Homology 3-spheres

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This document contains notes on the topology of homology 3-spheres, largely following Saveliev [Sav12]. Some solutions to exercises are also given. These notes were written as part of a reading course with Ciprian Manolescu in Spring 2020. (These notes are not self-contained, and assume some knowledge of results from my knot theory notes also from Spring 2020.)

Contents

Chapter 1

Constructing homology spheres

In this chapter we review some basic results from 3-dimensional topology, starting with Heegaard splittings. We then construct some homology spheres.

1.1 Heegaard splittings and the mapping class group

Let X, Y be manifolds with boundary, $\partial X \cong \partial Y$. Let $f : \partial X \to \partial Y$ be a homeomorphism. Then $X \sqcup_f Y$ is a new manifold without boundary. A *Heegaard splitting* is an example of this procedure, where a 3-manifold X is decomposed as $H_g \sqcup_f H'_g$ for *genus g handlebodies* H_g, H'_g.

Definition 1.1.1. A *handlebody of genus g* is a 3-manifold with boundary obtained by attaching g copies of 1-handles ($D^2 \times [-1, 1]$) to the 3-ball D^3. (The gluing homeomorphism attaches the disks $D^2 \times \{1, -1\}$ to disjoint disks in $\partial D^3 = \mathbb{S}^2$.)

The boundary of a handlebody of genus g is the unique (up to homeomorphism) closed surface of genus g, which we denote by Σ_g. Using this notion of handlebodies, we can define *Heegaard splittings*.

Definition 1.1.2. Let M be a closed 3-manifold. A *Heegaard splitting* of M of genus g is a decomposition $M = H_g \sqcup_f H'_g$, where H_g, H'_g are handlebodies of genus g, and $f : \partial H_g \to \partial H'_g$ is a homeomorphism.

It turns out that every 3-manifold admits a Heegaard splitting of some genus! This is a big step in understanding the topology of 3-manifolds.

Theorem 1.1.3. *Every closed orientable 3-manifold admits a Heegaard splitting.*

This is often proven using Morse theory, but here we give a proof using triangulations. Recall that a triangulation is a decomposition of a manifold into simplices, and forms an intermediate tier of structure between smooth manifolds and topological manifolds.

Definition 1.1.4. A triangulation of a topological space X is a simplicial complex K together with a homeomorphism $K \to X$.

Proof. Without further ado, we prove that 3-manifolds admit Heegaard splittings. Let M be a closed orientable 3-manifold, and T a triangulation of M. Each vertex of T has a neighbourhood homeomorphic to $0 \times D^3$, each edge a neighbourhood homeomorphic to $D^1 \times D^2$, each face, $D^2 \times D^1$, and each cell, $D^3 \times 0$. Taking appropriate intersections, M can be expressed as a union of these pieces glued along their boundaries. Let the neighbourhoods of vertices and edges define H_g, and faces and cells define H'_g. This is a Heegaard decomposition of M. □

Next we discuss ways to compare Heegaard splittings of a given 3-manifold. The most naive equivalence is the structure of an automorphism of M which restricts to homeomorphisms on the components H_g, H'_g. However, another natural equivalence is *stable equivalence.*

Definition 1.1.5. Let $M = H_g \cup H'_g$ be a Heegaard splitting. *Stabilisation* is the following procedure:

1. Attach an additional unknotted 1-handle h to H_g, to obtain H_{g+1}. Since H_g is a submanifold of M, "unkotted" is formalised by saying that the core of the 1-handle bounds an embedded disk D^2 in M.

2. Let h' denote a "thickening" of the embedded disk. Then $h \cup h' \cup H_g$ is homeomorphic to H_g (since $h \cup h'$ is just a boundary connected sum D^3 with H_g). Therefore M decomposes as $h \cup h' \cup H_g \cup H'_g$.

3. By studying the boundary of $h \cup h' \cup H_g$, we see that h' intersects H'_g along two disjoint disks. Therefore $H'_g \cup h'$ is a handlebody of genus $g+1$, which we denote by H'_{g+1}.

In summary we have $M = H_{g+1} \cup H'_{g+1}$, so from our genus g Heegaard splitting we can canonically obtain a genus $g+1$ Heegaard splitting. (This process is stabilisation.)

Definition 1.1.6. Two Heegaard splittings of a 3-manifold M are said to be *equivalent* if there is an automorphism of M which restricts to homeomorphisms on the components of the Heegaard splittings. Two Heegaard splittings are said to be *stably equivalent* if they are equivalent after stabilising each splitting some number of times.

Not all Heegaard splittings of a given 3-manifold are equivalent. For example, the 3-sphere admits a genus 0 Heegaard splitting - consider an embedded 2-sphere in the 3-sphere. By the Schönflies theorem, the 2-sphere separates the 3-sphere into two 3-balls.

On the other hand, the 3-sphere also admits a genus 1 Heegaard splitting. Take two solid tori with meridians and longitudes μ_1, μ_2 and λ_1, λ_2 respectively. Gluing the solid tori along a surface homeomorphism mapping μ_1 onto λ_2 and μ_2 onto λ_1 also results in a 3-sphere.

In fact, this process is exactly stablisation! The genus 1 Heegaard splitting of the 3-sphere described above is a stabilisation of the genus 0 Heegaard splitting. This is a general phenomenon, it turns out that all Heegaard splittings of a given manifold are stably equivalent.

Theorem 1.1.7. *Any two Heegaard splittings of a closed orientable 3-manifold are stably equivalent.*

This result is due to Singer. One way that this can be proven (as is done in Saveliev) is to first show that any two Heegaard splittings induced from triangulations are stably equivalent, and then show that any Heegaard splitting is stably equivalent to one induced from a triangulation.

To study the homeomorphism classes of manifolds that can arise from gluing handlebodies along their boundaries, we must understand surface homeomorphisms. First we observe that any two *isotopic* aurface homeomorphisms necessarily give rise to the same manifold.

Lemma 1.1.8. Suppose U, V are 3-manifolds with homeomorphic boundaries, and that $h_0, h_1 : \partial U \to \partial V$ are *isotopic* homeomorphisms. Then $U \sqcup_{h_0} V$ and $U \sqcup_{h_1} V$ are homeomorphic.

Recall that homeomorphisms are said to be isotopic if they are homotopic, and the homotopy gives a homeomorphism at every t. Thus to understand manifolds of the form $U \sqcup_f V$, or even just $U \sqcup_f U$, we wish to understand isotopy classes of automorphisms of ∂U. This is exactly the *mapping class group.*

Definition 1.1.9. The *mapping class group* of an oriented manifold M is

$$\operatorname{Mod}(M) := \operatorname{Aut}^+(M)/\operatorname{Aut}_0(M),$$

where $\operatorname{Aut}_0(M)$ denotes the isotopy class of the identity.

We describe some essential properties and examples of mapping class groups here. The most important property we use is that mapping class groups are generated from *Dehn twists.*

Definition 1.1.10. A *Dehn twist* is an automorphism of a surface F isotopic to the following map T:

- Let $A \subset F$ be an embedded annulus $\mathbb{S}^1 \times [0, 1]$.
- Define $T : F \to F$ to be the identity on $\Sigma - A$.

- Define T by $T(e^{i\theta}, t) = (e^{i(\theta - 2\pi t)}, t)$ on A.

Theorem 1.1.11 (Dehn-Lickorish theorem)**.** *For $g \geq 0$, the mapping class group $\mathrm{Mod}(\Sigma_g)$ is generated by $3g - 1$ Dehn twists about non-separating simple closed curves.*

Here Σ_g denotes the closed surface of genus g. Excellent exposition on the mapping class group (and the above result) is available in Farb and Margalit [FM12]. In fact, the following improvement can be made:

Theorem 1.1.12 (Wajnryb)**.** *For $g \geq 3$, the mapping class group of Σ_g (or Σ_g with one boundary component) is finitely presented by $2g + 1$ generators (corresponding to Dehn twists). The relations are described in [FM12].*

Theorem 1.1.13. *There is an isomorphism $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2, \mathbb{Z})$.*

Proof. We give a proof outline. Every pair $(p, q) \in \mathbb{Z}^2$ determines a straight arc $\alpha_{p,q}$ in $\mathbb{R}^2$ with endpoints at $(0, 0)$ and (p, q). If $p : \mathbb{R}^2 \to T^2$ is the usual universal cover, then every such arc descends to a closed curve on T^2. If $(p, q) \in \mathbb{Z}^2$ is *primitive*, i.e. if $\gcd(p, q) = 1$, then $\alpha_{p,q}$ descends to a simple closed curve.

Using the Jordan curve theorem in $\mathbb{R}^2$, one can show the following result:

Lemma 1.1.14. Homotopy classes of simple closed curves on T^2 are in bijective correspondence with primitive elements of $\pi_1(T^2) \cong \mathbb{Z}^2$.

To more closely study the relationships between simple closed curves, we use *intersection numbers*. If α, β denote two curves on a surface in general position, their *geometric intersection number* denoted $i(\alpha, \beta)$ is the minimal number of intersections between representatives of free homotopy classes of α and β. The *algebraic intersection number* denoted $\hat{i}(\alpha, \beta)$ is the signed count of intersections of two oriented simple closed curves α, β in general position. Using the previous lemma together with Bezout's identity, the following results can be obtained:

Lemma 1.1.15. Let α, β be simple closed curves on T^2. These correspond to primitive elements of $\mathbb{Z}^2$, which we denote by (p, q) and (p', q'). Then

$$i(\alpha, \beta) = \left| \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \right|, \quad \hat{i}(\alpha, \beta) = \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}.$$

Next we describe the isomorphism $\sigma : \mathrm{Mod}(T^2) \to \mathrm{SL}(2, \mathbb{Z})$. This is induced by a map

$$\sigma : \mathrm{Mod}(T^2) \to \mathrm{Aut}(\pi_1(T^2)) \cong \mathrm{GL}(2, \mathbb{Z}).$$

The latter map is the canonical map sending $f : T^2 \to T^2$ to $f_* : \pi_1(T^2) \to \pi_1(T^2)$. To see that this restricts to a map into $\mathrm{SL}(2, \mathbb{Z})$, we use the algebraic intersection number.

Suppose α, β are simple closed curves on T^2, and φ an orientation preserving automorphism of T^2. Then

$$\hat{i}(\alpha, \beta) = \hat{i}(\varphi \circ \alpha, \varphi \circ \beta).$$

If $A = \sigma(\varphi)$, one can show from the above that

$$\det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \det A \det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}.$$

By choosing α, β with intersection number non-zero, it follows that $\det A = 1$, so the image of σ lies in $\mathrm{SL}(2, \mathbb{Z})$.

It is straight forward to show that $\sigma : \mathrm{Mod}(T^2) \to \mathrm{SL}(2, \mathbb{Z})$ is surjective by constructing a map $\eta : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{Mod}(T^2)$ as follows: A matrix A defines an orientation preserving automorphism of the plane. This descends to an automorphism of T^2. Finally to see that $\sigma : \mathrm{Mod}(T^2) \to \mathrm{SL}(2, \mathbb{Z})$ is injective, we show that the map $\eta : \mathrm{SL}(2, \mathbb{Z}) \to \mathrm{Mod}(T^2)$ is surjective: lift arbitrary automorphisms ψ of T^2 to automorphisms $\widetilde{\psi}$ of $\mathbb{R}^2$, and show that the straight line homotopy from $\widetilde{\psi}$ to $\widetilde{(\eta \circ \sigma)}(\psi)$ is equivariant under Deck transformations. Therefore the straight line homotopy descends to an isotopy between ψ and $(\eta \circ \sigma)(\psi)$ in $\mathrm{Mod}(T^2)$. □

This correspondence result allows us to interpret elements of the mapping class group by their actions on simple closed curves. In particular, an element A of $\mathrm{SL}(2, \mathbb{Z})$ is determined uniquely by the images of $(1, 0)$ and $(0, 1)$ under A. But these elements of $\mathbb{Z}^2$ correspond exactly to the meridian and longitude of the torus! In summary, *any orientation preserving (or reversing) homeomorphism of $\Sigma_1 = T^2$ is completely determined by where it sends the meridian and longitude of the torus.*

Also observe that, $\mathrm{Mod}(T^2)$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

These correspond to Dehn twists about the meridian and longitude of T^2.

Note that the following result also holds (although it is not of interest for studying Heegaard splittings).

Theorem 1.1.16. *(Generalising the previous theorem), in the homotopy category, for any n, $\mathrm{Mod}(T^n) \cong \mathrm{SL}(n, \mathbb{Z})$.*

1.2 Lens spaces and Seifert manifolds

Here we begin classifying Heegaard splittings by *Heegaard genus*.

Definition 1.2.1. Let M be a closed oriented 3-manifold. The *Heegaard genus* of M is the minimal genus of a component H_g of a Heegaard splitting of M.

The first important examples are captured in the following theorem.

Theorem 1.2.2. *The unique closed 3-manifold with Heegaard genus 0 is the 3-sphere. The unique closed 3-manifolds with Heegaard genus* 1 *are either* $\mathbb{S}^1 \times \mathbb{S}^2$*, or lens spaces* $L(p, q)$ *for* p, q *coprime,* $p \geq 2$*, and* $1 \leq q \leq p - 1$*.*

Of course, lens spaces have yet to be defined. They will be introduced naturally while we give a proof outline of the theorem.

For the first part, we noted that (by Alexander's lemma) if M and M' are obtained by gluing two genus g handlebodies along homeomorphisms f and f' which are isotopic, then $M \cong M'$. Since $\mathrm{Mod}(\mathbb{S}^2) = 1$, any two homeomorphisms of $\mathbb{S}^2$ are isotopic, so any manifold with Heegaard genus 0 is homeomorphic to $\mathbb{S}^3$.

For the second part, suppose $M = H_1 \sqcup_f H_1'$, where $f : \partial H_1 \to \partial H_1'$ is a homeomorphism. Then the homeomorphism class of M is determined by the isotopy class of f. In fact, $\partial H_1 = T^2$, so the homeomorphism class of M is determined by an element of $\mathrm{SL}(2, \mathbb{Z}) = \mathrm{Mod}(T^2)$. Recall further that $A \in \mathrm{SL}(2, \mathbb{Z})$ is determined by where it sends $(1, 0)$ and $(0, 1)$, which correspond to meridians and longitudes. With these considerations in mind, we can classify 3-manifolds of Heegaard genus 1.

We establish some notation. Let μ_1, λ_1 be the meridian and longitude of ∂H_1, and μ_2, λ_2 the meridian and longitude of $\partial H_1'$. Let $f \in \mathrm{Mod}(T^2)$, and let $A = (a_{ij})$ be the matrix in $\mathrm{SL}(2, \mathbb{Z})$ representing f with respect to the bases μ_i, λ_i. By our previous remarks,

$$\mu_1 \mapsto a_{11}\mu_2 + a_{21}\lambda_2, \quad \lambda_1 \mapsto a_{12}\mu_2 + a_{22}\lambda_2$$

completely determines f. In fact, in the context of using f as a gluing map in a Heegaard splitting, the homeomorphism type of the result 3-manifold depends only on the image of μ_1!

Lemma 1.2.3. Suppose $M = H_1 \sqcup_f H_1'$. Then with notation as above, the homeomorphism type of M depends only on $f(\mu_1)$.

Proof. The idea is to glue H_1 to H_1' along f in two steps. First isolate a regular neighbourhood $\mathbb{S}^1 \times D^1$ of μ_1 in ∂H_1. Since μ_1 is a meridian, this extends to $D^2 \times D^1$ in H_1. We can write

$$M = ((D^2 \times D^1) \cup (H_1 - D^2 \times D^1)) \sqcup_f H_1'$$

Then $D^2 \times D^1$ is glued to H_1' along f according to the image of μ_1 (and the convention that gluing maps are always orientation reversing). It remains to glue $H_1 - D^2 \times D^1$ to $(D^2 \times D^1) \sqcup_f H_1'$. But both pieces now have boundaries homeomorphic to 2-spheres, so by the first argument, no more choices can be made (up to homeomorphism). □

We are now ready to define *lens spaces.* These are exactly the spaces obtained by the above gluing procedure.

Definition 1.2.4. The *lens space* $L(p, q)$ is the closed orientable 3-manifold obtained from a Heegaard splitting $H_1 \sqcup_f H_1'$, where f sends μ_1 to $-q\mu_2 + p\lambda_2$.

Remark. In the previous proof, we remarked that gluing maps are conventionally orientation reversing. This means each element $A \in \mathrm{SL}(2, \mathbb{Z})$ determines a gluing map τA, where

$$\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is implicitly employed from here on out without further mention.

We now attempt to classify lens spaces.

The meridian of a genus 1 handlebody is fixed, but longitudes are defined modulo meridians. Therefore one can replace λ_1 by $n\mu_1 + \lambda_1$ for any integer n. The corresponding change of basis for $A \in \mathrm{SL}(2, \mathbb{Z})$ has the effect of adding n times the first column of A to the second column. Similarly, replacing λ_2 with $n\mu_2 + \lambda_2$ corresponds to subtracting n times the second row from the first row.

Let $A \in \mathrm{SL}(2, \mathbb{Z})$ have entries

$$A = \begin{pmatrix} q & r \\ p & s \end{pmatrix},$$

in some basis $\{\mu_i, \lambda_i\}$.

- Case 1. $p = 0$. Then $qs = 1$, so without loss of generality $q = s = 1$. (The ambiguity in sign corresponds to the choice of orientation of meridians and longitudes.) The corresponding 3-manifold is determined by

  $$\mu_1 \mapsto -\mu_2.$$

  The meridians each bound embedded disks, and restricting the gluing map to regular neighbourhoods of the disks gives $\mathbb{S}^2 \times D^1$. Globally we obtain $\mathbb{S}^2 \times \mathbb{S}^1$.

