# A non-visual proof that higher homotopy groups are abelian

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This document proves that homotopy groups $\pi_n(X)$ are abelian, for $n \geq 2$. Often this result is proven with just a picture, so this is for those souls who desire a proof without any drawings whatsoever.

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# 1 Introduction

The most popular proof that homotopy groups $\pi_n(X)$ are abelian (for $n \geq 2$) is entirely visual. The idea is that the non-trivial parts of the domains of maps $f, g : \mathbb{S}^n \to X$ can be "shrunk down" so that there is enough space to slide $[f]$ over $[g]$. Then $[f][g] = [g][f]$. While this can be made rigourous, it can also be dissatisfying. For this reason, we give an entirely algebraic proof that homotopy groups are abelian, using the Eckmann-Hilton argument.

# 2 Eckmann-Hilton argument

The Eckmann-Hilton argument is the following result from algebra.

**Theorem 2.1.** *Let $\times$ and $\otimes$ be two unital binary operations on a set $X$. Suppose*

$$(a \times b) \otimes (c \times d) = (a \otimes c) \times (b \otimes d)$$

*for all $a, b, c, d \in X$. Then $\times$ and $\otimes$ are in fact the same operation, and are commutative and associative.*

*Proof.* We first show that $\times$ and $\otimes$ have the same unit. Let $1_\times$ and $1_\otimes$ denote their respective units. Then

$$1_\times = 1_\times \times 1_\times = (1_\times \otimes 1_\otimes) \times (1_\otimes \otimes 1_\times) = (1_\times \times 1_\otimes) \otimes (1_\otimes \times 1_\times) = 1_\otimes \otimes 1_\otimes = 1_\otimes.$$

We hereafter write 1 to denote the shared unit. Next we use this to prove that $\times$ and $\otimes$ are really the same operation. Let $a, b \in X$. Then

$$a \times b = (a \otimes 1) \times (1 \otimes b) = (a \times 1) \otimes (1 \times b) = a \otimes b.$$

It remains to show that $\times$ is commutative and associative. For commutativity, observe that

$$a \times b = (1 \otimes a) \times (b \otimes 1) = (1 \times b) \otimes (a \times 1) = b \otimes a = b \times a.$$

Finally for associativity, let $a, b, c \in X$. Then

$$(a \times b) \times c = (a \times b) \times (1 \times c) = (a \times 1) \times (b \times c) = a \times (b \times c).$$

$\square$

**Remark.** This result can be considered an algebraification of the "sliding over" argument, if we write $a \times b$ as $ab$, and $a \otimes b$ as $\substack{a \\ b}$.

## 3 Higher homotopy groups are abelian

We use the Eckmann-Hilton argument to prove that higher homotopy groups are abelian. First let us recall the relevant definitions.

**Definition 3.1.** Let $(X, x)$ be a path connected pointed space. The underlying set of $\pi_n(X)$ is the collection of homotopy classes of maps

$$f : \mathbb{S}^n \to X,$$

(where $\mathbb{S}^n$ is also pointed.) Equivalently, $\pi_n(X)$ consists of homotopy classes of maps

$$f : [0,1]^n \to X$$

for which $\partial[0,1]^n$ is mapped onto $x$.

**Proposition 3.2.** The set $\pi_n(X)$ forms a group (called the $n$th homotopy group of $X$) when equipped with the following product: let $[f], [g] \in \pi_n(X)$. Then define $[f] \star [g]$ to be the homotopy class of the map $f \star g$ defined by

$$(f \star g)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & t_1 \in [0, 1/2] \\ g(2t_1 - 1, t_2, \ldots, t_n) & t_1 \in [1/2, 1]. \end{cases}$$

Observe that $\star$ is unital, where the unit is the homotopy class of the constant map $[0,1]^n \mapsto x$. To use the Eckmann-Hilton argument, we define an alternative binary operation.

**Definition 3.3.** Let $[f], [g] \in \pi_n(X)$. Then define $[f] \times [g]$ to be the homotopy class of the map $f \times g$ defined by

$$(f \times g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, 2t_2, t_3 \ldots, t_n) & t_2 \in [0, 1/2] \\ g(t_1, 2t_2 - 1, t_3, \ldots, t_n) & t_2 \in [1/2, 1]. \end{cases}$$

It is clear that this operation is also unital, with the same unit. To make use of the Eckmann-Hilton argument, it remains to prove that for any $f, g, h, k$,

$$([f] \times [g]) \star ([h] \times [k]) = ([f] \star [h]) \times ([g] \star [k]).$$

The left hand side is defined to be the homotopy class of

$$(f \times g) \star (h \times k)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3 \ldots, t_n) & t_1 \leq 1/2, t_2 \leq 1/2 \\ g(2t_1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \leq 1/2, t_2 \geq 1/2 \\ h(2t_1 - 1, 2t_2, t_3 \ldots, t_n) & t_1 \geq 1/2, t_2 \leq 1/2 \\ k(2t_1 - 1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \geq 1/2, t_2 \geq 1/2. \end{cases}$$

The right hand side is the homotopy class of

$$(f \star h) \times (g \star k)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3 \ldots, t_n) & t_1 \leq 1/2, t_2 \leq 1/2 \\ h(2t_1 - 1, 2t_2, t_3 \ldots, t_n) & t_1 \geq 1/2, t_2 \leq 1/2 \\ g(2t_1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \leq 1/2, t_2 \geq 1/2 \\ k(2t_1 - 1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \geq 1/2, t_2 \geq 1/2. \end{cases}$$

For fixed $f, g, h, k$, these are not only homotopic, but exactly the same map! Either way, since these two maps are homotopic, the premise for the Eckmann-Hilton argument is satisfied. It follows that $\star$ is commutative, so higher homotopy groups are abelian.

**Remark.** This proof makes it clear why the fundamental group need not be abelian, even though higher homotopy groups are. We simply do not have enough space in $[0, 1]$ to carry out the same argument.

**Remark.** This proof was genuinely algebraic, no messy homotopies needed to be defined to establish that higher homotopy groups are abelian.