$$t : C_\bullet(M, \mathbb{Z}) \to C^\bullet(M, \mathbb{Z}), \quad t_k : C_k(M, \mathbb{Z}) \to C^{n-k}(M, \mathbb{Z}).$$

In particular, for each k there is an isomorphism $H_k(M, \mathbb{Z}) \cong H^{n-k}(M, \mathbb{Z})$.

First we must make sense of cohomology in the sense of Morse complexes. In singular homology, the chain complex is directly dualised to give singular cohomology. That is, each C_k is replaced with $C_k^* = \mathrm{Hom}_R(C_k, R)$, and the boundary maps are replaced with their adjoints. For Morse homology, it is cleaner to define the Morse cohomology by first negating the Morse function as follows:

Let f be a Morse function on M. Then the index k critical points of f are precisely the index $n-k$ critical points of $-f$, and $-f$ is itself a Morse function on M. For each k, this gives an isomorphism

$$C_k(f, R) \to C_{n-k}(-f, R).$$

We then define the co-complex $C^\bullet(f, R)$ to be the dual complex $C_{n-k}(-f, R)^*$. In summary, given a coefficient ring R, we have the following diagram where each t_k is an isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{n-k}(f,R) \arrow[r, "\partial^{n-k+1}"] & C^{n-k+1}(f,R) \arrow[r] & \cdots \\
\cdots & C_{n-k}(-f,R) \arrow[l] \arrow[u] & C_{n-k+1}(-f,R) \arrow[l, "\delta_{n-k+1}"'] \arrow[u] & \cdots \arrow[l] \\
\cdots \arrow[r] & C_k(f,R) \arrow[r, "\partial_k"] \arrow[u] \arrow[uu, bend left=60, "t_k"] & C_{k-1}(f,R) \arrow[r] \arrow[u] \arrow[uu, bend left=60, "t_{k-1}"] & \cdots
\end{tikzcd}$$

To prove theorem 4.3.2, it remains to show that t is a *chain map* (in each of the two cases).

Let X be a pseudo-gradient adapted to f, satisfying the Smale condition. The key is to compare the number of trajectories mod 2 for the mod 2 homology case; $n_X(b,a)$ to $n_{-X}(a,b)$, and signed trajectories $N_X(b,a)$ to $N_{-X}(a,b)$ for the integral homology case. Here we simply discuss the counting in the integral case, as mod 2 immediately follows.

Recall that $N_X(b,a)$ is a *signed count*. Given any trajectory $\gamma \in \mathcal{L}_X(b,a)$, its sign is determined by the orientation of $T_{\gamma(t)}\mathcal{M}_X(b,a)$, which is the difference of the orientations of $T_{\gamma(t)}W^s(b)$ and $T_{\gamma(t)}W^s(a)$. By assuming that M is oriented, orientations of stable manifolds induce orientations of unstable manifolds. In particular, the short exact sequence

$$0 \to T_{\gamma(t)}\mathcal{M}_{-X}(a,b) \to T_{\gamma(t)}W^u(a) \to T_{\gamma(t)}W^u(b) \to 0$$

induces an orientation on $T_{\gamma(t)}\mathcal{M}_{-X}(a,b)$. Moreover, the induced orientation did not depend on γ at any point, and the signed count of trajectories $N_X(b,a)$ agrees with $N_{-X}(a,b)$. More explicitly, one can show that the orientations of $T_\gamma\mathcal{L}_X(b,a)$ and $T_\gamma\mathcal{L}_{-X}(a,b)$ agree by "orientation chasing" the following diagram:

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[r] & T_{\gamma(t)}\mathcal{M}_X(b,a) \arrow[r] \arrow[d] & T_\gamma\mathcal{L}_X(b,a) \\
 & T_{\gamma(t)}W^u(b) & T_{\gamma(t)}M \arrow[l] & T_{\gamma(t)}W^s(b) \arrow[l] \arrow[d] & \\
 & T_{\gamma(t)}W^u(a) \arrow[u] \arrow[r] & T_{\gamma(t)}M \arrow[r] & T_{\gamma(t)}W^s(a) & \\
T_\gamma\mathcal{L}_{-X}(a,b) & T_{\gamma(t)}\mathcal{M}_{-X}(a,b) \arrow[l] \arrow[u] & \mathbb{R} \arrow[l] & &
\end{tikzcd}$$

We are now ready to prove the celebrated Poincaré duality.