- Case 2. $p \neq 0$. Then by changing bases, the matrix A is of the form

  $$A = \begin{pmatrix} q' & r' \\ p & s \end{pmatrix},$$

  where $0 \leq q' \leq p - 1$.

  - If $p = 1$, then $q' = 0$. Moreover, we can also eliminate the bottom right entry, so that in some basis

    $$A = \begin{pmatrix} 0 & r'' \\ 1 & 0 \end{pmatrix}.$$

    This forces $r'' = 1$. Therefore the gluing map exactly sends meridians to longitudes and longitudes to meridians. This is the stabilisation procedure, which familiarly gives us $\mathbb{S}^3$.

– If $p \geq 2$, more possibilities exist.

This gives us a weak classification of lens spaces:

Theorem 1.2.5. *Non-trivial Lens spaces are of the form $L(p,q)$, for p,q coprime, $p \geq 2$, and $0 \leq q \leq p-1$. If $p = 0$, then $L(p,q) \cong \mathbb{S}^1 \times \mathbb{S}^2$. If $p = 1$, then $L(p,q) = \mathbb{S}^3$.*

This completes our classification of 3-manifolds of Heegaard genus 0 and 1. Next we improve our classification of lens spaces, without proof (for the most part).

Theorem 1.2.6. *Let $L(p,q)$ and $L(p',q')$ be lens spaces. Then*

- *$L(p,q)$ and $L(p',q')$ are homotopy equivalent if and only if $p' = p$ and $qq' = \pm n^2 \mod p$ for some integer n.*
- *$L(p,q)$ and $L(p',q')$ are homeomorphic if and only if $p' = p$ and $q' = \pm q^{\pm 1} \mod p$.*

We don't prove the parts that depend on q, but determine the homology and fundamental group of Lens spaces. This turns out to depend only on p! In particular, we can distinguish lens spaces if they have distinct values of p, as above.

Proposition 1.2.7. Let $L(p,q)$ be a lens space. Then $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$. Moreover,

$$H_k(L(p,q)) = \begin{cases} \mathbb{Z} & k \in \{0,3\} \\ \mathbb{Z}/p\mathbb{Z} & k = 1 \\ 0 & k = 2. \end{cases}$$

Proof. To prove that $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$, we use the Seifert-van Kampen theorem. Explicitly we have

$$\pi_1(L(p,q)) = \pi_1(H_1) *_{\pi_1(\partial H_1)} \pi_1(H_1') = \langle \lambda_1 \rangle *_{\langle \mu_1, \lambda_1 \rangle} \langle \lambda_2 \rangle.$$

But the lens space $L(p,q)$ identifies μ_1 with $-q\mu_2 + p\lambda_2$. Therefore

$$\langle \lambda_1 \rangle *_{\langle \mu_1, \lambda_1 \rangle} \langle \lambda_2 \rangle = \langle \lambda_1 \rangle *_{\langle p\lambda_2, \lambda_1 \rangle} \langle \lambda_2 \rangle = \langle \lambda_2 \rangle / \langle p\lambda_2 \rangle = \mathbb{Z}/p\mathbb{Z}.$$

For the next claim, notice that lens spaces are orientable, so by Poincaré duality, $H_2 \cong H^1 = 0$. By Hurewicz's theorem H_1 is exactly π_1, and by connectedness and orientability H_0 and H_3 are both $\mathbb{Z}$. □

Next we introduce a generalisation of lens spaces called *Seifert manifolds* or *Seifert fibred spaces*. These provide examples of closed 3-manifolds with Heegaard genus 2.

Definition 1.2.8. A *Seifert manifold* is a closed orientable 3-manifold constructed as follows:

- Let F be a 2-sphere with the interiors of n disjoint disks D_i^2 removed. Then F is homotopic to $\bigvee_{i=1}^{n-1} \mathbb{S}^1$, so it has fundamental group the free group on $n-1$ generators. More visually, if x_i represents a curve in F homotopic to ∂D_i^2, then

$$\pi_1(F) = \langle x_1, \ldots, x_n \mid x_1 \cdots x_n = 1 \rangle.$$

- Consider the product $F \times \mathbb{S}^1$. This is a compact orientable 3-manifold whose boundary is $\bigcup_i \partial D_i^2 \times \mathbb{S}^1$, i.e. a disjoint union of tori. The fundamental group is presented by

$$\pi_1(F \times \mathbb{S}^1) = \langle x_1, \ldots, x_n, h \mid hx_i = x_i h, x_1 \cdots x_n = 1 \rangle,$$

where h represents the factor $\mathbb{S}^1$.

- To obtain a closed manifold, we "fill" the tori (boundary components) with solid tori. Consider pairs of coprime integers $\{(a_i, b_i) : a_i \geq 2, 1 \leq i \leq n\}$. As with lens spaces, we glue in solid tori by specifying that a meridian on the ith solid torus maps to a curve in $\partial D_i^2 \times \mathbb{S}^1$ isotopic to $a_i x_i + b_i h$.

The closed manifold obtained in this way is called a *Seifert manifold*. The image of $0 \times \mathbb{S}^1 \subset D^2 \times \mathbb{S}^1$ (under the gluing map) is called the *ith singular fibre*.

In particular, the above construction gives the *Seifert manifold* $M((a_1, b_1), \ldots, (a_n, b_n))$ *of genus 0 with n singular fibres*. The construction generalises to arbitrary initial orientable closed surfaces (rather than just the 2-sphere), so the genus refers to the initial surface.

We mentioned in the above construction that

$$\pi_1(F \times \mathbb{S}^1) = \langle x_1, \ldots, x_n, h \mid hx_i = x_i h, x_1 \cdots x_n = 1 \rangle.$$

By gluing in additional tori, we can compute the fundamental group of $M((a_1, b_1), \ldots, (a_n, b_n))$ by the Seifert-van Kampen theorem. Specifically,

$$\pi_1(M((a_1, b_1), \ldots, (a_n, b_n))) = \pi_1(F \times \mathbb{S}^1) *_{\langle \mu_1, \lambda_1 \rangle} \langle \lambda_1 \rangle * \cdots *_{\langle \mu_n, \lambda_n \rangle} \langle \lambda_n \rangle.$$

By the identification $\mu_i = a_i x_i + b_i h$, this gives

$$\pi_1(M((a_1, b_1), \ldots, (a_n, b_n))) = \langle x_1, \ldots, x_n, h \mid hx_i = x_i h, x_1 \cdots x_n = 1, x_i^{a_i} h^{b_i} = 1 \rangle.$$

In particular, for $n \geq 3$, the fundamental group is generally not abelian! Therefore the Seifert fibred space cannot have Heegaard genus 0 or 1.

Example. Let M be a Seifert manifold.

- If M has one singular fibre, M is a lens space by definition.
- If M has two singular fibres, M is also a lens space! We explore this in more detail below.

- If M has at least three singular fibres, M has non-abelian fundamental group, so is *not* a lens space.

In the 2000s Perelman proved the *elliptisation conjecture*, from which we can see why the second point is true.

Theorem 1.2.9. *Let M be a closed orientable manifold with finite fundamental group. Then M is a* spherical manifold*. This means $M \cong \mathbb{S}^3/\Gamma$, where Γ is a finite subgroup of* $\mathrm{SO}(4)$ *acting freely by rotations. Moreover, $\pi_1(M) \cong \Gamma$.*

Proof. This result is equivalent to the Poincaré conjecture (which is of course true, due to Perelman). It is clear how we can get from elliptisation to the Poincaré conjecture. For the converse, suppose M is orientable and closed with finite fundamental group. Then it has a universal cover

$$p : \widetilde{M} \to M.$$

Since the universal cover is orientable and connected, $H_1(\widetilde{M}) = H_3(\widetilde{M}) = \mathbb{Z}$. On the other hand, $H_1(\widetilde{M})$ vanishes since $\pi_1(\widetilde{M}) = 1$, and by Poincaré duality, so does $H_2(\widetilde{M})$. It follows that $\widetilde{M}$ is a simply connected homology sphere. By Whitehead's theorem, it is then a homotopy sphere. By the Poincaré conjecture, it is homeomorphic to a sphere. This means we have a covering map

$$p : \mathbb{S}^3 \to M.$$

Now $M \cong \mathbb{S}^3/\operatorname{Aut}(p)$, and we are done. □

If $M((a_1, b_1), (a_2, b_2))$ is a Seifert manifold with two singular fibres, then its fundamental group is

$$\pi_1(M) = \langle x_1, x_2, h \mid hx_i = x_i h, x_1 x_2 = 1, x_1^{a_1} h^{b_1} = 1, x_2^{a_2} h^{b_2} = 1 \rangle.$$

Using Tietze transformations, this gives

$$\pi_1(M) = \langle x_1, h \mid hx_1 = x_1 h, x_1^{a_1} h^{b_1} = 1, x_1^{-a_2} h^{b_2} = 1 \rangle.$$

This gives a finite abelian group on two generators, and using the fact that (a_i, b_i) are coprime, it follows from the Chinese remainder theorem that $\pi_1(M)$ is finite cyclic. Therefore by the elliptisation conjecture, $M = \mathbb{S}^3/\Gamma$ where Γ is some cyclic group of rotations. This is exactly an alternative definition that can be used to define lens spaces.

Proposition 1.2.10. A Seifert manifold of genus g is a circle bundle over an orbifold of genus g.

Proof. This is more of an intuitive statement. In the construction of a Seifert manifold $M((a_1, b_1), \ldots, (a_n, b_n))$ above, we start with $F = \mathbb{S}^2 - \operatorname{int}(\cup_i D_i)$ for some disjoint disks D_i. These form the regular points of the orbifold, and each disk will contain a single orbifold point. (More generally, a genus g Seifert surface is obtain by removing disks from $\#^g T^2$

instead of $\mathbb{S}^2$.) We then glue solid tori into the trivial bundle $\mathbb{S}^1 \times F$. Above each disk, the torus is" twisted" by b_i/a_i. This gives the degree of the orbifold point at the center of each disk. □

Remark. A significant benefit of this interpretation is that we can declare that an orientable closed 3-manifold M is a Seifert manifold if M is equipped with an $\mathbb{S}^1$ action, and the action is free away from some number of points. These points correspond to the orbifold points of the underlying orbifold surface.

This gives further intuition regarding the fundamental group of a Seifert manifold. Rather than using Seifert-van Kampen, we can determine the fundamental group by considering the *orbifold fundamental group.* Suppose Σ is an orbifold surface of genus 0, and orbifold points $p_1, \ldots, p_n$, with degrees $d_1, \ldots, d_n$. Then the fundamental group of Σ is generated by loops around each p_i, and each of these loops must have order at most d_i. Moreover, any $n-1$ loops determines the nth loop, so the *orbifold fundamental group* is

$$\pi_1(\Sigma) = \langle x_1, \ldots, x_n \mid x_1^{d_1} = 1, \ldots, x_n^{d_n} = 1, x_1 \cdots x_n = 1 \rangle.$$

Observe that these are generalisations of von Dyck groups! The circle bundle over this orbifold is trivial away from the orbifold points, so we obtain a central extension

$$0 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(\Sigma) \to 0.$$

1.3 Dehn surgery

Another powerful description of orientable closed 3-manifolds is that they are all obtained from the sphere by Dehn surgery along links. While Heegaard splittings give representations of 3-manifolds via *Heegaard diagrams*, surgery provides a representation via *surgery diagrams.* In this section we briefly describe Dehn surgery, although more detail is given in my notes on knot theory. We then describe lens spaces and Seifert manifolds in terms of surgery.

Theorem 1.3.1 (Lickorish-Wallace theorem)**.** *Every closed orientable 3-manifold is obtained by* integral surgery *on a link $L \subset \mathbb{S}^3$.*

We now describe integral and rational surgery along knots, to understand the statement of the above theorem (which is proven in, e.g. [Lic97]).

Let K be a knot in a closed orientable manifold M, and $N(K)$ a tubular neighbourhood of K. The *knot exterior* of K in M is $M - \operatorname{int} N(K)$. This has boundary $\mathbb{S}^1 \times \mathbb{S}^1$, while $N(K)$ itself is diffeomorphic to $D^2 \times \mathbb{S}^1$. *Dehn surgery* is the process of gluing $D^2 \times \mathbb{S}^1$ back into $M - \operatorname{int} N(K)$ along some surface homeomorphism.

Example. One canonical way of gluing a solid torus back into the knot exterior is to replace $D^2 \times \mathbb{S}^1$ with $\mathbb{S}^1 \times D^2$. This is an *integral surgery* along K, and exactly the type of surgery used in the Lickorish-Wallace theorem.

From the previous sections, we know that the manifold obtained from surgery is completely determined by the image of the meridian of $D^2 \times \mathbb{S}^1$ under the gluing homeomorphism. Let $M = \mathbb{S}^3$, and $X = M - \operatorname{int} N(K)$. Then $\partial X = \mathbb{S}^1 \times \mathbb{S}^1$ has a meridian m and longitude ℓ defined as follows:

- A meridian is a generator of $H_1(X)$. This is equivalently a meridian of the solid torus $N(K)$.
- The *canonical longitude* is the unique longitude of $N(K)$ which is homologically trivial in X.

Then m and ℓ are unique up to isotopy and choice of orientation.

We can fix orientations by requiring that (m, ℓ, n) is positively oriented, where n is a normal vector to m and ℓ pointing "inwards" into X from $N(K)$. With these orientations fixed, any meridian of a solid torus $D^2 \times \mathbb{S}^1$ is mapped by the gluing surface homeomorphism to $pm + q\ell$ for some p, q. Then the reduced fraction p/q is called the *surgery coefficient*.

Example. Suppose $q = 0$, so that $p/q = \infty = 1/0$. Then the meridian maps to a meridian, which is the same as gluing $N(K)$ back into X. Therefore 0-surgery returns $M = \mathbb{S}^3$.

Example. Suppose $p/q = 0 = 0/1$. Then the meridian of a solid torus is mapped to a longitude by the gluing homeomorphism. This an example of an *integral surgery*, as used in the Lickorish-Wallace theorem.

Example. More generally, *integral surgery* is any p/q surgery with $q = \pm 1$. That is, the meridian of $D^2 \times \mathbb{S}^1$ maps to a curve with traces out one loop longitudinally, but with p additional "twists". The value $p/q = \pm p$ is then the *framing* of the surgery along K. The notion of integral surgery is well defined for *any* M rather than just the 3-sphere, as we do not need a canonical choice of longitude.

With the last example, we can introduce the notion of a surgery diagram. This is a link diagram L with each component decorated by an integer. The integer specifies the surgery gradient for Dehn surgery along the component. The corresponding oriented closed 3-manifold is that obtained by integral surgeries along each component with the designated surgery gradient.

Example. The lens space $L(p, 1)$ for $p \geq 2$ has surgery diagram the unknot decorated with integer $-p$. To see this, observe that $L(p, 1)$ is defined by an orientation preserving surface homeomorphism

$$\begin{pmatrix} 1 & r \\ p & s \end{pmatrix}.$$

Changing the choice of longitude λ_1 (in the "domain") corresponds to adding multiples of the first column to the second. Therefore without loss of generality, our surface homeomorphism is

$$\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

Reversing the orientation, our gluing map is defined by

$$\mu_1 \mapsto -\mu_2 + p\lambda_2, \quad \lambda_1 \mapsto \lambda_2.$$

To view our second solid torus (with meridian and longitude μ_2, λ_2) as a trivial knot exterior, we simply "turn it inside out". Then $\mu_2 \mapsto \ell$ and $\lambda_2 \mapsto m$. Therefore our gluing map satisfies

$$\mu_1 \mapsto pm - \ell.$$

Our surgery coefficient is $p/(-1) = -p$, and the knot exterior is that of a trivial knot. Our claim follows.

Example. More generally, the lens space $L(p, q)$ has surgery diagram given by a chain of linked unknots (analogous to the Audi logo) each with framing x_i, where $[x_1, \ldots, x_n]$ is a continued fraction of p/q.

Example. The Seifert manifold $M((a_1, b_1), \ldots, (a_n, b_n))$ has a surgery diagram consisting of n unknots each linked to a central unknot, where the central unknot is decorated with 0, and the peripheral unknots are decorated with a_i/b_i. These fractions be replaced by chains of integers corresponding to continued fractions, as above.

1.4 Brieskorn homology spheres and Seifert homology spheres

An important class of manifolds called *Brieskorn manifolds* are obtained as links of *Brieskorn singularities*. These include the Poincaré homology sphere. These also have descriptions as Seifert manifolds.

Definition 1.4.1. A *Brieskorn singularity* is the zero set

$$\mathcal{Z}(z_1^{a_1} + \cdots + z_n^{a_n}) \subset \mathbb{C}^n.$$

Provided $z_1^{a_1} + \cdots + z_n^{a_n}$ has an isolated singularity at the origin, the *link of the singularity* is the intersection of the singularity with a small sphere containing it:

$$L = \mathcal{Z}(z_1^{a_1} + \cdots + z_n^{a_n}) \cap \mathbb{S}^{2n-1}.$$

This defines a smooth manifold of real dimension

$$2(n-1) - 1 = 2n - 3.$$

The smooth structure is independent of the radius of $\mathbb{S}^{2n-1}$, for sufficiently small radii.

Example. If $n = 2$, then the link of a Brieskorn singularity defines a link in $\mathbb{S}^3$ in the usual sense. If $n \geq 3$, we obtain codimension 2 links in $\mathbb{S}^{2n-1}$.

