A COMBINATORIAL APPROACH TO STUDYING LAGRANGIANS IN 4-MANIFOLDS

SHINTARO FUSHIDA-HARDY

Abstract. Smooth 4-dimensional topology is full of mysteries. One class of mysteries is to understand embedded surfaces in 4-manifolds. In this talk I'll describe ongoing work with Devashi Gulati and Laura Wakelin in which we use trisections to study Lagrangian surfaces in symplectic 4-manifolds.

1. Introduction

(1) I graduated with a BSc in maths and physics here at the University of Auckland in 2017. I then did a BSc with Honours in 2018, under the instruction of Rod Gover.
(2) Since then, I've been studying towards a PhD at Stanford University. In a couple of weeks I'll be starting my fourth year there - I should finish in about two years from now. Stanford is on the west coast of the United States of America - this is my first time back home, and I'm very happy to be back!
(3) I'm working in low dimensional topology, advised by Ciprian Manolescu. I spend my time thinking about surfaces embedded in 4-manifolds. My main guiding question has been the "minimal genus problem" which I'll mention soon.
(4) Today I'll talk about a side project I'm working on with two other postgrads - Devashi Gulati at the University of Georgia (in Georgia, USA), and Laura Wakelin at Imperial College (in London).

2. Four dimensions

Why four dimensions?

To illustrate why we care about 4-dimensions, we'll consider smooth homotopy spheres in each dimension.

Definition 2.1. In dimension n, a *smooth homotopy sphere* is a closed smooth n-dimensional manifold S such that

- $\pi_n(S) = \mathbb{Z}$,
- $\pi_m(S) = 0$ for $m \neq n$.

We have the following list:

(1) A 1-dimensional homotopy sphere is a circle.
(2) A 2-dimensional homotopy sphere is a 2-dimensional sphere.
(3) More generally, every 3, 5, 6, 12, 56, and 61 dimensional homotopy spheres are spheres.

(4) In every dimension at least 5, there are finitely many distinct homotopy spheres. In principle the number can be computed in each dimension, but the general algorithm is too inefficient to implement in practice.
(5) In dimension 4, the result is completely unknown: maybe the only homotopy sphere is the sphere, or maybe there are even infinitely many distinct homotopy spheres.

In fact, there is not a single closed topological 4-manifold for which all compatible smooth structures are known. In general, this is referred to as the problem of

classifying exotic smooth structures.

Hopefully this demonstrates how four dimensions is hugely complicated, and explains why a lot of mathematicians work in four dimensional topology.

Some difficult open problems closer to my heart are the following:

- The *exotic embedding problem*. Suppose S and S' are embedded 2-dimensional spheres in $\mathbb{S}^4$. Suppose there's a topological isotopy from S to S'. Then is there a smooth isotopy from S to S'?
- What are the knotted surfaces in $\mathbb{S}^4$?
- Given a homology class $a \in H_2(X)$ for some 4-manifold X, what is the minimum genus of an embedded surface $\Sigma \subset X$ such that $[\Sigma] = a$?

3. Symplectic topology

Rather than working in the smooth category, in this talk we'll consider some more geometric structure.

Definition 3.1. A *symplectic manifold* is a manifold X equipped with a closed non-degenerate 2-form ω (which is called a *symplectic form.*

Symplectic manifolds are necessarily even dimensional, for the same reason that complex manifolds are even dimensional.

Recall that a Riemannian manifold is essentially a manifold together with a symmetric bilinear form. Here we're equipping a manifold with an anti-symmetric bilinear form.

Example. The *standard symplectic structure* on $\mathbb{R}^4$ is

$$(\mathbb{R}^4, dx \wedge dy + dz \wedge dw).$$

Example. The simplest *closed* symplectic 4-manifold is $\mathbb{CP}^2$. We can think of this space in homogeneous coordinates as

$$\{[z_1 : z_2 : z_3] | z_i \in \mathbb{C}\}.$$

The symplectic form is given by

$$\omega = -\frac{i}{2}\partial\overline{\partial}\log(|z_1|^2 + |z_2|^2 + |z_3|^2).$$

Definition 3.2. A submanifold $L \subset (X, \omega)$ is called *Lagrangian* if it is a maximal submanifold on which ω vanishes. (This forces $\dim L = \frac{1}{2}\dim X$.)