Proof of theorem 4.3.2. As remarked earlier, it suffices to show that t is a chain map. That is, we must show that for any k,

$$t_{k-1} \circ \partial_k = \partial^{n-k+1} \circ t_k.$$

To this end, fix $b \in C_k(f)$ and $a \in C_{n-k+1}(-f)$. On one hand, we have

$$\begin{aligned}
((t_{k-1} \circ \partial_k)(b))(a) &= t_{k-1}\Big(\sum_{a_i \in \mathrm{Crit}_{k-1}(f)} N_X(b,a_i)a_i\Big)(a) \\
&= \sum_{a_i \in \mathrm{Crit}_{k-1}(f)} N_X(b,a_i)a_i^*(a) = N(b,a).
\end{aligned}$$

On the other hand,

$$\begin{aligned}((\partial^{n-k+1} \circ t_k)(b))(a) &= \partial^{n-k+1}(b^*)(a) \\ &= (b^* \circ \delta_{n-k+1})(a) \\ &= b^* \sum_{b_i \in \mathrm{Crit}_k(f)} N_{-X}(a, b_i) b_i = N_{-X}(a, b).\end{aligned}$$

As shown above, the orientation of M ensures that $N_X(b, a) = N_{-X}(a, b)$. Therefore t is a chain map, so the isomorphisms t_k extend to an isomorphism of chain complexes. $\square$

Chapter 5

The h-cobordism theorem

5.1 Smale's original proof of the generalised Poincaré conjecture

The *Poincaré conjecture* states that *every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.* This was open for a long time and was even a Millennium problem, but was settled in 2006 by Grigori Perelman. The proof was analytic, following Hamilton's Ricci flow programme.

An equivalent statement to the Poincaré conjecture is that *every 3-manifold with the homotopy type of the 3-sphere is homeomorphic to the 3-sphere.* To see this, we prove the following proposition.

Proposition 5.1.1. Let M be a closed simply connected 3-manifold. Then M is a homotopy 3-sphere.

Proof. By Whitehead's theorem, if there is a map $f : \mathbb{S}^3 \to M$ inducing isomorphisms on all homotopy groups, then f is a homotopy equivalence. Therefore we wish to find such a map. But $\mathbb{S}^3$ and M are simply connected, so by the relative form of Hurewicz's theorem, any $f : \mathbb{S}^3 \to M$ inducing isomorphisms on homology will induce isomorphisms on homotopy. Therefore to prove the proposition, it suffices to find a map $f : \mathbb{S}^3 \to M$ inducing isomorphisms on homology.

Since M is simply connected, $H_1(M)$ is trivial, so $H_2(M)$ is also trivial by Poincaré duality. Since M is 2-connected, Hurewicz's theorem applies: $\pi_3(M) \cong H_3(M) \cong \mathbb{Z}$. Thus let $f : \mathbb{S}^3 \to M$ be any generator of $\pi_3(M)$. Then f induces isomorphisms on all homology groups as required. Therefore M is a homotopy 3-sphere. $\square$

This gives a version of the Poincaré conjecture that easily generalises to higher dimensions:

Conjecture 5.1.2. Let an n-manifold M be a homotopy n-sphere. Then M is homeomorphic to $\mathbb{S}^n$.

As of 2006, this conjecture is settled for all dimensions. Dimensions $n \in \{0, 1, 2\}$ are trivial by the classification of n-manifolds for small n. Dimension 3 is the classical case proven by Perelman. Dimension 4 was solved by Michael Freedman in 1982. Dimensions 5 and above were settled by Stephen Smale in the 1960s.

We now describe Smale's beautifully simple original proof outline.

Theorem 5.1.3. *For $n \geq 5$, an n-manifold homotopic to the n-sphere is homeomorphic to the n-sphere.*

Proof. Most of the work goes into proving *theorem C* (from Smale's paper [Sma61]).

Let M be an n-manifold homotopic to the n-sphere, with $n \geq 5$. Then the Morse homology of M is isomorphic to that of the n-sphere. In particular, the sum of the ranks of its homology groups is 2. By the Morse inequalities, any Morse function on M has at least two critical points. By *Theorem C* this inequality is sharp: M admits a Morse function with two critical points. By Reeb's theorem, M is therefore homeomorphic to an n-sphere. □

5.2 Proving the Poincaré conjecture from the h-cobordism theorem

Remark. In this section we consider manifolds that are *not* smooth. (This is the only section of these notes in which we do this.) Therefore we explicitly say *smooth manifold* if a manifold is equipped with a smooth structure.