Example. Suppose $\gcd(p, q) = 1$. Then the link of the Brieskorn singularity $z_1^p + z_2^q = 0$ is the torus knot $T_{p,q}$.

1-dimensional knots were extensively studied using knot polynomials, one of which was the Alexander polynomial. We can extend the Alexander polynomial to isolated singularities (see [Mur17].):

Theorem 1.4.2. *Let f be an isolated singularity, and $K(f)$ the link of the singularity. Then there is a Laurent polynomial $\Delta_f(t)$ associated to $K(f)$ generalising the Alexander polynomial such that*

1. *If $K(f)$ is the (usual one dimensional) unknot, then $\Delta_f(t) = 1$.*
2. *$\Delta_f(t)$ is an isotopy invariant.*
3. *$\Delta_f(1) = \pm 1$ if and only if $K(f)$ is a homology sphere.*

In fact, this has an exact formula: if

$$f = z_1^{a_1} + \cdots + z_n^{a_n},$$

then

$$\Delta_f(t) = \prod_{0 < i_k < a_k} (t - \xi_0^{i_0} \cdots \xi_n^{i_n}), \quad \xi_k = e^{2\pi i / a_k}.$$

Example. Let $f = z_1^2 + z_2^3 + z_3^5$. By the above formula, the corresponding link of the singularity has Alexander polynomial

$$\Delta_f(t) = \prod_{i,j} (t + \zeta_3^i \zeta_5^j), \quad \xi_k = e^{2\pi i / k}, i \in \{1, 2\}, j \in \{1, 2, 3, 4\}.$$

But now $t^{30} - 1$ has roots $\zeta_3^i \zeta_5^j$ (along with others). The other roots are removed by considering $(t^{30} - 1)/(t^{15} - 1)(t^{10} - 1)(t^6 - 1)$. Unfortunately this removes too many roots! In this fashion we observe that

$$\Delta_f(t) = \frac{(t^{30} - 1)(t^5 - 1)(t^3 - 1)(t^2 - 1)}{(t^{15} - 1)(t^{10} - 1)(t^6 - 1)(t - 1)}.$$

Then $\Delta_f(1) = 1$ so the link of f is a homology sphere. In fact, the link of f is the *Poincaré homology sphere.*

Example. More generally, if p, q, r are positive pairwise coprime integers, then the link of $x^p + y^q + z^r$ has Alexander polynomial

$$\Delta(t) = \frac{(t^{pqr} - 1)(t^p - 1)(t^q - 1)(t^r - 1)}{(t^{pq} - 1)(t^{pr} - 1)(t^{qr} - 1)(t - 1)}.$$

If p, q, r are coprime, then $\Delta(1) = 1$, so that p, q, r determines a homology sphere.

Definition 1.4.3. A *Brieskorn 3-manifold*, denoted $M(p, q, r)$, is the link of the singularity

$$x^p + y^q + z^r = 0$$

for p, q, r positive integers.

Proposition 1.4.4. The Brieskorn 3-manifold $M(p, q, r)$ is a homology sphere if and only if p, q, r are pairwise coprime. In this case, it is denoted by $\Sigma(p, q, r)$, and called a *Brieskorn homology 3-sphere.*

Observe that $\Sigma(2, 3, 5)$ is the Poincaré homology sphere.

Proposition 1.4.5. If any of p, q, r are equal to 1, then $\Sigma(p, q, r)$ is homeomorphic to the usual 3-sphere.

This can be seen as a corollary of the following result, due to Milnor [Mil75].

Theorem 1.4.6. *If* $1/p + 1/q + 1/r \neq 1$*, then* $\pi_1(\Sigma(p, q, r))$ *is the commutator subgroup of a central extension* $\Gamma(p, q, r)$ *of the following von Dyck group*

$$D(p, q, r) = \langle a, b, c \mid a^p = b^q = c^r = abc = 1\rangle.$$

Specifically, the central extension is

$$\Gamma(p, q, r) = \langle a, b, c \mid a^p = b^q = c^r = abc\rangle.$$

Suppose one of $p, q, r = 1$. (Without loss of generality, say $r = 1$.) Then the corresponding central extension is

$$\Gamma(1, q, r) = \langle a, b, c \mid a^p = b^q = c = abc\rangle = \langle a, b \mid a^p = b^q, ab = 1\rangle = \langle a \mid a^p = a^{-q}\rangle.$$

Since $a^p = a{-}q$, $a^{p+q} = 1$. Therefore Γ is the finite cyclic group of order $p + q$. Note that $1/p + 1/q + 1/r \neq 1$ when one of p, q, r is equal to 1. Since the commutator subgroup of an abelian group is trivial, it follows from the previous theorem that $\Sigma(p, q, 1)$ has trivial fundamental group. Now by Whitehead's theorem and the Poincaré conjecture, $\Sigma(p, q, r)$ is the 3-sphere whenever one of p, q, r is 1.

Example. Which Brieskorn homology spheres have finite non-trivial fundamental groups? We answer this question using the following lemma.

Lemma 1.4.7. Let $G = D(p, q, r)$. The abelianisation $G/[G, G]$ is finite if and only if at most one of p, q, r is zero.

Proof. If at least two of them are zero, the von Dyck group is isomorphic to the free product of $\mathbb{Z}$ with a finite cyclic group. The abelianisation contains $\mathbb{Z}$. Conversely, if at most one of them is zero, then the abelianisation is generated by two commuting elements of finite order. □

Since we only consider $\Sigma(p, q, r)$ for $p, q, r \geq 1$ and coprime, it follows that the abelianisation is always finite. In particular, the derived subgroup $[G, G]$ is finite if and only if G is finite. But recall the following famous result:

Lemma 1.4.8. $D(p, q, r)$ is finite if and only if $1/p + 1/q + 1/r > 1$.

This brings us to our first case: consider $\Sigma(p, q, r)$ with $1/p + 1/q + 1/r > 1$. The only integral solutions are

$$\{2, 3, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 2, n : n \geq 2\}.$$

Of these, only $\{2, 3, 5\}$ consists of pairwise coprime integers. Therefore this is the unique non-trivial Brieskorn homology sphere with finite fundamental group! We find that

$$\Gamma(2, 3, 5) = \langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle$$

is the *binary icosahedral group*, which is exactly the fundamental group of the Poincaré homology sphere.

In the Euclidean case $1/p + 1/q + 1/r = 1$, the only integral solutions are not pairwise coprime, and therefore we do not obtain any homology spheres. In the hyperbolic case $1/p+1/q+1/r < 1$, there are many pairwise coprime solutions, and they all give Brieskorn spheres with infinite fundamental group or trivial fundamental group.

More generally, we can consider Brieskorn manifolds in higher dimensions. These can be used to construct exotic spheres!

Theorem 1.4.9. *Let $m > 1$. Then the manifold*

$$\Sigma = \{z \in \mathbb{C}^{2m+1} \mid z_0^3 + z_1^5 + z_2^2 + \cdots + z_{2m}^2 = 0, |z| = 1\}$$

is homeomorphic to $\mathbb{S}^{4m-1}$, but not diffeomorphic.

A *Seifert homology sphere* is a generalisation of Brieskorn homology spheres. Rather than just considering $x^p + y^q + z^r = 0$, we add some coefficients.

Definition 1.4.10. A *Seifert homology sphere* is defined as follows:

- Let $a_1, \ldots, a_n$ be positive integers, $n \geq 3$. Let $B = (b_{ij})$ be an $(n-2) \times n$ matrix with non-zero maximal minors.

- Consider the variety

$$V(a_1, \ldots, a_n) \coloneqq \{b_{i1} z_1^{a_1} + \cdots + b_{in} z_n^{a_n} = 0, i \in \{1, \ldots, n-2\}\} \subset \mathbb{C}^n.$$

  This is non-singular except possibly at the origin.

- Define

$$\Sigma(a_1, \ldots, a_n) = V(a_1, \ldots, a_n) \cap \mathbb{S}^{2n-1},$$

  for a small sphere $\mathbb{S}^{2n-1}$. The resulting manifold has dimension $2n-1-2(n-2) = 3$. Moreover, its homeomorphism type is independent of B.

- $\Sigma(a_1, \ldots, a_n)$ is a homology 3-sphere if and only if the a_i are pairwise coprime. In this case, it is called a *Seifert homology sphere*.

Example. Clearly a Brieskorn homology 3-sphere is a Seifert homology sphere with $n = 3$.

Seifert homology spheres also admit surgery descriptions, which we give after looking at an example.

Example. Consider the Seifert manifold $M((2,-1),(3,1),(5,1))$. This manifold has fundamental group

$$\pi_1(M) = \langle a, b, c, h \mid [a,h] = [b,h] = [c,h] = abc = a^2 h^{-1} = b^3 h = c^5 h = 1 \rangle,$$

following the Seifert-van Kampen argument from when Seifert manifolds were first defined. Using Tietze transformations, we have

$$\pi_1(M) = \langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle.$$

This is exactly the binary icosahedral group! In fact, $M((2,-1),(3,1),(5,1)) = \Sigma(2,3,5)$.

This correspondence is not a coincidence. Seifert homology spheres defined "algebraically" also have surgery descriptions.

Proposition 1.4.11. The Seifert homology sphere $\Sigma(a_1, \ldots, a_n)$ can be constructed as follows:

- Consider $M((a_1, b_1), \ldots, (a_n, b_n))$. As noted earlier, this has fundamental group

$$\langle x_1, \ldots, x_n, h \mid [x_i, h] = x_1, \ldots, x_n = x_i^{a_i} h^{b_i} = 1 \rangle.$$

Moreover, recall that the first homology is the abelianisation of the fundamental group: it is exactly the cokernel of the map $\alpha : \mathbb{Z}^{n+1} \to \mathbb{Z}^{n+1}$, which has matrix

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & a_2 & \cdots & 0 & b_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & a_n & b_n \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}$$

in the canonical basis used above.

- The first homology is trivial if and only if $\det A = \pm 1$. This happens exactly when

$$a_1 \cdots a_n \sum_{i=1}^{n} b_i/a_i = \pm 1.$$

- Given coprime $a_1, \ldots, a_n$, there exists a unique Seifert manifold $M = M((a_1, b_1), \ldots, (a_n, b_n))$ up to homeomorphism, such that M is a homology sphere. Equivalently, such that $H_1(M)$ is trivial. In this case, it is exactly a *Seifert homology sphere* $\Sigma(a_1, \ldots, a_n)$.

Proposition 1.4.12. Seifert homology spheres obtained as "Seifert manifolds that happen to be homology spheres" are exactly Seifert homology spheres obtained as links of singularities.

Proof. We don't give a proof of the general case, but instead show that a Brieskorn manifold agrees with the Seifert manifold surgery description above. To this end, we show that a Brieskorn manifold $M(p, q, r)$ admits a circle action which acts freely at all but three orbits. The failure of freeness at these points records the degrees of the orbifold points.

Let $M(p, q, r)$ denote

$$\{(x, y, z) : x^p + y^q + z^r = 0\} \cap \mathbb{S}^5.$$

Then a canonical circle action $\mathbb{S}^1 \circlearrowright M(p, q, r)$ is defined by

$$e^{i\theta}(x, y, z) = (e^{i\theta bc}x, e^{i\theta ac}y, e^{i\theta ab}z).$$

This can be verified to be well defined. Moreover, it is free on all about three orbits: suppose $x = 0$. Then

$$y^q + z^r = 0, \quad |y|^2 + |z|^2 = \varepsilon.$$

This defines a circle in $M(p, q, r)$, which is exactly an orbit of the circle action. Moreover, on this orbit, we have

$$e^{i\theta}(0, y, z) = (0, e^{i\theta ac}y, e^{i\theta ab}z),$$

so restricting to this orbit, the action is an $a:1$ map. Similarly on $y=0$ and $z=0$, we obtain $b:1$ and $c:1$ maps.

Consider the quotient map

$$\pi : M(p,q,r) \to M(p,q,r)/\mathbb{S}^1.$$

This descends to a manifold away from the orbits $x=0, y=0$, and $z=0$. At these points, the quotient descends to "orbifold points" with degrees p,q, and r. The fibre of the quotient map is globally $\mathbb{S}^1$. Therefore this quotient map realises $M(p,q,r)$ as an $\mathbb{S}^1$ bundle over an orbifold, with orbifold points of degrees p,q,r, as required. $\square$

Chapter 2

Rokhlin invariant

In the previous chapter, we explored several examples of homology 3-spheres, together with constructions of 3-manifolds in general. In this chapter we define and explore an invariant of homology 3-spheres called the *Rokhlin invariant*. This is defined in terms of the signature of a compact smooth 4 manifold whose boundary is our given 3 manifold. Its well-definedness depends on Rokhlin's theorem, which we state in a general form with reference to the Arf invariant.

2.1 The Arf invariant of a knot

We defined and established the well-definedness of the Arf invariant in my notes on knot theory following Lickorish. Therefore we start by listing definitions and results without proofs.

Definition 2.1.1. Let k be a field, and V a finite dimensional vector space over k. A *quadratic form* is a map $\varphi : V \to k$ such that $\varphi(ax) = a^2\varphi(x)$ for all $x \in V, a \in k$, and such that

$$(x, y) \mapsto \varphi(x + y) - \varphi(x) - \varphi(y)$$

is a symmetric bilinear form. φ is said to be *non-degenerate* if the associated bilinear form is non-degenerate.

Remark. When k is of characteristic 2, the property $\varphi(ax) = a^2\varphi(x)$ for all $x \in V, a \in k$ is implied by the requirement that $\varphi(x+y)-\varphi(x)-\varphi(y)$ be bilinear. Explicitly, by bilinearity,

$$0\varphi(x) = 0 = \varphi(x + 0y) - \varphi(x) - \varphi(0y) = \varphi(0y).$$

On the other hand, it is immediate that $1\varphi(x) = \varphi(1x)$.

Definition 2.1.2. Two quadratic forms $\varphi, \psi : V \to k$ are *equivalent* if there exists $A \in \mathrm{GL}(V)$ such that $\varphi(Ax) = \psi(x)$ for all $x \in V$.

Theorem 2.1.3. *Let $\varphi : V \to \mathbb{Z}/2\mathbb{Z}$ be a non-degenerate quadratic form. Then φ belongs to one of exactly two equivalence classes:*

$$\psi_1(x_1e_1 + \cdots + y_nf_n) = x_1y_1 + \cdots + x_ny_n$$
$$\psi_2(x_1e_1 + \cdots + y_nf_n) = x_1y_1 + \cdots + x_ny_n + x_n^2 + y_n^2.$$

If φ is equivalent to ψ_1, φ is of type I. *If φ is equivalent to ψ_2, φ is of* type II.

Definition 2.1.4. The *Arf invariant* of a non-degenerate quadratic form $\varphi : V \to \mathbb{Z}/2\mathbb{Z}$, denoted $c(\varphi)$, is defined by

$$c(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is of type I} \\ 1 & \text{if } \varphi \text{ is of type II.} \end{cases}$$

Proposition 2.1.5. Let $\varphi : V \to \mathbb{Z}/2\mathbb{Z}$ be a non-degenerate quadratic form. The following values are equal:

1. The Arf invariant $c(\varphi)$ of φ.
2. The value 0 or 1 attained more often by φ as it ranges over the 2^{2n} elements of V.
3. The value $\sum_{i=1}^{n} \varphi(e_i)\varphi(f_i)$ where $\{e_1, f_1, \ldots, e_n, f_n\}$ is any symplectic basis.

We now define the Arf invariant of a knot. Let $K \subset \Sigma$ be a knot embedded in an integral homology 3-sphere Σ. K admits a Seifert surface F in Σ, and this has a Seifert form α. This bilinear form defines a non-degenerate quadratic form

$$q : H_1(F; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

by $q(x) = \alpha(x, x) \mod 2$.

Definition 2.1.6. The *Arf invariant* of a knot K is the Arf invariant of the quadratic form q defined above.

Proposition 2.1.7. The Arf invariant satisfies the following properties:

1. $\mathrm{Arf}(0_1) = 0$.
2. $\mathrm{Arf}(K_1 + K_2) = \mathrm{Arf}(K_1) + \mathrm{Arf}(K_2)$.
3. The Arf invariant is a concordance invariant of knots (and links).
4. $\mathrm{Arf}(K) = \frac{1}{2}\Delta''_K(1) \mod 2$, where Δ_K is the Alexander polynomial of K (with the Conway normalisation).

Proof. The proof follows the following four steps.

1. Let S be a Seifert matrix for K. Then for $Q = S + S^T$, we write

$$a^2 Q = P^T D P$$

where P is an integral matrix with odd determinant, a is an odd integer, and

$$D = \begin{pmatrix} 2p_1 & c_1 \\ c_1 & 2q_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 2p_g & c_g \\ c_g & 2q_g \end{pmatrix}$$

for c_i odd.