Example. In the standard symplectic $\mathbb{R}^4$, the (x,z)-plane is a Lagrangian submanifold. This is because given any pair of vector fields u and v in the plane, one can compute that $\omega(u,v) = 0$.

Why do we care about Lagrangians? I'll motivate them a little through physics. The main idea is that the *Hamiltonian formulation* of classical mechanics corresponds precisely with symplectic geometry. (The Hamiltonian formulation is when mechanics is encoded by a configuration space X of points (q,p) corresponding to position and momentum, and a *Hamiltonian* $H : X \to \mathbb{R}$ encoding the energy of each configuration.) In the correspondence, *Lagrangian submanifolds* correspond to all of the possible momenta of a given position.

What is the question I'm thinking about in my collaboration?

Question. Let X be a closed symplectic 4-manifold. What are all of the Lagrangian surfaces inside X?

First we need to know how we're counting them! two common methods are the following:

(1) *Smooth isotopy.*
(2) *Lagrangian isotopy.* (Smooth isotopy through Lagrangian submanifolds.)
(3) *Hamiltonian isotopy.* Lagrangains L and L' are *Hamiltonian isotopic* if there's an isotopy $X \times [0,1] \to X$ arising from some *Hamiltonian flow*, under which L maps to L'.

Classifying Lagrangians is very difficult, but here are some known and unknown results.

(1) Up to Lagrangian isotopy, there is a unique Lagrangian torus in $\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$. (Rizell-Goodman-Ivrii)
(2) Lagrangian spheres in $T^*\mathbb{S}^2$ are Hamiltonian isotopic to the 0-section. Similarly for $T^*\mathbb{T}^2$. (Hind, Rizell-Goodman-Ivrii)
(3) Up to Hamiltonian isotopy, we have no classification of all Lagrangians in any symplectic 4-manifold.

Our approach for classifying Lagrangians, introduced by Sarah Blackwell in her PhD thesis for $\mathbb{CP}^2$, is the following:

(1) Trisect (or multisect) the ambient manifold X.
(2) Embedded surfaces are determined up to isotopy by the intersections with the spine of the trisection.
(3) This cuts down dimensions, and counting Lagrangians in X should correspond to counting *grid diagrams.*

4. Trisections

The last big idea we need to introduce is trisections.

Definition 4.1. Let X be a closed oriented 4-manifold. A *trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that

- each X_i is of the form $\natural^k \mathbb{S}^1 \times B^3$ for some k,
- each $H_i = X_i \cap X_{i+1}$ is of the form $\natural^g \mathbb{S}^1 \times B^2$ for some g,

- and $\Sigma = X_1 \cap X_2 \cap X_3$ is a surface of genus g.

Theorem 4.2. *Given any closed oriented 4-manifold X, there is always a trisection of X. These are unique up to certain stabilisation move. Trisections are determined uniquely by their* spines $H_1 \cup H_2 \cup H_3$.

Example. $\mathbb{S}^4$ has a genus 0 trisection built up from lower dimensional balls and spheres.

Example. $\mathbb{CP}^2$ has a genus 1 trisection.

Theorem 4.3. *Importantly, we can encode trisections by* trisection diagrams *which consist of curves on a surface. I won't define these rigorously, but I'll explain them through examples.*

Example. Trisection diagram for $\mathbb{S}^4$.

Example. Trisection diagram for $\mathbb{CP}^2$.

We can also embed surfaces in a special way inside trisections.

Definition 4.4. Let $\mathcal{K} \subset X$ be an embedded surface in a 4-manifold. $\mathcal{K}$ is said to be in *bridge position* if the following hold:

- Each $\mathcal{K}_i = \mathcal{K} \cap X_i$ is a disjoint union of boundary-parallel disks.
- Each $\tau_i = \mathcal{K} \cap H_i$ is a disjoint union of boundary-parallel arcs.