Soon after the proof that was geared towards the generalised Poincaré conjecture, Smale observed that his results could be made a lot more general by stating them in terms of *cobordisms.*

Definition 5.2.1. A *cobordism* is a compact manifold with boundary W whose boundary decomposes as $\partial W = V_0 \sqcup V_1$, where V_0 and V_1 are themselves embedded smooth manifolds.

An example of a cobordism is a *pair of pants with a mysterious hole* as in figure 5.1.

Definition 5.2.2. Two n-manifolds V_0, V_1 (without boundary) are said to be *cobordant* if there is a cobordism W such that the boundary of W is the disjoint union of V_1 and V_2.

Observe that any cobordism W is equipped with natural inclusion maps $\iota_0 : V_0 \hookrightarrow W, \iota_1 : V_1 \hookrightarrow W$.

Definition 5.2.3. A cobordism W between V_0 and V_1 is said to be an *h-cobordism* if ι_0, ι_1 are homotopy equivalences. The h stands for homotopy.

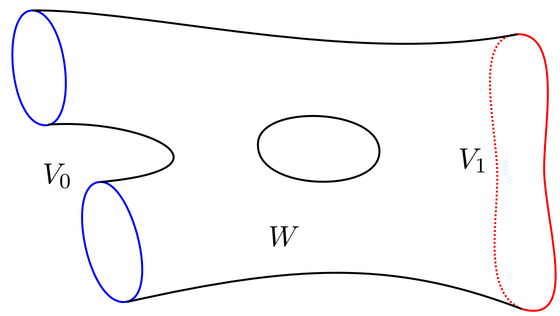

Figure 5.1: Example of a cobordism.

Example. The cobordism in figure 5.1 is clearly not an h-cobordism, since V_0 and V_1 do not have the same homotopy type. However, the cylinder $\mathbb{S}^1 \times [0, 1]$ is a cobordism between two circles which is easily seen to be an h-cobordism.

Theorem 5.2.4 (h-cobordism theorem)**.** *Let n be at least 6, and W a compact n-dimensional simply connected smooth h-cobordism between simply connected smooth $(n-1)$-manifolds V_0 and V_1. Then W is diffeomorphic to $V_0 \times [0, 1]$.*

In the next section, we will gain some insight as to why it is necessary that the dimension of W be at least 6. For now we assume that the h-cobordism theorem holds, and prove the Poincaré conjecture for dimensions at least 6.

Theorem 5.2.5 (Smooth Poincaré conjecture, $n \geq 6$)**.** *For $n \geq 6$, a smooth n-manifold homotopic to the n-sphere is homeomorphic to the n-sphere.*

Proof. Assume the h-cobordism theorem. The proof follows figure 5.2.

Suppose M is a smooth n-manifold ($n \geq 6$) with the homotopy type of an n-sphere. Any two distinct points are contained in disjoint disks D_0^n and D_1^n. By cutting along the boundary of the disks, we obtain a decomposition of M as shown in figure 5.2. Precisely, we write $M = D_0^n \cup W \cup D_1^n$, where $W = M \setminus \mathrm{int}(D_0^n \sqcup D_1^n)$.

Observe that W is a cobordism between spheres $\mathbb{S}_0^{n-1}$ and $\mathbb{S}_1^{n-1}$. We prove in the subsequent lemma that $\iota_0 : \mathbb{S}_0^{n-1} \hookrightarrow W$ is a homotopy equivalence. (The same result holds for $\mathbb{S}_1^{n-1}$). Therefore by the h-cobordism theorem, W is diffeomorphic (and in particular homeomorphic) to $\mathbb{S}^{n-1} \times [0, 1]$, with the homeomorphism denoted by f in the figure.

f restricts to homeomorphisms on the boundary, e.g. $g_0 : \mathbb{S}_0^{n-1} \to \mathbb{S}^{n-1}$ as shown in the figure. But any homeomorphism of a sphere induces a homeomorphism of disks $D_0^n \to D^n$ by the *Alexander trick*. (One can simply take the radial extension of the homeomorphism.) Therefore we have homeomorphisms $g_0, g_1 : D_0^n, D_1^n \to D^n$ which agree with f on overlaps. The map $M \to D^n \cup (\mathbb{S}^{n-1} \times [0, 1]) \cup D^n \cong \mathbb{S}^n$ defined piecewise by g_0, f, and g is therefore a homeomorphism. ∎