2. Express the Arf invariant $\operatorname{Arf}(K)$ in terms of the diagonal entries of D. Explicitly,

$$\operatorname{Arf}(K) = \sum_{j=1}^{g} p_j q_j \mod 2.$$

3. Relate the Arf invariant to the Alexander polynomial: $\Delta_K(-1) = 1 + 4\operatorname{Arf}(K)$.

4. Observe that $\Delta_K(-1) = 1 + 2\Delta_K''(1) \mod 8$.

1. Let S be a $2g \times 2g$ Seifert matrix of a Seifert surface F of K. Write $Q = S + S^T$. Modulo 2, Q is the intersection form of F. Therefore writing $Q = (a_{ij})$, we know that $a_{11} = 0 \mod 2$, and $a_{12} = 1 \mod 2$. Write

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

We further write Q as a block matrix

$$Q = \begin{pmatrix} A & L^T \\ L & B \end{pmatrix}.$$

Note that A and B are symmetric, and $\det A = a_{12}^2 = 1 \mod 2$, so A is invertible over $\mathbb{Q}$. Now if

$$R = \begin{pmatrix} I & A^{-1}L^T \\ 0 & I \end{pmatrix},$$

then R has determinant 1, and $Q = R^T(A \oplus (B - LA^{-1}L^T))R$. The matrix $B - LA^{-1}L^T$ is itself even with odd determinant. Therefore by induction, $Q = R^T D R$ where D has the desired form. R is in general not integral, as it may have odd denominators in its rational terms. Let a be the greatest common divisor of the denominators, so that

$$a^2 Q = (aR)^T D (aR) = P^T D P.$$

2. The expression $a^2Q = (aR)^T D(aR) = P^T DP$ asserts that there is a basis $\{a_j, b_j\}$ of $H_1(F;\mathbb{Z})$ for which

$$a^2Q(a_i, a_j) = 2p_i\delta_{ij}, \quad a^2Q(b_i, b_j) = 2q_i\delta_{ij}, \quad a^2Q(a_i, b_j) = c_i\delta_{ij}.$$

Therefore the $\{a_j, b_j\}$ descend to a symplectic basis of $H_1(F;\mathbb{Z}/2\mathbb{Z})$. Moreover, this basis satisfies

$$q(a_j) = \frac{1}{2}Q(a_j, a_j) = p_j \mod 2, \quad q(b_j) = \frac{1}{2}Q(b_j, b_j) = q_j \mod 2.$$

This is because $q(x) = S(x, x) \mod 2 = \frac{1}{2}(S + S^T)(x, x) \mod 2 = \frac{1}{2}Q(x, x) \mod 2$. It follows that

$$\operatorname{Arf}(K) = \operatorname{Arf}(q) = \sum_{j=1}^{g} q(a_i)q(b_i) = \sum_{j=1}^{g} p_i q_i \mod 2.$$

3. To relate the Arf invariant to the Alexander polynomial, recall that

$$\Delta_K(t) = \det(t^{1/2}S - t^{-1/2}S^T).$$

Therefore $\Delta_K(-1) = \det(iQ)$. On the other hand, $a^2Q = P^T DP$, so

$$(a^2)^{2g} \det(iQ) = (\det P)^2 \det(iD) = (\det P)^2 \prod_{j=1}^{g}(c_j^2 - 4p_j q_j).$$

Note that $a^{2g}, \det P$, and c_j are all odd. But if $t = 2k + 1$ is an odd integer, then $t^2 = 4k(k + 1) + 1$, so $t^2 = 1 \mod 8$. It follows that,

$$\Delta_K(-1) = \det(iQ) = \prod_{j=1}^{g}(1 - 4p_j q_j) \mod 8 = 1 + 4\operatorname{Arf}(K).$$

4. Finally, recall that the Alexander polynomial can be expressed as

$$\Delta_K(t) = a_0 + a_1(t + t^{-1}) + a_2(t^2 - t^{-2}) + \cdots.$$

Moreover, $\Delta_K(1) = 1$. Using these two facts, we can calculate that $\Delta''_K(1) = 2\sum_j j^2 a_j$. Therefore

$$1 + 2\Delta''_K(1) \mod 8 = 1 + 4\sum_j j^2 a_j \mod 8.$$

On the other hand, evaluation of $\Delta_K(-1)$ using the standard form above gives $\Delta_K(-1) = 1 - 4\sum_{j \text{ odd}} a_j$. Therefore

$$\Delta_K(-1) = 1 + 2\Delta''_K(1) \mod 8.$$

To complete the proof, we combine the conclusions of points 3 and 4 to obtain $1 + 4\operatorname{Arf}(K) = 1 + 2\Delta''_K(1) \mod 8$, so that in particular

$$\operatorname{Arf}(K) = \frac{1}{2}\Delta''_K(1) \mod 2.$$

□

2.2 Rokhlin's theorem

Before discussing the Rokhlin invariant, it remains to prove Rokhlin's theorem (which will ensure that our invariant is well defined).

Theorem 2.2.1 (Rokhlin)**.** *Let M be a simply connected oriented smooth 4-manifold, and F a closed oriented surface smoothly embedded in M. If F is* characteristic*, then*

$$\frac{1}{8}(\sigma M - F \cdot F) = \mathrm{Arf}(M, F) \mod 2.$$

Before proving this theorem, or even interpreting it, we give some corollaries to motivate this section.

Corollary 2.2.2. *There exist topological 4-manifolds that admit no smooth structures.*

Proof. By Freedman's theorem, there exists a unique simply connected closed topological 4-manifold M whose intersection form is E_8. But E_8 has signature 8, so 16 does not divide σM. Therefore M cannot be smooth by Rokhlin's theorem. □

Corollary 2.2.3. *Let Σ be an oriented homology 3-sphere, and M a smooth simply-connected oriented 4-manifold with even intersection form, with boundary Σ. Then the* Rokhlin invariant

$$\mu(\Sigma) = \frac{1}{8}\sigma M \mod 2$$

is well defined.

We do not give a full proof here, as he have yet to justify the existence of an M as above. However, we can prove that $\mu(\Sigma)$ is independent of M. Suppose M_1, M_2 are two smooth simply-connected oriented 4-manifolds with even intersection form, with boundary Σ. Then gluing M_1 to M_2 along an orientation reversing diffeomorphism of Σ, $M = M_1 \sqcup_\Sigma -M_2$ is a simply connected oriented smooth 4-manifold with signature $\sigma M_1 - \sigma M_2$. By Rokhlin's theorem, $\sigma M = 0 \mod 16$, so σM_1 must agree with σM_2 modulo 16. Dividing each by 8, they agree modulo 2 as required.

To understand the theorem, we now define the relevant concepts.

Definition 2.2.4. Let M be a simply connected oriented closed smooth manifold, and $Q_M : H_2(M;\mathbb{Z}) \otimes H_2(M;\mathbb{Z}) \to \mathbb{Z}$ its intersection form. Write $Q_M(a,b) = a \cdot b$. A closed oriented surface F smoothly embedded in M is called *characteristic* if

$$F \cdot x = x \cdot x \mod 2$$

for all $x \in H_2(M;\mathbb{Z})$.

We hereafter write $a \cdot b$ to mean $Q_M(a,b)$ without further mention. Given a characteristic surface, we can define an Arf invariant.

Definition 2.2.5. Let $F \subset M$ be characteristic. Then $\mathrm{Arf}(M, F)$ is defined as follows:

1. Let $\gamma \in H_1(F; \mathbb{Z}/2\mathbb{Z})$. This is represented by an embedded circle $\gamma \subset F$.

2. Observe that γ represents the trivial homology in $H_1(M; \mathbb{Z})$ (since the whole first homology vanishes). Therefore γ bounds a connected orientable surface D embedded in M. This can be taken to be transverse to F.

3. Let D' be a push-off of D, deformed to ensure transversality. (In particular, $\partial D' \cap \partial D = \varnothing$.) Define
$$\widetilde{q} : H_1(F; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}, \quad \widetilde{q}(\gamma) = D \cdot D' + D \cdot F \mod 2.$$

4. With the above definition, $\widetilde{q}$ is a well defined quadratic form, and $\mathrm{Arf}(\widetilde{q}) = \mathrm{Arf}(M, F)$.

We take for granted that this is well defined, but give a key point as to why the definition works, and why it relies on F being characteristic. Suppose D_1, D_2 are two choices of orientable surface bound by γ. Then $S = D_1 \sqcup_\gamma D_2$ is a smoothly embedded closed surface, and represents some homology class. Similarly let $S' = D_1' \sqcup_{\gamma'} D_2'$. Since F is characteristic, modulo two we have

$$D_1 \cdot D_1' + D_2 \cdot D_2' = S \cdot S' = S \cdot S = S \cdot F = D_1 \cdot F + D_2 \cdot F.$$

Therefore
$$D_1 \cdot D_1' + D_1 \cdot F = D_2 \cdot D_2' + D_2 \cdot F \mod 2.$$

The definition also agrees the Arf invariant of a knot in the following way:

Proposition 2.2.6. Let $F \subset M$ be a characteristic surface. Suppose $\Sigma \subset M$ be a homology 3-sphere, separating F as $F' \sqcup_K D^2$, where K is the knot $F \cap \Sigma \subset \Sigma$. Then

$$\mathrm{Arf}(K) = \mathrm{Arf}(M, F).$$

Proof. We in fact prove a better result, which implies the result above. Let $q : H_1(F'; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ be the form defining the Arf invariant $\mathrm{Arf}(K) = \mathrm{Arf}(q)$. Let $\widetilde{q} : H_1(F; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ be defined as above. The natural isomorphism $\iota_* : H_1(F'; \mathbb{Z}/2\mathbb{Z}) \to H_1(F; \mathbb{Z}/2\mathbb{Z})$ makes the following diagram commute:

$$\begin{array}{ccc} H_1(F'; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\quad i_* \quad} & H_1(F; \mathbb{Z}/2\mathbb{Z}) \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle \widetilde{q}} & \\ & \mathbb{Z}/2\mathbb{Z} & \end{array}$$

Let $i : F' \to F$ be the inclusion map, so that i_* is the induced map. To see that this is an isomorphism, we use the long exact sequence of homology. Specifically, $H_1(F, F'; \mathbb{Z}/2\mathbb{Z}) = F_1(\mathbb{S}^2; \mathbb{Z}/2\mathbb{Z}) = 0$, and the connecting map $\partial : H_2(F, F'; \mathbb{Z}/2\mathbb{Z}) \cong H_2(\mathbb{S}^2; \mathbb{Z}) \to H_1(F'; \mathbb{Z})$ is the zero map. Therefore $i_* : H_1(F'; \mathbb{Z}/2\mathbb{Z}) \to H_1(F; \mathbb{Z}/2\mathbb{Z})$ is an isomorphism.

Therefore to understand the image of $\widetilde{q} : H_1(F; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, it suffices to study homology classes represented by circles γ in F'. Let D be a Seifert surface of γ in Σ, which does not intersect D^2. Then

$$\widetilde{q}(\gamma) = D \cdot D' + D \cdot F = \operatorname{lk}(\gamma, \gamma') + \operatorname{lk}(\gamma, K) \mod 2.$$

Firstly we have $\operatorname{lk}(\gamma, \gamma') = \operatorname{lk}(\gamma, \gamma^+)$ modulo 2, where γ^+ is a positive push-off. Of course the latter is the definition of q, so $\operatorname{lk}(\gamma, \gamma')$ agrees with $q(\gamma)$ modulo 2. Secondly we have $\operatorname{lk}(\gamma, K) = 0$, since K is homologous to $\partial N(K) \cap F'$. It follows that

$$\operatorname{lk}(\gamma, \gamma') + \operatorname{lk}(\gamma, K) = q(\gamma) \mod 2.$$

Therefore the above diagram commutes. In particular,

$$\operatorname{Arf}(K) = \operatorname{Arf}(M, F).$$

□

We now give a proof outline of Rokhlin's theorem. We first state some theorems which are used in the proof.

Theorem 2.2.7 (Wall)**.** *Suppose M and N are simply connected closed oriented smooth 4-manifolds. If their intersection forms are equivalent, then M and N are* stably diffeomorphic*. That is, there exists $k \geq 0$ such that $M \# k(\mathbb{S}^2 \times \mathbb{S}^2)$ is diffeomorphic to $N \# k(\mathbb{S}^2 \times \mathbb{S}^2)$.*

Lemma 2.2.8. The algebraic curve $x_0 x_1^{s-1} + x_2^s = 0$ in $\mathbb{CP}^2$ is homeomorphic to $\mathbb{S}^2$ and represents the homology class $s[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$.

We are now ready to prove Rokhlin's theorem.

Theorem 2.2.9 (Rokhlin)**.** *Let M be a simply connected oriented smooth 4-manifold, and F a closed oriented surface smoothly embedded in M. If F is* characteristic*, then*

$$\frac{1}{8}(\sigma M - F \cdot F) = \operatorname{Arf}(M, F) \mod 2.$$

Proof. A proof outline is as follows.

1. Observe that $M \# \ell_1 \mathbb{CP}^2 \# \ell_2 \overline{\mathbb{CP}^2} = a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$ for some ℓ_1, ℓ_2, a, b.

2. Show that it suffices to prove the formula for characteristic surfaces in $\mathbb{CP}^2$.

3. Within $\mathbb{CP}^2$, relate the Arf invariant $\mathrm{Arf}(\mathbb{CP}^2, F)$ to that of a knot, $\mathrm{Arf}(K)$. The latter can be computed, and we find that it gives the desired result.

1. Consider the manifold $M\#\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. This has intersection form $Q_1 = Q_M \oplus (1) \oplus (-1)$. This is indefinite and odd, so by the classification of unimodular forms, is equivalent to

$$p(1) \oplus q(-1), \quad p = b_+(M) + 1, q = b_-(M) + 1.$$

On the other hand, the manifold $N = p\mathbb{CP}^2\#q\overline{\mathbb{CP}^2}$ has the same intersection form. By Wall's theorem, M and N are stably diffeomorphic. But we also know that

$$\mathbb{CP}^2\#(\mathbb{S}^2 \times \mathbb{S}^2) = \overline{\mathbb{CP}^2}\#2\mathbb{CP}^2, \quad \overline{\mathbb{CP}^2}\#(\mathbb{S}^2 \times \mathbb{S}^2) = 2\overline{\mathbb{CP}^2}\#\mathbb{CP}^2.$$

Therefore for some ℓ_1, ℓ_2, and $a = \ell_1 + b_+(M), b = \ell_2 + b_-(M)$,

$$M\#\ell_1\mathbb{CP}^2\#\ell_2\overline{\mathbb{CP}^2} = a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}.$$

2. We now show that Rokhlin's theorem for $\mathbb{CP}^2$ implies the general theorem. Consider the pairs $(M_1, F_1), (M_2, F_2)$, and $(M_1\#M_2, F_1 \sqcup F_2)$. Then

- $\sigma(M_1\#M_2) = \sigma M_1 + \sigma M_2$,
- $(F_1 \sqcup F_2) \cdot (F_1 \sqcup F_2) = F_1 \cdot F_1 + F_2 \cdot F_2 + 2F_1 \cdot F_2 = F_1 \cdot F_1 + F_2 \cdot F_2$.
- Observe that $H_1(F_1 \sqcup F_2; \mathbb{Z}/2\mathbb{Z}) = H_1(F_1; \mathbb{Z}/2\mathbb{Z}) \oplus H_1(F_2; \mathbb{Z}/2\mathbb{Z})$. The quadratic form $\widetilde{q} : H_1(F_1 \sqcup F_2; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is the direct sum of the two quadratic forms associated to F_1 and F_2. It follows that $\mathrm{Arf}(M_1\#M_2, F_1 \sqcup F_2) = \mathrm{Arf}(M_1, F_1) + \mathrm{Arf}(M_2, F_2)$ modulo 2.

Combining these three properties, we find that the formula

$$\frac{1}{8}(\sigma M - F \cdot F) = \mathrm{Arf}(M, F) \mod 2$$

holds for all three of the pairs above, provided it holds for two of them.

Now let $F \subset M$ be characteristic, $\eta \in H_2(\mathbb{CP}^2)$ a generator of H_2 respresented by $\mathbb{CP}^1 \subset \mathbb{CP}^2$, and $\overline{\eta} \in H_2(\overline{\mathbb{CP}^2})$ a generator represented by $\overline{\mathbb{CP}}^1$. Let

$$F^s = F + \ell_1\eta + \ell_2\eta_2 \in H_2(M\#\ell_1\mathbb{CP}^2\#\ell_2\overline{\mathbb{CP}^2}).$$

We show that this is characteristic. Let

$$x \in H_2(M\#\ell_1\mathbb{CP}^2\#\ell_2\overline{\mathbb{CP}^2}) \cong H_2(M) \oplus \bigoplus_{i=1}^{\ell_1} H_2(\mathbb{CP}^2) \oplus \bigoplus_{j=1}^{\ell_2} H_2(\overline{\mathbb{CP}^2}).$$

We can write x as

$$x_M + a_1\eta_{c,1} + \cdots + a_{\ell_1}\eta_{c,\ell_1} + b_1\overline{\eta}_{\overline{c},1} + \cdots + b_{\ell_2}\overline{\eta}_{\overline{c},\ell_2}.$$

Since an integer is even if and only if its square is even,

$$a_1^2 + \cdots + a_{\ell_1}^2 = a_1 + \cdots + a_n \mod 2, \quad b_1^2 + \cdots + b_{\ell_2}^2 = b_1 + \cdots + b_{\ell_2} \mod 2.$$

It follows that F^s is characteristic, since

$$\begin{aligned} F^s \cdot x &= (F + \ell_1\eta_1 + \ell_2\eta_2) \cdot (x_M + a_1\eta_{c,1} + \cdots + a_{\ell_1}\eta_{c,\ell_1} + b_1\overline{\eta}_{\overline{c},1} + \cdots + b_{\ell_2}\overline{\eta}_{\overline{c},\ell_2}) \\ &= F \cdot x_M + a_1 + \cdots + a_{\ell_1} - b_1 - \cdots - b_{\ell_2} \\ &= x_M \cdot x_M + a_1^2 + \cdots + a_{\ell_1}^2 - b_1^2 - \cdots - b_{\ell_2}^2 \mod 2 \\ &= x \cdot x \mod 2. \end{aligned}$$

By the previous remark concerning the two-out-of-three property, to show that

$$\frac{1}{8}(\sigma M - F \cdot F) = \mathrm{Arf}(M, F) \mod 2,$$

it suffices to show that F^s, η, and $\overline{\eta}$ satisfy the corresponding properties. Moreover, under the stable diffeomorphism of M and N, F^s is sent to a class in $a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$. Therefore it suffices to show that the above formula holds for characteristic surfaces in $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$. The formula is invariant under orientation reversal, so we need only verify the formula for characteristic surfaces in $\mathbb{CP}^2$.