Theorem 4.5. *Every surface $\mathcal{K} \subset X$ can be smoothly isotoped to lie in bridge position in any trisection of X. A surface in bridge position is determined by its spine (i.e. its intersection with the spine of the trisection of X).*

Example. Draw shadow diagram of $\mathbb{CP}^1 \subset \mathbb{CP}^2$. Explain how it builds up into a surface in a 4-manifold.

5. Trisections, but symplectic

Finally, the question is whether or not we can encode pairs (X, L) of embedded Lagrangian surfaces inside symplectic manifolds with diagrams as above that not only encode the smooth topology of our manifolds but also the geometry. This might be possible! It's the crux of my research with Devashi Gulati and Laura Wakelin.

To describe this I need to introduce a couple more definitions! These are essentially the analogues of Lagrangian surfaces in symplectic manifolds in odd dimensions rather than even dimensions.

Definition 5.1. A *contact 3-manifold* is a 3-manifold Y equipped with a non-integrable plane field ξ (called the contact distribution). (This means we're specifying a plane at every point in Y, and locally ξ is not the tangent bundle of any embedded surface.) Often contact manifolds are described as (Y, α) where α is a 1-form whose kernel is the contact distribution.

Example. $\mathbb{R}^3$ equipped with the 1-form $\alpha = dz - y dx$ is the *standard contact structure* on $\mathbb{R}^3$. If we consider the half space $\{(x, y, z, w) \mid w \geq 1\}$ in $\mathbb{R}^4$, the standard symplectic structure is $\omega = d\alpha$ on the boundary.

The analogue of Lagrangians in symplectic manifolds is Legendrian knots in contact manifolds. These are knots in Y that are everywhere tangent to the contact distribution.

Example. Draw example of a Legendrian unknot in a drawing of a contact distribution.

Legendrian knots are combinatorial objects: the *front projection* is a knot diagram with two additional properties:

(1) There are no vertical tangencies, instead there are cusps.
(2) In any overpass, the overpass has a larger gradient than the underpass.

The key idea connecting all of this to our original question of classifying Lagrangians is that if (X, L) is a Lagrangian surface embedded in a symplectic manifold X, and X has a compatible contact boundary Y, then L intersects Y along a Legendrian knot. With this in mind, we're finally ready to bring everything together.

Definition 5.2. A *Weinstein trisection* of a closed symplectic manifold X is a trisection (X_1, X_2, X_3) in which each X_i is a Weinstein domain. This essentially means the boundary of each X_i is a compatible contact manifold ∂X.

Theorem 5.3 (Lambert-Cole, Meier, Starkston)**.** *Every closed symplectic manifold admits a Weinstein trisection.*

6. The classification programme

We're now ready to outline the programme for classifying Lagrangian surfaces in some symplectic manifolds.

(1) Suppose (X, L) is a Lagrangian surface embedded in a symplectic manifold.
(2) Weinstein trisect the manifold X.
(3) Lagrangian isotope the surface L, so that it's also in *bridge position* in the trisection. Now L is determined by its intersection with the spine of the trisection.
(4) Since L is a Lagrangian and the trisection is a Weinstein trisection, the spine of L is a collection of three Legendrian knots (links), satisfying some properties.

The idea is that classifying these triples of Legendrian links should correspond to classifying Lagrangians. Classifying Legendrian links is an easier problem because it's combinatorial. For all of this to work, we need two directions to hold:

(1) Every set of three Legendrians (with certain properties) must uniquely determine a Lagrangian, up to Lagrangian isotopy.
(2) Every Lagrangian must have a corresponding set of three Legendrians that encodes it.

In my collaboration with Devashi and Laura, we've established the first fact, but have yet to verify the second. We have a concrete example showing that the second statement is actually false in general:

Example. There is only one valid triple of Legendrian links in the spine of $\mathbb{S}^2 \times \mathbb{S}^2$ encodes a Lagrangian torus in $\mathbb{S}^2 \times \mathbb{S}^2$. However, $\mathbb{S}^2 \times \mathbb{S}^2$ also contains a Lagangrian sphere.

Our hope is that there's some condition we can determine which tells us when Lagrangians really arise from these triples of Legendrians.

In summary, we have a programme for possibly classifying some Lagrangian surfaces in certain symplectic 4-manifolds. It's still on-going but we're having a lot of fun!