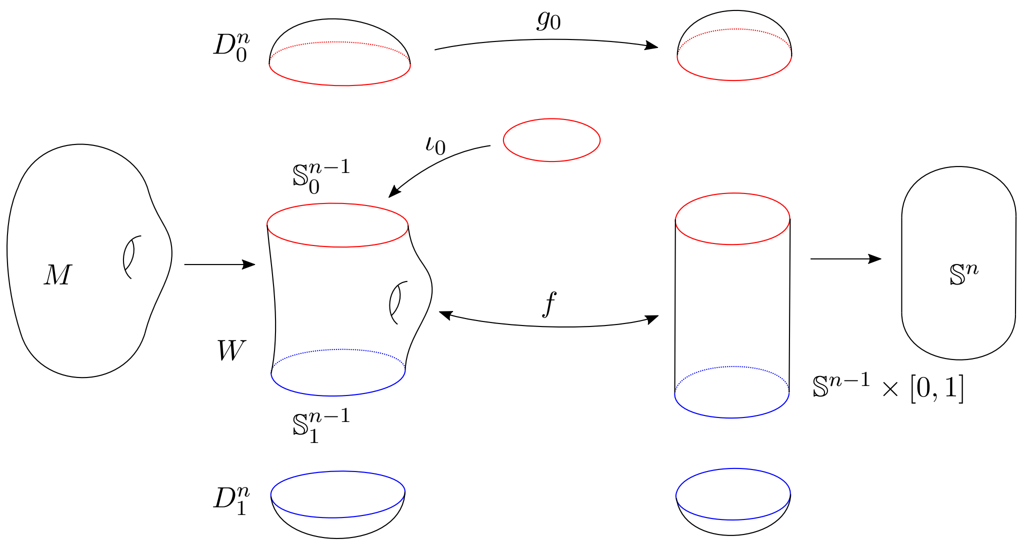

Figure 5.2: Proof of the Poincaré conjecture (in dimensions at least 6).

In the above proof we left out a crucial and (in my opinion subtle) step. We wish to argue that the inclusion ι_0 factors through $\mathbb{S}^{n-1} \times [0,1]$ via f, so ι_0 is a composition of two homotopy equivalences and hence a homotopy equivalence. This is not true since f will not preserve injectivity. To conclude that ι_0 is a homotopy equivalence, we use the Excision theorem for homology.

Lemma 5.2.6. Let M be a topological n-manifold homotopic to $\mathbb{S}^n$. Let W be as in figure 5.2, i.e. a cobordism between $\mathbb{S}_0^{n-1}$ and $\mathbb{S}_1^{n-1}$ obtained by removing the interiors of two n-disks. Then $\mathbb{S}_0^{n-1} \hookrightarrow W$ is a homotopy equivalence.

Proof. Both W and $\mathbb{S}_0^{n-1}$ are simply connected. Therefore (as remarked at the start of this chapter) the following version of Whitehead's theorem holds: *Any map $\iota : \mathbb{S}_0^{n-1} \to W$ inducing isomorphisms on homology is a homotopy equivalence.* Therefore we prove that the inclusion map induces isomorphisms on homology.

For notational brevity we hereafter write S_0, S_1, D_0, and D_1 to denote $\mathbb{S}_0^{n-1}$ and so on. Fix $j \geq 0$. To prevent complications when $j = 0$, in this argument H denotes the reduced homology. By the homology long exact sequence, it suffices to show that $H_j(W, S_0) = 0$. By the excision theorem, $H_j(W, S_0) \cong H_j(W \cup D_0, D_0)$. By the homology long exact sequence of $H_j(W \cup D_0, D_0)$, since $H_j(D_0)$ is trivial, $H_j(W \cup D_0, D_0) \cong H_j(W \cup D_0)$. Again by the excision theorem, $H_j(M, W \cup D_0) \cong H_j(D_1, S_1)$. By the homology long exact sequence, $\cong H_j(D_1, S_1) \cong H_j(\mathbb{S}^n) \cong H_j(M)$. But with $H_j(M, W \cup D_0)$ isomorphic to $H_j(M)$, the homology long exact sequence implies that $H_j(W \cup D_0)$ is trivial, and hence $H_j(W, S_0)$ is trivial. $\square$

5.3 Proof outline of the h-cobordism theorem

We now give an overview of the proof of the (smooth) h-cobordism theorem, following Milnor's famous lecture notes [Mil65].