3. In summary, we must verify that

$$\frac{1}{8}(\sigma\mathbb{CP}^2 - s\eta \cdot s\eta) = \mathrm{Arf}(\mathbb{CP}^2, s\eta) \mod 2,$$

for all $s \in \mathbb{Z}$ is chosen such that $s\eta$ is characteristic. Let $k \in \mathbb{Z}$ be arbitrary. Then, modulo 2, we have

$$s\eta \cdot k\eta = sk = k^2 = k\eta \cdot k\eta$$

if and only if s is odd. Therefore it suffices to verify the above formula for $s\eta$ where s is odd. By the previous lemma, $s\eta$ is represented by the algebraic curve

$$C = \{[x_0 : x_1 : x_2] : x_0 x_1^{s-1} + x_2^s\} \subset \mathbb{CP}^2.$$

A small 3-sphere centered at $[1 : 0 : 0]$ intersects C along the $(s, s-1)$ torus knot $T_{s,s-1}$. By the earlier proposition relating the Arf invariant of a surface in a four manifold to that of a knot, we have

$$\mathrm{Arf}(T_{s,s-1}) = \mathrm{Arf}(\mathbb{CP}^2, s\eta).$$

We can in fact compute the Arf invariant of a torus knot! The Jones polynomial is given by

$$V_{T_{s,s-1}}(t) = t^{(s-1)(s-2)/2}\frac{1 - t^s - t^{s+1} + t^{2s-1}}{1 - t^2}.$$

Recall from Lickorish (or my notes from earlier in the quarter) that the Arf invariant is given by

$$(-1)^{\mathrm{Arf}(K)} = V_K(i).$$

Therefore the Arf invariant of a torus knot $T_{s,s-1}$, for s odd, can be determined by case work. The general formula, for $s = 2k+1$, is

$$(-1)^{\mathrm{Arf}(T_{s,s-1})} = i^{k(2k-1)}\frac{1 - i(-1)^k + i - (-1)^{k+1}}{2}.$$

Evaluation of this expression for the different values of k modulo 4 gives

$$\mathrm{Arf}(T_{s,s-1}) = \begin{cases} 0 & s \in \{1,7\} \mod 8 \\ 1 & s \in \{3,5\} \mod 8. \end{cases}$$

On the other hand, we consider the expression $(1-s^2)/8$ for odd s. If s is 1 or 7 modulo 8, then $(1-s^2)/8$ is 0 modulo 2. If s is 3 or 5 modulo 8, then $(1-s^2)/8$ is 1 modulo 2. It follows that

$$\mathrm{Arf}(T_{s,s-1}) = \frac{1-s^2}{8} \mod 2.$$

But observe that 1 is the signature of $\mathbb{CP}^2$, and $s^2 = s\eta \cdot s\eta$. Therefore, modulo 2, we have

$$\frac{1}{8}(\sigma\mathbb{CP}^2 - s\eta \cdot s\eta) = \frac{1-s^2}{8} = \mathrm{Arf}(T_{s,s-1}) = \mathrm{Arf}(\mathbb{CP}^2, s\eta)$$

as required. This proves Rokhlin's theorem for $\mathbb{CP}^2$, so by earlier considerations we have proven it in general. □

An immediate corollary (mentioned in the motivation) is obtained by taking F to be empty. If M is an even four manifold, then $x \cdot x = 0$ modulo 2 for all $x \in H_2(M;\mathbb{Z})$. Therefore the empty surface is characteristic. This gives

Theorem 2.2.10 (Rokhlin)**.** *Let M be a closed even simply connected smooth 4-manifold. Then 16 divides the signature of M.*

A generalisation of Rokhlin's theorem to topological manifolds is the following:

Theorem 2.2.11 (Rokhlin)**.** *Let M be a simply connected oriented topological 4-manifold, and F a closed oriented surface smoothly embedded in M. If F is* characteristic*, then*

$$\frac{1}{8}(\sigma M - F \cdot F) = \mathrm{Arf}(M, F) + \kappa(M) \mod 2.$$

Here $\kappa(M) \in H^4(M;\mathbb{Z}/2\mathbb{Z})$ is the Kirby-Siebenmann invariant*.*

The main use of this invariant is that it detects smoothability of 4-manifolds. If a closed simply-connected topological 4-manifold M admits a smooth structure, then $\kappa(M)$ must vanish. This is clear by rearranging the above formula.

If M is even, the empty surface is characteristic, so we have

$$\frac{1}{8}\sigma M = \kappa(M) \mod 2.$$

Therefore, as we have already established, M admits a smooth structure only if 16 divides σM.

2.3 The Rokhlin invariant and homology cobordism group

Another consequence of Rokhlin's theorem, as mentioned at the start of the previous section, is that the *Rokhlin invariant* is well defined.

Definition 2.3.1. Let Σ be a homology 3-sphere. The *Rokhlin invariant* of Σ is defined by

$$\mu(\Sigma) = \frac{1}{8}\sigma W \mod 2,$$

where W is a simply connected even oriented smooth 4-manifold with boundary Σ.

This is well defined by Rokhlin's theorem, since if W' is another simply connected even oriented smooth 4-manifold with boundary Σ, then $W \sqcup_{\Sigma} (-W')$ has signature 0 mod 16 by Rokhlin's theorem.

Example. The usual 3-sphere has Rokhlin invariant

$$\mu(\mathbb{S}^3) = 0.$$

This is because the 3-sphere bounds the 4-ball, which has trivial second homology and hence trivial intersection form. Thus it has signature 0.

Example. The Poincaré homology sphere has Rokhlin invariant

$$\mu(\Sigma(2, 3, 5)) = 1.$$

This is because the Poincaré homology sphere bounds a compact simply-connected smooth 4-manifold M with intersection form E_8. This has signature 8.

For a long time the Rokhlin invariant was the only invariant that was understood in the study of the *homology cobordism group*, which we now define and investigate.

Definition 2.3.2. Let Σ_1, Σ_2 be integral homology 3-spheres. Σ_1 and Σ_2 are said to be *homology cobordant*, or H-cobordant, if there exists a smooth compact oriented 4-manifold W with boundary $\partial W = -\Sigma_1 \sqcup \Sigma_2$, such that the inclusions of Σ_i into W induce isomorphisms in homology.

The collection of equivalence classes of oriented homology 3-spheres under homology cobordism is called the *homology cobordism group*, denoted by Θ^3.

Proposition 2.3.3. The homology cobordism group is an abelian group, under the operation of connected sums.

Proof. We do not give a proof, but simply describe the group. It is clear that, if Θ^3 is a group, then it is abelian. The identity element is the standard 3-sphere $\mathbb{S}^3$. Inverses are given by reversing orientation. This is proven in an exercise. □

Proposition 2.3.4. Σ is homology cobordant to $\mathbb{S}^3$ if and only if there exists an oriented smooth compact 4-manifold with boundary Σ and the homology of a point.

Proof. Suppose Σ is homology cobordant to $\mathbb{S}^3$. Then capping $\mathbb{S}^3$ gives an oriented smooth compact 4-manifold with boundary Σ and the homology of a point. Conversely, given the latter condition, one can cut along and embedded $\mathbb{S}^3$ to obtain the desired cobordism. □

The Arf invariant was a cobordism invariant of knots. We find that the analogous result holds here: the Rokhlin invariant is an invariant of homology cobordism.

Theorem 2.3.5. *The Rokhlin invariant defines a surjective homomorphism $\Theta^3 \to \mathbb{Z}/2\mathbb{Z}$.*

We use the following lemma, which is a generalisation of a special case of Rokhlin's theorem from the previous section.

Lemma 2.3.6. If M is an even smooth compact oriented 4-manifold, then 16 divides its signature.

Using this theorem, we will show that the Rokhlin theorem is well defined.

Proof. The proof of the theorem amounts to proving three things: firstly that the Rokhlin invariant is a homology cobordism invariant, secondly that the Rokhlin invariant is non-trivial, and thirdly that it behaves correctly under connected sums. This second claim is already understood to be true, by the earlier examples.

For the first claim, suppose Σ_1 and Σ_2 are homology cobordant. This is equivalent to the requirement that $\Sigma_1 \# (-\Sigma_2)$ bounds a compact oriented smooth 4-manifold W with $H_*(W;\mathbb{Z}) = H_*(D^4;\mathbb{Z})$. Therefore $\Sigma_1 \# (-\Sigma_2)$ bounds a smooth oriented even compact manifold W. By Rokhlin's theorem (as stated above),

$$\mu(\Sigma_1) - \mu(\Sigma_2) = \mu(\Sigma_1 \# (-\Sigma_2)) = 0.$$

This shows that the map is well defined as a function.

Finally to see that μ defines a group homomorphism, simply note that

$$\mu(\Sigma_1 \# \Sigma_2) = \mu(\Sigma_1) + \mu(\Sigma_2)$$

as used above. This is because intersection form Q of a boundary connected sum $M_1 \natural M_2$ is the direct sum of intersection forms: $Q = Q_{M_1} \oplus Q_{M_2}$. (This is because boundary connected sums involve only 3-cells, which do not affect second homology.) $\square$

In the 70s it was conjectured that $\mu : \Theta^3 \to \mathbb{Z}/2\mathbb{Z}$ was an isomorphism. However, Donaldson's diagonalisability theorem provides a counter example.

Example. Let Σ denote the Poincaré homology sphere, $\Sigma(2,3,5)$. This bounds a smooth E_8 manifold W. For each integer m, $m\Sigma$ is a homology sphere which bounds the boundary-connected sum mW of m copies of W. This has intersection form mE_8. Suppose $m\Sigma$ is homology cobordant to $\mathbb{S}^3$, for some $m \geq 1$. Then there is a 4-manifold W' homologous to D^4 with boundary $m\Sigma$. But now

$$mW \sqcup_{m\Sigma} W'$$

is a smooth closed 4-manifold with intersection form mE_8. This is definite and even, but any even form cannot possibly be diagonalisable, contradicting Donaldson's theorem [Don87]. Therefore Σ has infinite order.

In fact, the homology cobordism group is not only infinite, but infinitely generated. The homology 3-spheres $\Sigma(p, q, pqk - 1)$ are linearly independent over $\mathbb{Z}$ in Θ^3.

Are there elements of finite order? It is known that there are no elements of order 2 [Man13a]. In fact, surprisingly this is equivalent to a problem concerning triangulations.

Theorem 2.3.7. *There are no elements of order 2 in Θ^3. Equivalently, in every dimension at least 5, there exist topological manifolds that admit no simplicial triangulations.*

I wonder if it is known if there exist any finite order elements? A google search returned no results.

2.4 Exercises

Exercise 2.4.1. (Saveliev 11.5.3) Prove that for any homology sphere Σ, $\Sigma \#(-\Sigma)$ is homology cobordant to zero.

Solution: Σ is homology cobordant to itself, via $W = \Sigma \times [0, 1]$. Consider a path γ joining Σ to the other copy, through W. This has a regular neighbourhood $N = D^3 \times \gamma$, whose boundary is $\mathbb{S}^2 \times \gamma$.

Consider $W - \operatorname{int} N$. This has boundary $\Sigma\#(\mathbb{S}^2 \times \gamma)\#(-\Sigma) = \Sigma\#(-\Sigma)$. Next we compute $H_*(W - \operatorname{int} N; \mathbb{Z})$. By the homology long exact sequence, we have an exact sequence

$$\cdots \to H_i(W - \operatorname{int} N; \mathbb{Z}) \to H_i(W; \mathbb{Z}) \to \widehat{H}_i(\mathbb{S}^3 \times [0,1]; \mathbb{Z}) \to H_{i-1}(W - \operatorname{int} N; \mathbb{Z}) \to \cdots .$$

This is because relative homology is just reduced homology of the quotient. It follows immediately that $H_i(W - \operatorname{int} N; \mathbb{Z}) = H_i(D^4; \mathbb{Z})$. By an earlier proposition, this is equivalent to the statement that $\Sigma\#(-\Sigma)$ is homology cobordant to $\mathbb{S}^3$. △

Chapter 3

The Triangulation conjecture (is false)

The *triangulation conjecture* is the statement that all topological manifolds can be triangulated, in other words, all topological manifolds arise as the geometric realisation of a simplicial complex. This is true in dimensions 1, 2, and 3. However, it has been known to be false in dimension 4 by the work of Casson and Freedman, and we outline the more recent result here that it is false in all dimensions at least 5. This chapter of the notes is primarily sourced from Manolescu *The Conley index, gauge theory, and triangulations* [Man13b], but also borrows some definitions from Lurie [Lur09] (used in section 1) and Sato [Sat72] (used in section 3).

3.1 Simplicial triangulations vs PL structures

Here we review different structures that can be equipped on manifolds and investigate how they relate to each other. We begin with a review of the categories **Man**, $\mathbf{Man}^{\mathbf{PL}}$, and $\mathbf{Man}^{\infty}$.

Definition 3.1.1. **Man** consists of topological manifolds, with continuous maps as morphisms. $\mathbf{Man}^{\infty}$ consists of manifolds equipped with smooth structures, with morphisms smooth maps. Finally, $\mathbf{Man}^{\mathbf{PL}}$ consists of manifolds equipped with *piecewise linear* structures, with morphisms piecewise-linear maps.

While topological and smooth manifolds are familiar, piecewise linear manifolds are less so. We now work through some definitions to describe piecewise linear maps and piecewise linear manifolds.

Definition 3.1.2. A *k-simplex* in $\mathbb{R}^n$ is the convex hull of $k+1$ *geometrically independent* points in $\mathbb{R}^n$. That is, whenever $\sum_i c_i x_i = 0$, and $\sum_i c_i = 0$, then all of the c_i must vanish.

Given a simplex σ determined by $S = \{x_1, \ldots, x_k\}$, a *face* of σ is the convex hull of any subset of S. (This includes the empty simplex.)

A collection of K simplices is called a *simplicial complex* provided they glue correctly. More precisely,

1. If $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.
2. Any two simplices σ, τ in K intersect along a face of σ and a face of τ.
3. K is *locally finite*, i.e. given any point x in a simplex in K, there is a neighbourhood of x in $\mathbb{R}^n$ meeting finitely many simplices in K.

Definition 3.1.3. The *underlying polyhedron* of a simplicial complex K is the underlying topological space $|K| \subset \mathbb{R}^n$. Explicitly,

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

Conversely, K is called a *(simplicial) triangulation* of $|K|$. Any subset of $\mathbb{R}^n$ that admits a triangulation is called a *polyhedron*.

Next we define the corresponding notion of maps between polyhedra.

Definition 3.1.4. Let $P \subset \mathbb{R}^n$ be a polyhedron. A map $f : P \to \mathbb{R}^m$ is called *linear* if it is the restriction of an affine map $\mathbb{R}^n \to \mathbb{R}^m$. f is called *piecewise linear* (or PL) if there is a triangulation K of P so that $f|_\sigma$ is linear for each $\sigma \in K$. Finally, for P, Q polyhedra, a map $f : P \to Q$ is called PL if the induced map $f : P \to \mathbb{R}^m$ is PL.

In general a polyhedron is not a topological manifold, since there are no constraints on dimensions. To ensure that a polyhedron is a manifold, it suffices to declare that it is locally Euclidean.

Definition 3.1.5. Let P be a polyhedron. P is a *PL manifold* if there exists some n so that P is locally PL homeomorphic to $\mathbb{R}^n$. That is, for each $x \in P$, there exists a neighbourhood of x in P which is homeomorphic to $\mathbb{R}^n$, with the homeomorphism given by a PL map.

Note that the inverse of a PL homeomorphism is also PL, so this notion of PL-homeomorphism is symmetric. We are now ready to define the category $\mathbf{Man}^{\mathbf{PL}}$. The simplest definition is as follows:

Definition 3.1.6. The objects of the category $\mathbf{Man}^{\mathbf{PL}}$ are PL manifolds, and the morphisms are PL maps.

Equivalently, a PL manifold can be described as a topological manifold equipped with a combinatorial structure. We describe what this means, and give a proof outline of the equivalence.

Definition 3.1.7. Let M be an n-manifold. A *combinatorial structure* on M is a simplicial complex K such that $|K|$ is homeomorphic to M, and the link of each vertex of K is homeomorphic to $\mathbb{S}^{n-1}$.

The *link* of a vertex is essentially the union of sub-simplices that encloses the vertex. Formally, we have the following definitions given a simplicial complex K:

- Given $\sigma \in K$, the *star* of σ is

  $$\mathrm{Star}(\sigma) = \{\tau \subset \alpha \in K : \sigma \subset \alpha\}.$$

  More generally, the star of $S \subset K$ is the union of the stars of simplices in K.

- Given $\sigma \in K$, the *link* of σ is everything in the star disjoint from σ, i.e.

  $$\mathrm{lk}(\sigma) = \{\tau \in \mathrm{Star}(\sigma) : \tau \cap \sigma = \varnothing\}.$$

With this notation set up, we can prove the following:

Proposition 3.1.8. A PL manifold canonically determines a combinatorial structure on the underlying topological manifold, and vice versa.