Definition 5.3.1. The *Morse number* $\mu(M)$ of a manifold M is the minimum of the number of critical points of a Morse function on M.

By the Morse inequalities, for closed manifolds this is bounded below by the topological complexity (homology groups) of M, and is always at least 2. For a compact manifold with boundary, the global extrema need not be critical points (in the sense of having vanishing derivative), so manifolds with boundary may have Morse number 0.

Theorem 5.3.2. *Let $(W; V_0, V_1)$ be a cobordism. If $\mu(W) = 0$, then W is a product cobordism, i.e. $W \cong V_0 \times [0,1]$.*

The above theorem is related to a theorem from section two, in which we showed that moving from one sub-level set M^a to another M^b doesn't change the diffeomorphism class provided there are no critical points in $f^{-1}([0,1])$. The proof is also similar. We make use of this theorem to prove the h-cobordism theorem, by showing that for sufficiently high dimensions, the Morse number of a simply connected cobordism between simply connected manifolds is zero. More practically, the goal is to start with a Morse function on W, and continue to modify it to eliminate critical points.

Some motivation for how we might eliminate critical points is the observation that critical points of index λ and $\lambda + 1$ might cancel out, as in 5.3. In this example, we see

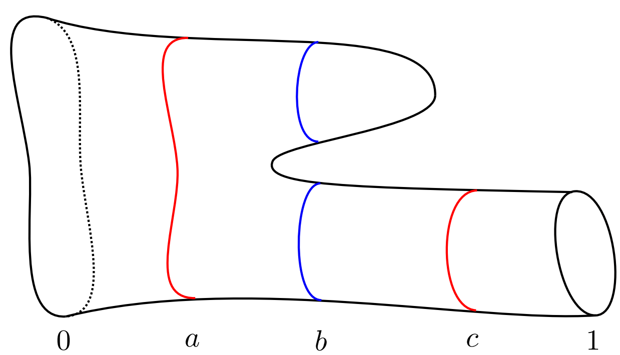

Figure 5.3: Example of cancellation.

that the sublevel set at a and c are diffeomorphic, even though they are not diffeomorphic to the sublevel set at b. The idea is that the index 1 critical point between a and b has been cancelled by the index 2 critical point between b and c. A precise statement for when this occurs is the *first cancellation theorem*:

Theorem 5.3.3 (First cancellation theorem)**.** *Suppose W is a cobordism equipped with a Morse-Smale pair (f, X) with exactly two critical points c and c', of index k and $k+1$, such that $f(c) < f(c')$. If $\mathcal{L}(c', c)$ consists of a single point, then the cobordism is a product cobordism. If fact, the pseudo-gradient field X can be modified on an arbitrarily small neighbourhood of the trajectory from c' to c to produce a new pseudo-gradient field, which corresponds to a new Morse function f' on W with no critical points that agrees with f near ∂W.*

In order to make use of this theorem, we must be able to guarantee that $\mathcal{L}(c', c)$ consists of a single point along with the hypothesis of critical points occurring in the correct order. The second of these is the *rearrangement theorem*, which comes in two forms.

Theorem 5.3.4 (Rearrangement theorem, version one)**.** *Any cobordism W of dimension n can be expressed as a composition of cobordisms $W = U_0 U_1 \cdots U_n$, where each cobordism U_k admits a Morse function with only one critical level, and all critical points of index k.*

This can be phrased without reference to decompositions of cobordisms, and instead in terms of *self-indexing Morse functions*.

Theorem 5.3.5 (Rearrangement theorem, version two)**.** *Any cobordism (W, V_0, V_1) can be equipped with a self-indexing Morse function. Explicitly, this means a Morse function f satisfying*

- $f(V_0) = -1/2, f(V_1) = n + 1/2.$
- $f(c) = \operatorname{ind}(c)$*, for each critical point c.*

We are now in good shape, as all that remains to apply the first cancellation theorem is to show that under certain conditions, $\mathcal{L}(c', c)$ is a singleton. Unfortunately this is a very difficult condition to guarantee. This is where the condition of W being simply connected and has dimension at least 6 becomes necessary.