Proof. We take for granted the following fact: if P is a polyhedron and K is a triangulation of p, then for any vertex $x \in K$, the homeomorphism class of $\mathrm{lk}(x)$ is independent of the choice of triangulation of P.

First suppose P is a topological manifold equipped with a combinatorial structure K. Let $x \in P$. If x is a vertex of K, then $\mathrm{lk}(x)$ is homeomorphic to $\mathbb{S}^{n-1}$. The star of x can be identified with the cone of $\mathrm{lk}(x)$, so it is a piecewise linear disk D^n. This descends to a piecewise linear homeomorphism of a neighbourhood of x in P with an open disk in $\mathbb{R}^n$. If x is not a vertex of K, then K can be modified so that x is indeed a vertex and $\mathrm{lk}(x)$ is still homeomorphic to $\mathbb{S}^{n-1}$. In any case, P is locally PL homeomorphic to $\mathbb{R}^n$ as required.

Conversely, suppose P is a PL manifold. Then any $x \in P$ has a neighbourhood which is PL homeomorphic to $\mathbb{R}^n$. The image of x under this homeomorphism can be taken to be the origin, and then $\mathrm{lk}(x)$ is homeomorphic to $\partial\Delta^n \cong \mathbb{S}^{n-1}$. In particular any triangulation of P gives a combinatorial structure. □

It turns out that the condition of links being spheres is non-trivial. Suppose M is a topological manifold that admits a triangulation. Then for low dimensions, it is true that any triangulation has links homeomorphic to spheres, but in high dimensions this ceases to be true. That is, all PL manifolds admit triangulations, but not all manifolds with triangulations are PL manifolds. In summary, we have the following inclusions:

$$\text{topological manifold} \supset \text{triangulable manifold} \supset \text{PL manifold} \supset \text{smooth manifold}.$$

3.2 The triangulation conjecture and related results

The *triangulation conjecture* states that all manifolds admit a triangulation. This is false, more explicitly in the following ways:

- Dimensions $n \leq 3$: true (Radó, Moise).
- Dimension $n = 4$: false (Casson, Freedman).
- Dimension $n \geq 5$: false for each such n (Manolescu).

For dimensions at most three, it is essentially a classical result. All topological manifolds of dimension up to three admit unique smooth structures, so by "sandwiching", they admit unique triangulations.

For dimension $n = 4$, we note that by Freedman's classification of topological 4-manifolds, the E_8-manifold is not smoothable. In dimension 4 PL and smooth structures are equivalent. Moreover, the link of any triangulation of a 4-manifold is guaranteed to be a homotopy 3-sphere, so by the Poincaré conjecture, triangulations are equivalent to PL structures. Therefore the E_8-manifold cannot be triangulated.

The case with $n \geq 5$ will be discussed in the most detail. Before this, we mention the *Hauptvermutung*.

Conjecture 3.2.1. Any two triangulations of a triangulable space have subdivisions that are combinatorially equivalent. (False, Milnor.)

In fact, it is also false if we restrict to manifolds, due to Kirby and Siebenmann.

Conjecture 3.2.2. Any two combinatorial structures on a manifold have subdivisons that are combinatorially equivalent. (False, Kirby-Siebenmann.)

Moreover, Kirby and Siebmann also disproved the "existence" statement: there exist manifolds of dimension at least 5 that admit not PL structures. The Hauptvermutung was settled several decades ago, but the triangulation conjecture for dimensions at least five was more recent. This result has two parts, which we explore in each of the subsequent sections.

But one final thing to note - we have yet to establish that triangulations are genuinely distinct from combinatorial structures! Do there exist triangulations which aren't combinatorial? That is, do there exist triangulations of manifolds whose links of vertices are not homeomorphic to spheres? Yes, this indeed true, and an example exists in five dimensions. We use the *double suspension theorem*:

Theorem 3.2.3 (Cannon, Edwards)**.** *Let M be a homology n-sphere (for n at least 3). Then the double suspension $\Sigma^2 M$ is homeomorphic to the standard $n+2$ sphere $\mathbb{S}^{2n}$.*

Recall that the suspension ΣX of a topological space X is $X \times [0,1]$, with $X \times \{0\}$ and $X \times \{1\}$ each identified to distinct points. Equivalently, ΣX is obtained by taking two cones over X. This has a simplicial version: given a simplicial complex K, the cone over K, denoted $C_K(x)$, is the simplicial complex consisting of all simplices in K, along with simplices spanned by $\{x\} \cup \sigma$ for $\sigma \in K$. Then $\Sigma K = C_K(x) \cup C_K(y)$.

To see how the double suspension theorem gives an example of a simplicial complex which isn't combinatorial, consider M to be the Poincaré homology sphere. Then $\Sigma^2 M$ is the 5-sphere $\mathbb{S}^5$, which certainly is a manifold and admits a combinatorial triangulation.

However, we claim that if K is a triangulation of M, then $\Sigma^2 K$ is not a combinatorial triangulation of $\Sigma^2 M$. Writing

$$\Sigma^2 K = C_{\Sigma K}(x) \cup C_{\Sigma K}(y),$$

the link of x is exactly ΣK. Therefore it suffices to show that $\Sigma K \cong \Sigma M$ is not a 4-sphere.

This follows from Van Kampen's theorem. If ΣM were a manifold, then (since it is four dimensional) it follows from Van Kampen's theorem that

$$\begin{aligned}\pi_1(\Sigma M - \{a,b\}) &= \pi_1(\Sigma M - (B_1^4 \sqcup B_2^4)) \\ &= \pi_1(\Sigma M - (B_1^4 \sqcup B_2^4)) *_1 1 \\ &= \pi_1(\Sigma M - (B_1^4 \sqcup B_2^4)) *_{\pi_1(\mathbb{S}_1^3 \sqcup \mathbb{S}_1^3)} \pi_1(B_1^4 \sqcup B_1^4) = \pi_1(\Sigma M).\end{aligned}$$

But taking a, b to be the cone points of the suspension ΣK, $\Sigma K - \{a,b\}$ is homotopic to K. Since K is not simply connected, neither is M. Therefore M cannot be the 4-sphere.

3.3 The connection between Θ^3 and triangulations in high dimensions

Theorem 3.3.1. *Let Θ^3 denote the integral homology cobordism group (of homology 3-spheres). Then the triangulation conjecture holds (in each dimension at least 5) if and only if Θ^3 has an element of order 2 with non-trivial Rokhlin invariant.*

More naturally we consider the following short exact sequence:

$$0 \to \ker \mu \to \Theta^3 \xrightarrow{\mu} \mathbb{Z}/2\mathbb{Z} \to 0. \tag{3.1}$$

Here μ denotes the Rokhlin invariant map, which we introduced in the previous chapter. This sequence is easily seen to split if and only if Θ^3 contains an element of order two with non-trivial Rokhlin invariant.

We now relate the above short exact sequence to a long exact sequence in cohomology:

$$\cdots \to H^4(M; \ker \mu) \to H^4(M; \Theta^3) \xrightarrow{\mu_*} H^4(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\delta} H^5(M; \ker \mu) \to \cdots.$$

To construct this long exact sequence, fix a manifold M and consider its singular complex $C_i(M)$. This a complex consisting solely of free abelian groups, i.e. free (and hence projective $\mathbb{Z}$-modules). Therefore the functor $\mathrm{Hom}_{\mathbb{Z}}(C_i(M), -)$ is exact. Applying this to the previous short exact sequence, we obtain a short exact sequence of complexes

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C_i(M), \ker \mu) \to \mathrm{Hom}_{\mathbb{Z}}(C_i(M); \Theta^3) \to \mathrm{Hom}_{\mathbb{Z}}(C_i(M); \mathbb{Z}/2\mathbb{Z}) \to 0.$$

The homology long exact sequence corresponding to this short exact sequence of complexes is exactly the one claimed above. The map δ is called the *Bockstein homomorphism*.

To understand theorem 3.3.1, we relate the splitting of equation 3.1 to a condition on the homology long exact sequence. The long exact sequence is then related to triangulations. To this end, we define an invariant of triangulations:

Definition 3.3.2. Let K be a triangulation of a closed n-manifold M (with $n \geq 5$). Let K_{n-4} be the set of $(n-4)$-simplices of K, and $C_{n-4}(K)$ the corresponding space of integral simplicial $(n-4)$-chains. A homomorphism $\lambda : C_{n-4}(K) \to \Theta^3$ is determined uniquely by the data $\lambda(\sigma) = \mathrm{lk}(\sigma)$ for each $\sigma \in K_{n-4}$. The *Sullivan-Cohen-Sato class* is then the cohomology class:

$$c(K) = [\lambda] \in H^4(M; \Theta^3).$$

Note that if $\sigma \in K_{n-4}$, then $\mathrm{lk}(\sigma)$ is indeed a homology 3-sphere. To establish that the above is well defined, it remains to show that λ is really a cocycle. Equivalently, (if it is really a cocycle and M is orientable), by Poincaré duality $c(K)$ can be thought of as

$$\sum_{\sigma \in K_{n-4}} [\lambda(\sigma)]\sigma \in H_{n-4}(M; \Theta^3).$$

Therefore we give a proof outline that the above class is really a cycle. We must show that $\sum_{\sigma \in K_{n-4}} [\lambda(\sigma)] d\sigma = 0$. To this end, let $\mu \in K_{n-5}$ be an arbitrary $(n-5)$-simplex. Then μ is a face of a collection of $(n-4)$-simplices σ_i. The link of μ is a homology 4-sphere. The links of σ_i are contained in $\mathrm{lk}(\mu)$. In fact, more precisely, a cone over each link of σ_i is embedded in $\mathrm{lk}(\mu)$. Removing a cone point x_i from each, $\mathrm{lk}(\mu) - \{x_i\}$ gives exactly the desired cobordism of links of σ_i to the usual 3-sphere. Using this, one can show that $\sum_{\sigma \in K_{n-4}} [\lambda(\sigma)] d\sigma = 0$ as required.

To relate this class to the existence (or lack-thereof) of triangulations, we use the *Kirby-Siebenmann invariant.*

Proposition 3.3.3. Let M be a closed manifold of dimension at least 5. Then the Kirby-Siebenmann invariant $\kappa(M) \in H^4(M; \mathbb{Z}/2\mathbb{Z})$ vanishes if and only if M admits a combinatorial triangulation.

For manifolds that admit simplicial triangulations, the Sullivan-Cohen-Sato class actually determines the Kirby-Siebenmann class, in the following way:

Proposition 3.3.4. Let M be a closed oriented manifold of dimension at least 5, with a triangulation K. Then $\mu_* c(K) = \kappa(M) \in H^4(M; \mathbb{Z}/2\mathbb{Z})$.

For any such M, K, we necessarily have that $\delta(\mu_* c(K)) = \delta(\kappa(M)) = 0$, simply because the long exact sequence of cohomology is a cochain complex. In general M need not be triangulable. In that case, we can still consider the expression $\delta(\kappa(M))$. If this expression is non-zero, then we certainly cannot equip M with a triangulation, as it will contradict $\delta(\mu_* c(K)) = \delta(\kappa(M))$. Using obstruction theory, one can conversely show that if $\delta(\kappa(M)) = 0$, then M can be triangulated. In summary we have the following:

Proposition 3.3.5. Let M be a closed oriented manifold of dimension at least 5. Then $\delta(\kappa(M)) = 0$ if and only if M can be triangulated.

To relate this result back to the structure of Θ^3, it remains to show that the short exact sequence 3.1 splits if and only if $\delta(\kappa(M)) = 0$ for all M with dimension at least 5.

Proposition 3.3.6. If the short exact sequence 3.1 splits, then all manifolds of dimension at least 5 are triangulable.

Proof. Suppose the short exact sequence splits, so we have

$$\Theta^3 \cong \ker \mu \times \mathbb{Z}/2\mathbb{Z}.$$

The Bockstein homomorphism $\delta : H^4(M; \mathbb{Z}/2\mathbb{Z}) \to H^5(M; \ker \mu)$ is defined to be the connecting homomorphism. Choose an arbitrary element $[c]$ in $H^4(M; \mathbb{Z}/2\mathbb{Z})$. Then the connecting homomorphism sends it to $[da] = 0$ in $H^5(M; \ker \mu)$, by chasing the following diagram.

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^i(M, \ker \mu) & \longrightarrow & C^i(M, \Theta^3) & \longrightarrow & C^i(M, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{i+1}(M, \ker \mu) & \longrightarrow & C^{i+1}(M, \Theta^3) & \longrightarrow & C^{i+1}(M, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$\begin{array}{ccccc}
 & & (a, c) & \longmapsto & c \\
 & & \downarrow & & \downarrow \\
da & \longmapsto & (da, 0) & \longmapsto & 0
\end{array}
\qquad c \stackrel{\delta}{\longmapsto} da$$

This is because $C^i(M, \Theta^3) \cong C^i(M, \ker \mu) \times C^i(M, \mathbb{Z}/2\mathbb{Z})$. But if the Bockstein homomorphism is trivial, then $\delta(\kappa(M))$ always vanishes! Therefore any M is triangulable. $\square$

The converse is also true. It turns out that if the short exact sequence 3.1 doesn't split, then there exist manifolds of all dimension at least 5 which are not triangulable. It turns out that it suffices to consider a 5 dimensional example. To do this, we must first learn a little about Steenrod algebras.

Definition 3.3.7. The *Steenrod algebra* A_2 is the algebra over $\mathbb{Z}/2\mathbb{Z}$ consisting of all *stable cohomology operations* for mod 2 cohomology.

A *cohomology operation* is a natural transformation $F^i : H^n(-;\mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-;\mathbb{Z}/2\mathbb{Z})$. It is said to be *stable* if it commutes with suspension. That is, we require the following diagram to commute:

$$\begin{array}{ccc} H^n(X;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{F^i} & H^{n+i}(X;\mathbb{Z}/2\mathbb{Z}) \\ \downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma} \\ H^{n+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})) & \xrightarrow{F^i} & H^{n+i+1}(\Sigma Z;\mathbb{Z}/2\mathbb{Z}) \end{array}$$

For example, the "cup square" $S : H^n \to H^{2n}$ defined by $x \mapsto x \smile x$ is not stable, because cup products on suspensions are trivial. However, we have the following:

Theorem 3.3.8. *The* Steenrod algebra A_2 *is generated by the* Steenrod squares $\mathrm{Sq}^n : H^m \to H^{m+n}$. *These are characterised by the following axioms:*

- *Each* Sq^n *is a natural transformation. More concretely, for any* X, Y*, any* $f : X \to Y$*, and* $y \in H^m(Y;\mathbb{Z}/2\mathbb{Z})$*, we have*
$$f^*(\mathrm{Sq}^n(y)) = \mathrm{Sq}^n(f^*(y)).$$
- Sq^0 *is the identity homomorphism.*
- $\mathrm{Sq}^n(x) = x \smile x$ *for* $x \in H^n(X;\mathbb{Z}/2\mathbb{Z})$.
- *If* $n > \deg(x)$*, then* $\mathrm{Sq}^n(x) = 0$.
- *The* Cartan formula. *Namely, for any* x, y*, we have*
$$\mathrm{Sq}^n(x \smile y) = \sum_{i+j=n} (\mathrm{Sq}^i\, x) \smile (\mathrm{Sq}^j\, y).$$

Example. The Steenrod square Sq^1 is the Bockstein homomorphism β of the exact sequence
$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

It suffices to verify that β satisfies the axioms listed above. First we verify naturality. Let $f : X \to Y$ be arbitrary, and $\lambda \in C^i(Y; \mathbb{Z}/2\mathbb{Z})$ a cocycle. Observe that we can write

$$\beta(\lambda) = d^*\widehat{\lambda}$$

where $\widehat{\lambda}$ is a lift of $\lambda \in C^i(Y; \mathbb{Z}/2\mathbb{Z})$ to $C^i(Y; \mathbb{Z}/4\mathbb{Z})$, and d^* is the induced map on the cocomplex $C^*(Y; \mathbb{Z}/4\mathbb{Z})$. The equality above comes from an explicit description of the connecting homomorphism in the long exact sequence of cohomology, i.e. the Bockstein homomorphism. Let $\widetilde{f_i}$ be the induced maps $C_i(X) \to C_i(Y)$. This gives

$$f^*(\beta[\lambda]) = \beta([\lambda]) \circ \widetilde{f}_{i+1} = \widehat{\lambda} \circ d \circ \widetilde{f}_{i+1} = \widehat{\lambda} \circ \widetilde{f}_i \circ d = d^*(f^*\widehat{\lambda}) = \beta(f^*[\lambda]).$$

This verifies naturality as required.

The second condition - that Sq^0 is the identity - is irrelevant (for now).

The third condition requires us to verify that $\beta(x) = x \smile x$ for $\lambda \in H^1(X; \mathbb{Z}/2\mathbb{Z})$. We first reduce the problem to an easier one: by Brown's representability theorem, any $x \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ is the pullback of some $y \in H^1(K(\mathbb{Z}/2\mathbb{Z}, 1); \mathbb{Z}/2\mathbb{Z}) = H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$. Therefore it suffices to verify the result for $X = \mathbb{RP}^2$ (since both β and cup products commute with pullbacks). But now the cup product and Bockstein homomorphism agree trivially (that is, they both vanish).