Theorem 5.3.6 (Second cancellation theorem)**.** *Suppose (W, V_0, V_1) is a cobordism with W, V_0, V_1 simply connected and $\dim W = n$. Suppose W is equipped with a Morse-Smale pair (f, X) with exactly two critical points, of index λ and $\lambda + 1$. If $3 \leq \lambda \leq n - 3$, and the* signed count *$N_X(c', c)$ (as used in the Morse homology) is ± 1, then $W \cong V_0 \times [0, 1]$. More explicitly, given these hypotheses, the pseudo-gradient field of f can be altered between the critical points so that the un-signed count is 1 (so that the first cancellation theorem applies).*

Observe that a corollary of this theorem is that the analogous result holds with $2 \leq \lambda \leq n - 4$, by replacing f with $-f$. In each case, we observe that n must be at least 6 for the result to hold. We use the second cancellation theorem to eliminate critical points "of middle index".

Theorem 5.3.7 (Elimination of critical points of middle index). *Suppose* $(W; V_0, V_1)$ *is a cobordism with* $\dim W = n \geq 6$*, equipped with a Morse-Smale pair* (f, X) *with no critical points of index* $0, 1, n-1$*, or* n*. Assume moreover that* W, V_0*, and* V_1 *are simply connected, and* $H_*(W, V_0; \mathbb{Z}) = 0$*. Then* $(W; V_0, V_1)$ *is a product cobordism.*

Proof. We give a proof outline, using the Morse complex along with the rearrangement and second cancellation theorems. The idea is that we use the rearrangement theorem to decompose W as

$$W = U_2 U_3 \cdots U_{n-2},$$

where each U_k consists only of critical points of index k, all with the same level. Now perturb two critical points c and c', of index k and $k+1$ respectively, so that

$$U_k U_{k+1} \cong U'_k U_c U_{c'} U'_{k+1},$$

where U_c and $U_{c'}$ consist of single critical points, c and c'. Now the condition on homology ensures exactness of the Morse complex, from which we can conclude that the intersection number $N_X(c', c)$ is ± 1 (by inspecting the definition of the boundary map). By applying the second cancellation theorem to $U_c U_{c'}$, we conclude that $U_c U_{c'}$ is a product cobordism. Inductively we conclude that W is a product cobordism. □

It now remains to prove that critical points of low and high index can be eliminated. Again by replacing f with $-f$ (i.e. Poincaré duality), it suffices to eliminate critical points of middle index.

Theorem 5.3.8 (Elimination of critical points of low index). *Consider a cobordism* $(W; V_0, V_1)$*, with* $\dim W = n$*, equipped with a self-indexing Morse-Smale pair* (f, X)*. The following hold:*

- *If* $H_0(W, V_0; \mathbb{Z}) = 0$*, then critical points of index* 0 *can be cancelled against an equal number of critical points of index* 1*.*
- *Suppose* W, V_0 *are simply connected, and* $n \geq 5$*. If there are no critical points of index* 0*, one can insert an auxiliary critical point of index 2 and 3 for each critical point of index 1, so that the critical point of index 2 cancels the critical point of index 1. (Effectively, one can trade critical points of index 1 for those of index 3.)*

Once this theorem is established, the h-cobordism theorem is almost immediate:

Theorem 5.3.9 (h-cobordism theorem). *Suppose* $(W; V_0, V_1)$ *is an h-cobordism with* W, V_0, V_1 *simply connected, and* $\dim W \geq 6$*. Then* W *is a product cobordism.*

Proof. Since W is an h-cobordism, $H_*(W, V_0; \mathbb{Z})$ vanishes. Equip W with a self-indexing Morse-Smale pair (f, X). By the *elimination of critical points of low index*, we first eliminate all critical points of index 0. If there are any critical points of index 1, these are traded for critical points of index 3. Replacing f with $-f$, we similarly eliminate critical points of index n and $n-1$. Finally by *elimination of critical points of middle index*, the desired result follows. □

5.4 Poincaré conjecture and h-cobordism theorem in different categories

It is interesting to study the Poincaré conjecture in different categories. We have noted that the Poincaré conjecture holds for smooth manifolds, in the sense that a smooth homotopy n-sphere is homeomorphic to an n-sphere. However, we have not discussed the conjecture strictly in the category of topological, piecewise linear or smooth manifolds.

Definition 5.4.1. A *piecewise linear (PL) manifold* (also called a *combinatorial manifold*) is a topological manifold equipped with a PL structure. That is, the transition maps are piece-wise linear. Details are given here: [Bry01].

We hereafter denote the category of topological manifolds by $\mathbf{Man}$, PL manifolds by $\mathbf{Man^{PL}}$, smooth manifolds by $\mathbf{Man}^\infty$, and analytic manifolds by $\mathbf{Man}^\omega$. Precisely, the objects in these categories are the corresponding types of manifolds, and the morphisms are continuous, PL, smooth maps, and analytic maps respectively.

Theorem 5.4.2. $\mathbf{Man}^\infty$ *and* $\mathbf{Man}^\omega$ *are equivalent. More precisely, every smooth manifold admits a unique analytic structure.*

This result is due to Grauert and Morrey, see [Shi64]. It follows that the validity of the Poincaré conjecture or h-cobordism theorems in $\mathbf{Man}^\omega$ is the same as that in $\mathbf{Man}^\infty$, and for this reason we no longer mention $\mathbf{Man}^\omega$ (until the end of the section where results are summarised).

Theorem 5.4.3. *The h-cobordism theorem (for dimensions at least 6) holds in* $\mathbf{Man}, \mathbf{Man^{PL}}$*, and* $\mathbf{Man}^\infty$*.*

Proof. The h-cobordism theorem in $\mathbf{Man}^\infty$ is the subject of this chapter, and is the celebrated result proven in [Mil65]. The h-cobordism theorem holds in $\mathbf{Man^{PL}}$, with a proof given in [RS82]. Finally the h-cobordism theorem holds in $\mathbf{Man}$ as discussed in [KS77]. □

Corollary 5.4.4. *The Poincaré conjecture for dimensions at least 6 holds in* $\mathbf{Man^{PL}}$ *and* $\mathbf{Man}$*.*

Proof. Recall the proof that smooth homotopy n-spheres are homeomorphic to n-spheres. Since the h-cobordism theorem holds in $\mathbf{Man}$, an identical proof applies. For $\mathbf{Man^{PL}}$, the key is that the Alexander trick still holds (as radially extending a piecewise linear map gives a piecewise linear map).

However, this proof does not hold in $\mathbf{Man}^\infty$, as the centre of the disk is manifestly singular. □

Theorem 5.4.5. *In dimension 5, the h-cobordism theorem holds in* $\mathbf{Man}$ *but not in* $\mathbf{Man}^\infty$ *or* $\mathbf{Man^{PL}}$*.*

Proof. The affirmative result due to Freedman is discussed in his book, [FQ90]. Similarly the negative result due to Donaldson is discussed in [DK97]. (Moreover, in dimension 4, $\mathbf{Man}^{\mathbf{PL}}$ and $\mathbf{Man}^{\infty}$ are equivalent.) □

Theorem 5.4.6. *In dimensions at most 3,* $\mathbf{Man}, \mathbf{Man}^{\mathbf{PL}}$*, and* $\mathbf{Man}^{\infty}$ *are equivalent.*

More precisely, this follows from the result that every topological manifold of dimension at most 3 admits a smooth triangulation, unique up to diffeomorphism. (This is due to Bing and Moise, [Moi77].) By Perelman's proof of the Poincaré conjecture for dimension 3 in $\mathbf{Man}^{\infty}$, it follows that the Poincaré conjecture holds in all three categories. Similarly it holds in all three categories in lower dimensions, and likewise for the h-cobordism theorem.

By a theorem of Whitehead, every smooth manifold is canonically a PL manifold. Moreover, every topological n-sphere admits a smooth structure. Therefore if a topological n-sphere admits a *unique* smooth structure, the Poincaré conjecture in $\mathbf{Man}$ implies the result for both $\mathbf{Man}^{\mathbf{PL}}$ and $\mathbf{Man}^{\infty}$. On the other hand, whenever multiple smooth structures exist on the n-sphere, the Poincaré conjecture is false in $\mathbf{Man}^{\infty}$ for dimension n. It is conjectured that all spheres beyond dimension 61 admit at least one exotic smooth structure, which would guarantee that the smooth Poincaré conjecture is always false in dimensions above 61 (See [WX17]).