For the fourth condition, we must show that if $\deg x = 0$, then $\beta x = 0$. That is, we must show that the Bockstein homomorphism $\beta : H^0(X; \mathbb{Z}/2\mathbb{Z}) \to H^1(X; \mathbb{Z}/2\mathbb{Z})$ is trivial. But this is immediate from the long exact sequence of cohomology, which gives

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\beta} H^1(X; \mathbb{Z}/2\mathbb{Z}).$$

Finally for the fifth condition, we require that

$$\beta(x \smile y) = x \smile \beta(y) + \beta(x) \smile y.$$

This again follows from an explicit formula of the Bockstein homomorphism, since

$$d^*(x \smile y) = d^*x \smile y + (-1)^{\deg x} x \smile d^*y.$$

Note that everything is mod 2, so the sign is in fact irrelevant. The result follows.

To see how this is relevant, we now prove the following result which relates the first Steenrod square to triangulability.

Proposition 3.3.9. Suppose M is a manifold with $\mathrm{Sq}^1(\kappa(M)) \neq 0$, and suppose the short exact sequence 3.1 doesn't split. Then M is not triangulable.

Proof. We prove the contrapositive. Suppose the short exact sequence 3.1 doesn't split, and M is triangulable. We will show that $\mathrm{Sq}^1(\kappa(M)) = 0$. Let K be a triangulation of M, and L the subgroup of Θ^3 generated by links of K homologous to 3-spheres. (That is, all links of simplices $\sigma \in K_{n-4}$, where $\dim M = n$.) Consider the following diagram:

$$\begin{array}{ccccccccc} & & & & L & \xrightarrow{\iota} & \Theta^3 & & \\ & & & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mu} & & \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Here μ is the Rokhlin map, and ι is the inclusion map. Our initial aim is to construct a map φ so that the diagram commutes. By assumption, the short exact sequence 3.1 doesn't split, so any $\Sigma \in L$ with $\mu(\Sigma) = 1$ does not have order 2. By the structure theorem of finitely generated modules over a PID, we can write

$$L = \langle \Sigma_1 \rangle \oplus \cdots \oplus \langle \Sigma_k \rangle$$

for some homology 3-spheres Σ_i. (Here we are abusing notation - really the generators should be homology cobordism classes of homology 3-spheres.) We now define $\varphi : L \to \mathbb{Z}/4\mathbb{Z}$ as follows:

- If $\mu(\Sigma_i) = 0$, then $\varphi(\Sigma_i) = 0$.
- If $\mu(\Sigma_i) = 1$, and there is an isomorphism $\alpha : \langle \Sigma_i \rangle \to \mathbb{Z}$, then define $\varphi(\Sigma_i) = \alpha(\Sigma_i) \mod 4$.
- If $\mu(\Sigma_i) = 1$, and $\langle \Sigma_i \rangle$ is not infinite cyclic, then by the Chinese remainder theorem, Σ_i has order p^k for some prime p and $k \geq 1$. Then p^k is necessarily even, because

  $$0 = \mu(\mathbb{S}^3) = \mu(\Sigma_i^{p^k}) = \mu(\Sigma_i)p^k = p^k \mod 2.$$

  This forces p to be an even prime, so $p = 2$. Since Σ_i cannot have order 2 by assumption, there is an isomorphism $\alpha : \langle \Sigma_i \rangle \to \mathbb{Z}/2^k\mathbb{Z}$ for $k \geq 2$. We define $\varphi(\Sigma_i) = \alpha(\Sigma_i) \mod 4$.

This completes the definition of φ, and it is clear from the construction that it makes the above diagram commute.

Using the above diagram, we are now ready to verify our claim. Applying the homology functor, we obtain a diagram

$$\begin{array}{ccccccccc} & & H^4(M;L) & \xrightarrow{\iota^*} & H^4(M;\Theta^3) & & & & \\ & & \downarrow{\scriptstyle\varphi^*} & & \downarrow{\scriptstyle\mu^*} & & & & \\ \cdots & \longrightarrow & H^4(M;\mathbb{Z}/4\mathbb{Z}) & \xrightarrow{\pi^*} & H^4(M;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\mathrm{Sq}^1} & H^5(M;\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \cdots \end{array}$$

where we have used the fact that Sq^1 is the Bockstein homomorphism. Let $c(K) \in H^4(M;\Theta^3)$ denote the Sullivan-Cohen-Sato class defined earlier. Then $\mu^* c(K) = \kappa(M)$, the Kirby-Siebenmann class. Since L is generated by the links of $\sigma \in K_{n-4}$, $c(K)$ lifts to $\widetilde{c}(K) \in H^4(M;L)$. This gives

$$\mathrm{Sq}^1(\kappa(M)) = (\mathrm{Sq}^1 \circ \mu^* \circ \iota^*)(\widetilde{c}(K)) = (\mathrm{Sq}^1 \circ \pi^* \circ \varphi^*)(\widetilde{c}(K)) = 0,$$

because $\mathrm{Sq}^1 \circ \pi^* = 0$ by virtue of being a complex. Therefore $\mathrm{Sq}^1(\kappa(M))$ vanishes as required. $\square$

In fact, the following proposition shows that we need only find a five dimensional example of a manifold with $\mathrm{Sq}^1(\kappa M)$ non-zero!

Proposition 3.3.10. Let $T^m = \prod_m \mathbb{S}^1$. If M is a 5-manifold with $\mathrm{Sq}^1(\kappa(M)) \neq 0$, then for each $n \geq 5$, $N = M \times T^{n-5}$ is an n-manifold with $\mathrm{Sq}^1(\kappa(N)) \neq 0$.

Proof. Let $p : M \times T^{n-5} \to M$ be the projection map. Then $\kappa(M \times T^{n-5}) = p^*\kappa(M)$. Consider the following diagram:

$$\begin{array}{ccc} H^4(M;\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\mathrm{Sq}^1} & H^5(M;\mathbb{Z}/2\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^4(M \times T^{n-5};\mathbb{Z}/2\mathbb{Z})) & \xrightarrow{\mathrm{Sq}^1} & H^5(M \times T^{n-5};\mathbb{Z}/2\mathbb{Z})) \end{array}$$

The vertical maps are inclusion maps by the Künneth formula. Therefore

$$\mathrm{Sq}^1(\kappa(N)) = \mathrm{Sq}^1 p^*(\kappa(M)) = p^* \mathrm{Sq}^1 \kappa(M) \neq 0.$$

$\square$

Finally we construct an example of a 5-manifold M such that $\mathrm{Sq}^1(\kappa(M)) \neq 0$. This example is due to Kronheimer, and uses Freedman's classification of 4-manifolds.

Proposition 3.3.11. There exists a closed 5-manifold M such that $\mathrm{Sq}^1(\kappa(M)) \neq 0$.

Proof. Recall Freedman's classification of 4-manifolds: *For every unimodular symmetric bilinear form Q, there exists a simply connected closed topological 4-manifold X whose intersection form is Q. If Q is even, X is unique up to homeomorphism with $\kappa(X) = \sigma Q/8$* mod 2. *If Q is odd, there are exactly two homeomorphism types of such an X_i, with $\kappa(X_1) = 0$ and $\kappa(X_2) = 1$.*

With this in mind, there exists a 4-manifold X such that $Q_X = (1)\oplus(-1)$ and $\kappa(X) = 1$. Since Q_X is isomorphic to $-Q_X$, again by Freedman's theorem, there exists an orientation reversing homeomorphism $f : X \to X$. Let M be the mapping torus $X \times [0,1]/(x,1) \sim (f(x),0)$. This is a 5-manifold, and one can show that $\mathrm{Sq}^1(\kappa(M)) \neq 0$.

On one hand, $\kappa(M) \in H^4(M;\mathbb{Z}/2\mathbb{Z})$ is non-trivial by construction, with Poincaré dual a section of the bundle $M \to \mathbb{S}^1$. By Wu's formula, we know that

$$\mathrm{Sq}^1(\kappa(M)) = \kappa(M) \smile v_1$$

where v_1 is the first *Wu class.* This is defined implicitly by

$$0 + v_1 = \mathrm{Sq}^1(v_0) + \mathrm{Sq}^0(v_1) = w_1(TM).$$

Note that $\mathrm{Sq}^1(v_0)$ is trivial since $\deg v_0 < 1$. Therefore

$$\mathrm{Sq}^1(\kappa(M)) = \kappa(M) \smile w_1(TM) = 1 \neq 0.$$

To see that $\kappa(M) \smile w_1(TM)$ is non-zero, we noted earlier that $\kappa(M)$ has Poincaré dual a section of the bundle $M \to \mathbb{S}^1$. On the other hand, $w_1(TM)$ has Poincaré dual X. These intersect transversely at a point, giving the desired result. □

The previous three positions combine to give the following result:

Proposition 3.3.12. If the short exact sequence 3.1 doesn't split, then in each dimension at least 5, there exist non-triangulable (closed) manifolds.

We already proved the converse at an earlier time. In summary, we've established the main theorem of this section:

Theorem 3.3.13. *All manifolds of dimension at least 5 are triangulable if and only if* Θ^3 *contains an element of order 2 with non-trivial Rokhlin invariant.*

3.4 Disproving the triangulation conjecture

As mentioned earlier, the triangulation conjecture is in fact false. This is because Θ^3 *does not* contain any elements of order 2 with non-trivial Rokhlin invariant. How do we show that Θ^3 contains no such elements? The proof of Manolescu uses a certain Floer homology theory.

The strategy is as follows: we are familiar with the Rokhlin invariant map

$$\mu : \Theta^3 \to \mathbb{Z}/2\mathbb{Z}.$$

Suppose where was a lift $\widetilde{\mu}$ of μ to the integers, such that $\mu = \widetilde{\mu} \mod 2$, and $\widetilde{\mu}(-\Sigma) = -\widetilde{\mu}(\Sigma)$. Then for any homology sphere Σ of order 2,

$$\widetilde{\mu}(\Sigma) = \widetilde{\mu}(-\Sigma) = -\widetilde{\mu}(\Sigma) \Longrightarrow \widetilde{\mu}(\Sigma) = 0.$$

Therefore $\mu(\Sigma) = 0$, so there can be no order two elements of Θ^3 with non-trivial Rokhlin invariant. This was exactly Manolescu's strategy.

Proposition 3.4.1. There exists a lift of μ to a map $\widetilde{\mu} : \Theta^3 \to \mathbb{Z}$ such that its reduction mod 2 is the Rokhlin map, and $\widetilde{\mu}(-\Sigma) = -\widetilde{\mu}(\Sigma)$.

The strategy is to define a certain homology theory

$$\Sigma \mapsto SWFH_*^{\mathrm{Pin}(2)}(\Sigma)$$

for each homology 3-sphere Σ. This is called the Pin(2)-equivariant Seiberg-Witten Floer homology. It turns out that an integer associated to this homology theory gives exactly the lift we desire. Specifically,

$$\widetilde{\mu}(\Sigma) = \frac{1}{2}(\min\{\deg(\text{middle } v\text{-tower in } SWFH_*^{\mathrm{Pin}(2)}(\Sigma))\} - 1).$$

The homology is equipped with a map v which changes the index by 4. This gives four "towers", namely for each index modulo 4. The first tower consists of elements in degrees $2\mu(\Sigma) \mod 4$. The second, elements in degrees $2\mu(\Sigma)+1 \mod 4$, and the third, elements in degrees $2\mu(\Sigma)+2 \mod 4$. It turns out that the fourth tower is always trivial. We call these v-towers, so that $\widetilde{\mu}(\Sigma)$ is defined to be the $(B-1)/2$, where B is lowest degree that is realised by an element of the 2nd tower.

One can show that such a homology theory really exists, that $\widetilde{\mu}$ is a well defined invariant of homology cobordism classes of homology 3-spheres, and that it satisfies both of the desired properties listed above.

In summary, the proof relies on the existence of an invariant β satisfying the following properties:

1. $\beta(\Sigma)$ is an invariant of integral homology 3-spheres which is also invariant under homology cobordisms

2. $\beta(-\Sigma) = -\beta(\Sigma)$, where $-\Sigma$ is Σ with its orientation reversed.

3. $\beta(\Sigma)$ reduces mod 2 to the Rokhlin invariant.

We note that invariants which *almost* satisfy these properties were constructed at earlier times. Firstly, the *Casson invariant*. This satisfies properties 2 and 3 above, but not 1. Secondly, the *Froyshøv invariant* (which is also defined via Seiberg-Witten Floer homology) satisfies properties 1 and 3, but not 2. In the next chapter we explore the Casson invariant.

Chapter 4

The Casson invariant

4.1 When does rational surgery preserve homology?

The Casson invariant (which we introduce in the next section) is defined on integral homology spheres. It is characterised by how it behaves under rational surgeries, so we first explore how homology groups behave under rational surgery, and when rational surgery even makes sense.

First we show that rational surgery makes sense on integral homology 3-spheres. We write $\mathcal{S}$ to denote the set of integral homology 3-spheres. (Note that as sets $\Theta^3 = \mathcal{S}/\sim$.)

Proposition 4.1.1. Let $\Sigma \in \mathcal{S}$, and $K \in \Sigma$ a knot. Then K has well defined meridians and canonical longitudes. In particular, rational surgery is well defined.

Proof. Recall that a *meridian* of K is a generator of $H_1(\Sigma - N)$, where N is a regular neighbourhood of K. A *canonical longitude* is the unique longitude of K in ∂N which is homologically trivial in $\Sigma - N$. For notational brevity, we write $X = \Sigma - N$.

To establish the existence and uniqueness of these classes, we first calculate $H_i(X)$. We already know that $H_i(X) = 0$ for $i \geq 4$, while $H_i(X) = \mathbb{Z}$ for $i = 0, 3$. By Poincaré duality, it remains to determine $H_1(X)$. This can be determined by the Mayer-Vietoris sequence:

$$\cdots \to H_2(\Sigma) \to H_1(\partial N) \to H_1(X) \oplus H_1(N) \to H_1(\Sigma) \to \cdots .$$

Since Σ is a homology sphere, $H_1(\Sigma) = H_2(\Sigma) = 0$. On the other hand, ∂N is a 2-torus, so $H_1(\partial N) = \mathbb{Z}^2$. N is a solid torus, so $H_1(N) = \mathbb{Z}$. Therefore we have an exact sequence

$$0 \to \mathbb{Z}^2 \to H_1(X) \oplus \mathbb{Z} \to 0.$$

It follows that $H_1(X) = \mathbb{Z}$, so it has two generators, and just one up to sign. This defines the meridian of K.

Next we show that K has a canonical longitude. We consider the long exact sequence of relative homology:

$$\cdots \to H_1(\partial N) \to H_1(X) \to H_1(X, \partial N) \to H_0(\partial N) \to \cdots .$$

The pair $(X, \partial N)$ has the same homology as (Σ, K) by retracting ∂N onto K radially. Since Σ is a homology sphere, $H_1(\Sigma, K) = H_1(X, \partial N)$ is trivial. Therefore the above is an exact sequence

$$\mathbb{Z}^2 \to \mathbb{Z} \to 0 \to 0,$$

where $\mathbb{Z}^2$ is generated by a meridian $(1, 0)$ and some longitude $(n, 1)$. The first map in the above exact sequence maps (p, q) onto p, so there is a unique longitude (up to sign) mapping to the trivial class.

In summary this shows the existence of meridians and canonical longitudes which are unique up to sign. Fixing signs as in chapter 1, rational surgery is well defined. □

This shows that rational surgery is well defined on homology spheres! But we also want to know when a homology sphere remains a homology sphere, following rational surgery.

Example. Recall that the lens space $L(p, q)$ is obtained via $-p/q$-surgery on the unknot in $\mathbb{S}^3$. Moreover, we calculated that

$$H_i(L(p,q)) = \begin{cases} \mathbb{Z} & i \in \{0, 3\} \\ \mathbb{Z}/p\mathbb{Z} & k = 1 \\ 0 & k = 2. \end{cases}$$

Therefore $L(p, q)$ is a homology sphere if and only if $p = \pm 1$.

This result can be made a little more general. The above homological results hold for rational surgery in homology spheres!

Proposition 4.1.2. Let Σ be a homology sphere, and $p, q \in \mathbb{Z}$ coprime. Then p/q-surgery along a knot $K \subset \Sigma$ produces a homology sphere if and only if $p = \pm 1$. More precisely,

$$H_1(\Sigma') = \mathbb{Z}/p\mathbb{Z}$$

where Σ' is the result of p/q-surgery along K in Σ.

Proof. As in the previous proof, it suffices to study $H_1(\Sigma')$, where Σ' is the manifold obtained by p/q-surgery along K in Σ.

Explicitly, $\Sigma' = (\Sigma - N) \sqcup_{\varphi} D^2 \times \mathbb{S}^1$, where φ is defined by sending the curve $\partial D^2 \times 1$ in $\partial(D^2 \times \mathbb{S}^1)$ to $\gamma = p\mu + q\lambda$, where μ and λ are the meridian and canonical longitudes of K.

The Mayer-Vietoris sequence now gives

$$\cdots \to H_2(\Sigma') \to H_1(\partial N) \xrightarrow{(i_*,j_*)} H_1(\Sigma - N) \oplus H_1(D^2 \times \mathbb{S}^1) \xrightarrow{k_* - l_*} H_1(\Sigma') \to 0.$$

$H_1(\partial N)$ has two generators, m and ℓ, which are the "canonical" meridian and longitude by considering the solid torus N which it bounds. The maps i, j, k, l are the expected inclusions.

Under (i_*, j_*), the class $[m]$ is mapped to $[\gamma] + [m] = p[\mu]$. The other generator $[\ell]$ is mapped to $c + [\ell]$ for some class c which we can take to be trivial. Therefore we have a surjective map

$$\mathbb{Z} \oplus \mathbb{Z} \to H_1(\Sigma')$$

with kernel $\langle p[\mu], [\ell] \rangle$. By the first isomorphism theorem, $H_1(\Sigma')$ is isomorphic to

$$(\mathbb{Z} \oplus \mathbb{Z})/(p\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}.$$

Therefore Σ' is a homology sphere if and only if $p = \pm 1$. □

Since we are working with integers, this means that when doing surgery on homology spheres to obtain new homology spheres, we need only consider $1/m$-surgery for $m \in \mathbb{Z}$.