Since the 5-sphere has a unique smooth structure, the above theorem proves the 5-dimensional Poincaré conjecture in all three categories in consideration, even though the h-cobordism theorem fails in two of them.

In dimension 4, the Poincaré conjecture and h-cobordism theorem are equivalent to each other (in each of $\mathbf{Man}, \mathbf{Man}^{\mathbf{PL}}$, and $\mathbf{Man}^{\infty}$). In particular in $\mathbf{Man}^{\mathbf{PL}}$ and $\mathbf{Man}^{\infty}$, they are equivalent to the existence of an exotic smooth structure on the 4-sphere. This is a difficult problem, and is still open.

Proposition 5.4.7. Fix a category C consisting of 4-manifolds in either $\mathbf{Man}, \mathbf{Man}^{\mathbf{PL}}$, or $\mathbf{Man}^{\infty}$. Then the Poincaré conjecture in C is equivalent to the h-cobordism theorem in C.

Proof. Since $\mathbf{Man}^{\mathbf{PL}}$ and $\mathbf{Man}^{\infty}$ are equivalent in dimension 4, without loss of generality suppose C consists of 4-dimensional topological manifolds or PL manifolds.

Suppose the h-cobordism theorem holds in C. The earlier proof of the *Poincaré conjecture for dimensions at least 6* holds in C, so the Poincaré conjecture holds in C.

Next suppose the Poincaré conjecture holds in C. The proof will proceed in a similar way to the proof of the Poincaré conjecture assuming the h-cobordism theorem. Let W be a 4 dimensional compact simply connected h-cobordism between simply connected 3-manifolds V_0 and V_1. Then V_0 and V_1 are necessarily closed and simply connected, so by the Poincaré conjecture in dimension 3, they are C-isomorphic to 3-spheres. It follows that there are 4-cells D_0 and D_1 that can be glued along V_0 and V_1 respectively; let $M = D_0 \sqcup_{V_0} W \sqcup_{V_1} D_1$. By Whitehead's theorem, if we can prove that M has the same homology as a 4-sphere, then M is homotopy equivalent to the 4-sphere. By the Poincaré conjecture in C, it will follow that M is C-isomorphic to the 4-sphere. Therefore by removing V_0 and V_1, the h-cobordism theorem will follow.

It remains to prove that for each i, $H_i(M) = H_i(\mathbb{S}^4)$. Here H denotes the *reduced homology*, to avoid a complication when $i = 1$. By the long exact sequence of homology, $H_i(M) \cong H_i(M, D_1)$. By excision, the latter is isomorphic to $H_i(W \cup D_0, V_1)$. Next observe that $H_i(W \cup D_0) \cong H_i(W \cup D_0, D_0)$ again by the long exact sequence of homology. Another application of excision gives $H_i(W \cup D_0, D_0) \cong H_i(W, V_0)$. By virtue of W being an h-cobordism, $H_i(W, V_0)$ must vanish. This establishes that $H_i(W \cup D_0)$ vanishes, so by the long exact sequence of homology, we find that $H_{i+1}(M) = H_i(V_1) = H_{i+1}(\mathbb{S}^4)$ as required. $\square$

Finally, recall that Freedman famously proved the Poincaré conjecture for **Man** in dimension 4, for which he was awarded a Field's medal. In summary, as of the time of writing (the era of COVID-19 lockdowns), the status of the h-cobordism theorem and Poincaré conjecture in various dimensions is as follows:

Poincaré conjecture

Dim	**Man**	$\mathbf{Man}^{\mathbf{PL}}$	$\mathbf{Man}^\infty, \mathbf{Man}^\omega$
1,2,3	True	True	True
4	True	Open	Open
5,6	True	True	True
7+	True	True	Usually false

h-cobordism theorem

Dim	**Man**	$\mathbf{Man}^{\mathbf{PL}}$	$\mathbf{Man}^\infty, \mathbf{Man}^\omega$
1,2,3	True	True	True
4	True	Open	Open
5	True	False	False
6+	True	True	True

Finally we remark that it is not particularly interesting to consider this question in the categories of almost-complex or complex manifolds, since the the *only* spheres that exist in these categories are $\mathbb{S}^2$ and $\mathbb{S}^6$ for the former, and $\mathbb{S}^2$ for the latter. (It is not currently known if $\mathbb{S}^6$ admits a complex structure.) Exposition on almost complex spheres and complex spheres can be found in [KP17].

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