Definition 4.1.3. Let Σ be a homology sphere and $K \subset \Sigma$ a knot. We write $\Sigma + \frac{1}{m}K$ to denote the homology sphere resulting from $1/m$-surgery along K in Σ.

We have now established the essential background to study the Casson invariant.

4.2 The Casson invariant: uniqueness and other properties

As remarked at the end of the previous chapter, the Casson invariant is an invariant of homology 3-spheres which lifts the Rokhlin invariant to the integers. Writing $\mathcal{S}$ to denote the set of integral homology 3-spheres, the Casson invariant can be classified as follows:

Definition 4.2.1. A *Casson invariant* is a map $\lambda : \mathcal{S} \to \mathbb{Z}$ satisfying the following properties:

(1) $\lambda(\mathbb{S}^3) = 0$.

(2) For any homology 3-sphere Σ and knot $K \subset \Sigma$, the difference

$$\lambda\Big(\Sigma + \frac{1}{m+1}K\Big) - \lambda\Big(\Sigma + \frac{1}{m}K\Big)$$

is independent of m.

(3) For any homology 3-sphere Σ, *boundary link* $K \cup L \subset \Sigma$, and $m, n \in \mathbb{N}$,

$$\lambda\Big(\Sigma + \frac{1}{m+1}K + \frac{1}{n+1}L\Big) - \lambda\Big(\Sigma + \frac{1}{m}K + \frac{1}{n+1}L\Big) - \lambda\Big(\Sigma + \frac{1}{m+1}K + \frac{1}{n}L\Big) + \lambda\Big(\Sigma + \frac{1}{m}K + \frac{1}{n}L\Big) = 0.$$

By a boundary link, we mean a link which admits a Seifert surface consisting of disjoint pieces, one for each component of the link.

Remark. If $K \cup L$ is a boundary link in Σ, then $\mathrm{lk}(K, L) = 0$. Whenever $\mathrm{lk}(K, L) = 0$, the surgery $\Sigma + (p/q)K + (r/s)L$ is independent of the order in which the surgery is carried out and gives a homology sphere.

Proposition 4.2.2. The Casson invariant described above exists and is unique up to scalar multiple.

This will be the main result that we discuss in the coming sections. Uniqueness is easier than existence, so we start by proving uniqueness. We will also observe that the Casson invariant can truly be made unique by adding the following condition:

(0) $\lambda(\mathcal{S})$ is not contained in any proper subgroup of $\mathbb{Z}$.

For notational brevity, we write

$$\lambda'(K) = \lambda'(\Sigma, K) = \lambda(\Sigma + \frac{1}{m+1}K) - \lambda(\Sigma + \frac{1}{m}K),$$

$$\lambda''(K, L) = \lambda''(\Sigma, K, L) = \lambda\Big(\Sigma + \frac{1}{m+1}K + \frac{1}{n+1}L\Big) - \lambda\Big(\Sigma + \frac{1}{m}K + \frac{1}{n+1}L\Big) - \lambda\Big(\Sigma + \frac{1}{m+1}K + \frac{1}{n}L\Big) + \lambda\Big(\Sigma + \frac{1}{m}K + \frac{1}{n}L\Big).$$

If λ' is a well defined knot invariant (i.e. independent of m), then so is λ''. Therefore the three basic properties of the Casson invariant are as follows:

(1) $\lambda(\mathbb{S}^3) = 0$,

(2) $\lambda'(\Sigma, K)$ is well defined,

(3) $\lambda''(\Sigma, K, L) = 0$ for $K \cup L$ a boundary link.

To describe our proof of uniqueness, we introduce some additional properties of the Casson invariant:

(4) $\lambda'(\Sigma, 3_1) = \pm 1$ for any homology sphere Σ.

(5) $\lambda'(\Sigma, K) = \frac{1}{2}\Delta''_K(1)\lambda'(\Sigma, 3_1)$ for any $K \subset \Sigma$.

Uniqueness will then follow the following outline:

- Show that (0), (1), (2), and (3) imply (4) and (5).
- Show that (0), (1), (2), (4), and (5) imply uniqueness.

Note that the Casson invariant also satisfies the following useful properties:

(6) $\lambda(-\Sigma) = -\lambda(\Sigma)$ where $-\Sigma$ denotes Σ with the opposite orientation.

(7) $\lambda(\Sigma_1 \# \Sigma_2) = \lambda(\Sigma_1) + \lambda(\Sigma_2)$.

(8) $\lambda(\Sigma) = \mu(\Sigma) \mod 2$, where μ is the Rokhlin invariant.

Lemma 4.2.3.

$$(0), (1), (2), (3) \implies (4), (5).$$

Proof. We give a proof outline. First we reduce the problem to knots in $\mathbb{S}^3$. Specifically, for any $K \subset \Sigma$, we show that there exists $L \subset \mathbb{S}^3$ such that

$$\lambda'(\Sigma, K) = \lambda'(\mathbb{S}^3, L), \quad \Delta_K(t) = \Delta_L(t).$$

This uses property (3) of the Casson invariant.

Next we verify that $\lambda'(-)$ and $\frac{1}{2}\Delta''_-(1)$ change by the same rule in a crossing change. More precisely, in the *knot* picture, any two knot diagrams are related by changing the over-under data of crossings in diagrams. In the *surgery* picture, let $K \subset \mathbb{S}^3$ be a knot, and D a disk intersecting K at two points. Swapping crossing data of K corresponds to ± 1-surgery along the boundary of D. Note that ± 1-surgery along the unknot leaves the ambient space diffeomorphic to $\mathbb{S}^3$. This operation is called a *twist* across D. Let $c = \partial D$. Then the knot obtained from the twist is denoted K_c.

Suppose we have another disk D', with boundary c', so that twists of K along D or D' are disjoint. This gives knots $K, K_c, K_{c'}$, and $K_{cc'}$ which we now consider. In the surgery picture, it follows from property (3) that

$$\lambda''(\mathbb{S}^3, K, c) = \lambda''(\mathbb{S}^3 + c', K, c).$$

In the knot theoretic picture, we have

$$\frac{1}{2}\Delta''_{K_c}(1) - \frac{1}{2}\Delta''_K(1) = \frac{1}{2}\Delta''_{K_{cc'}}(1) - \frac{1}{2}\Delta''_{K_{c'}}(1).$$

The statement with the Alexander polynomial comes from skein relations.

Using the above, one can show that $\lambda'(K)$ is proportional to $\lambda'(3_1)$ for any K, so we conclude property (4) from property (0). Finally we calculate the proportionality constant and conclude property (5). ∎

Lemma 4.2.4.

$$(1),(2),(4),(5) \implies \text{uniqueness.}$$

Proof. Let Σ be a homology sphere. Then there exist knots $K_1, \dots, K_n$ with pairwise vanishing linking number, and $\varepsilon_i = \pm 1$ such that

$$\Sigma = \mathbb{S}^3 + \varepsilon_1 K_1 + \cdots + \varepsilon_n K_n.$$

Define $\Sigma_i = \mathbb{S}^3 + \varepsilon_1 K_1 + \cdots + \varepsilon_i K_i$, so that $\Sigma_0 = \mathbb{S}^3$ and $\Sigma_n = \Sigma$. Then

$$\lambda(\Sigma_i) - \lambda(\Sigma_{i-1}) = \varepsilon_i \lambda'(K_i)$$

by the definition of λ'. Using property property (1), we then have

$$\lambda(\Sigma) = \sum_{i=1}^{n} (\lambda(\Sigma_i) - \lambda(\Sigma_{i-1})) + \lambda(\Sigma_0) = \sum_{i=1}^{n} \varepsilon_i \lambda'(K_i).$$

But now from properties (4) and (5), we have

$$\lambda(\Sigma) = \pm\Big(\sum_{i=1}^{n} \frac{\varepsilon_i}{2} \Delta''_{K_i,\Sigma_i}(1) \Big)$$

where the sign depends on $\lambda'(3_1)$. ∎

Combining the previous two lemmas, we have the following result:

Theorem 4.2.5. *The Casson invariant is uniquely determined by properties (0), (1), (2), and (3).*

4.3 Construction of the Casson invariant

We have yet to show that the Casson invariant exists! This can be constructed as a certain count of a moduli space of SU(2)-valued representations of the fundamental group of homology spheres. Here we will construct the Casson invariant without explaining why certain choices are being made, and in later sections explain what goes wrong if we change our choices. The outline is as follows:

1. For any manifold M consider the space of representations

   $$R(M) = \mathrm{Hom}(\pi_1 M, \mathrm{SU}(2)).$$

   This is a topological space (equipped with the compact-open topology). SO(3) acts on $R(M)$ by conjugation, and $\mathcal{R}(M) \coloneqq R^{\mathrm{irr}}(M)/\,\mathrm{SO}(3)$ is called the *representation space* of M. (Note that $R(M)/\,\mathrm{SU}(2)$ is a *character variety.*) Here $R^{\mathrm{irr}}(M) \subset R(M)$ is the subspace of irreducible representations.

2. For M a handlebody of genus $g \geq 1$, $\mathcal{R}(M)$ is a smooth open manifold of dimension $3g-3$ (and empty if $g=1$). For F a closed oriented surface of genus $g \geq 1$, $\mathcal{R}(F)$ is a smooth open manifold of dimension $6g-6$.

3. For Σ a homology sphere, we can find a Heegaard decomposition $\Sigma = M_1 \sqcup_F M_2$. Then $\mathcal{R}(\Sigma) = \mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ is a compact manifold of dimension 0 in $\mathcal{R}(F)$. By orienting $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ we obtain an algebraic count of this intersection, from which we define the *Casson invariant*:

$$\lambda(\Sigma, M_1, M_2) = \frac{(-1)^g}{2} \#(\mathcal{R}(M_1) \cap \mathcal{R}(M_2)).$$

4. One can show that the invariant is independent of the choice of Heegaard splitting.

We now work through these steps in some more detail, in the corresponding subsections.

4.3.1 Step 1: representation spaces in general

Let M be a manifold. We equip $R(M) = \mathrm{Hom}(\pi_1 M, \mathrm{SU}(2))$ with the compact-open topology. Explicitly, $\pi_1 M$ is endowed with the discrete topology, and $\mathrm{SU}(2)$ with the usual topology making it homeomorphic to $\mathbb{S}^3$. Then the topology of $R(M)$ is generated by the open sets

$$U_{K,V} \coloneqq \{f : \pi_1 M \to \mathrm{SU}(2) : f(K) \subset V, K \subset \pi_1 M \text{ compact}, V \subset \mathrm{SU}(2) \text{ open}\}.$$

But $\pi_1 M$ is discrete so compact subsets are exactly finite subsets. Therefore the topology of $R(M)$ is generated by the sub-basis of functions

$$U_{g,V} \coloneqq \{f : \pi_1 M \to \mathrm{SU}(2) : f(g) \in V, V \subset \mathrm{SU}(2) \text{ open}\}.$$

Equivalently, $R(M)$ is equipped with the subspace topology of $R(M) \subset \mathrm{SU}(2)^{\pi_1 M}$, where the latter is equipped with the product topology. By Tychonoff's theorem, if $R(M)$ is closed in $\mathrm{SU}(2)^{\pi_1 M}$, it must also be compact. Indeed $R(M)$ is closed, so it is compact.

Definition 4.3.1. A morphism $f \in R(M)$ is called a *representation*. The representation determines a corresponding $\mathbb{C}[\pi_1 M]$-*module* V. Explicitly, V is isomorphic to $\mathbb{C}^2$ as a $\mathbb{C}$-vector space, but has an additional action of $\pi_1 M$ defined by $g \cdot v = f(g)v$. The module V (along with the representation f) are said to *reducible* if V has a non-zero proper $\mathbb{C}[\pi_1 M]$-submodule. Otherwise they are *reducible*.

In our context of $\mathrm{SU}(2)$-valued representations, we have the following characterisation of reducible representations.

Proposition 4.3.2. For any M, $f \in R(M)$ is reducible if and only if f factors through $U(1) \cong \mathbb{S}^1$.

Proof. By definition, we see that f is reducible if and only if there is a one dimensional subspace U of $\mathbb{C}^2$ such that $f(g)U \subset U$ for all $g \in \pi_1 M$. This means that f is reducible if and only if there is a basis $\{u, v\}$ of $\mathbb{C}^2$ so that each $f(g)$ maps v to $\lambda_g v$ for some scalar λ_g. In this basis, for each g we have

$$f(g) = \begin{pmatrix} a_g & b_g \\ 0 & \lambda_g \end{pmatrix}.$$

But SU(2) consists of unitary matrices of determinant 1, so the above matrix is constrained by

$$a\lambda = 1, \quad |a|^2 + |b|^2 = 1, \quad b\overline{\lambda} = 0, \quad |\lambda|^2 = 1.$$

The third and fourth constraints give $b = 0$, while the first and second give $a = \overline{\lambda}, |a|^2 = 1$. Therefore the image of f lies in

$$U(1) \cong \mathbb{S}^1 \cong \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{S}^1 \right\}.$$

□

Definition 4.3.3. We write $R^{\text{red}}(M)$ to denote the subset of $R(M)$ consisting of reducible representations, and $R^{\text{irr}}(M)$ to denote the irreducible representations.

By the above proposition, we see that $R^{\text{red}}(M) \subset R(M)$ is closed and hence compact. To eventually develop the Casson invariant, we want to control the reducible representations. These are of the following types:

- The trivial representation $\theta \in R^{\text{red}}(M)$; $\theta(g) = 1$.
- The central representations $f \in R^{\text{red}}(M)$; $f(g) = \{\pm 1\}$ (but f non-trivial). (These are *central* since the centre of SU(n) is generated by ζI for ζ an nth root of unity, which in this case is the group $\mathbb{Z}/2\mathbb{Z}$.)
- The other reducibles, which are exactly non-trivial and non-central representations that factor through $U(1)$.

Observe that $R(M)$ admits a natural group action - conjugation by SU(2). This does not act freely in general, and is particularly bad on the reducibles:

- The stabiliser of the SU(2) action on the trivial representation is all of SU(2).
- The stabiliser of the action on a central representation is also all of SU(2).
- The stabiliser of the action on any other reducible representation is $U(1) \subset \text{SU}(2)$.

It's natural to wonder if the conjugation action of SU(2) is free on $R^{\text{irr}}(M)$. Unfortunately, this is not the case, as $-A$ and A will always conjugate to the same element. But factoring SU(2) by $\{\pm 1\}$ gives the rotation group SO(3), and this acts freely on $R^{\text{irr}}(M)$.

Definition 4.3.4. The action of SO(3) on the irreducible representations $R^{\text{irr}}(M)$ is *free*. The quotient $R^{\text{irr}}/\operatorname{SO}(3)$, denoted by $\mathcal{R}(M)$, is called the *representation space* of M. It is equipped with the quotient topology.

In the next subsection we will establish that the representation spaces of surfaces and handlebodies are smooth open manifolds and compute their dimensions. Before proceeding with these special cases, we must better understand the local structure of representation spaces. Concretely, we would like to know when $R^{\text{irr}}(M)$ is closed and hence compact, so we desire a method of detecting when reducible representations are isolated. This requires the local theory of manifolds: tangent spaces. We will soon see that the relevant tangent spaces can be understood via group cohomology, which we now review.

Definition 4.3.5. Let G be a group, and M a G-module. (That is, an abelian group with a G action, each $g \in G$ acting as an automorphism.) Then the group of *inhomogeneous n-cochains* is the group of set theoretic functions $C^n(G, M) = M^{G^n}$. These form a cochain complex, with boundary map $d^{n+1} : C^n(G, M) \to C^{n+1}(G, M)$ defined by

$$
\begin{aligned}
(d^{n+1}\varphi)(g_1, \ldots, g_{n+1}) = g_1 \cdot \varphi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) \\
+ (-1)^{n+1} \varphi(g_1, \ldots, g_n).
\end{aligned}
$$

The *n-cocycles* are $Z^n(G; M) = \ker d^{n+1}$, and *n-coboundaries* $B^n(G; M) = \operatorname{im} d^n$. The *group cohomology* of G with coefficients in M is $H^n(G; M) = Z^n(G; M)/B^n(G; M)$.

In our case, we will consider just the first cohomology of $\pi_1 M$ with coefficients in $\mathfrak{su}(2)$. Recall the adjoint action $\operatorname{Ad} : G \to \operatorname{Aut}(\mathfrak{g})$, defined by $\operatorname{Ad} : g \mapsto \operatorname{Ad}_g$, where Ad_g is the derivative (at the origin) of the conjugation automorphism of g on G. This turns the Lie algebra $\mathfrak{su}(2)$ into an SU(2)-module. But now, given a representation $f : \pi_1 M \to \operatorname{SU}(2)$, we can pull back the module structure by f so that $\mathfrak{su}(2)$ is a $\pi_1 M$-module.

By inspecting the definition of the boundary map, we have:

$$
\begin{aligned}
Z^1_f(\pi_1 M; \mathfrak{su}(2)) &= \{\varphi : \pi_1 M \to \mathfrak{su}(2) : \varphi(xy) = \varphi(x) + \operatorname{Ad}_{f(x)} \varphi(y)\}, \\
B^1_f(\pi_1 M; \mathfrak{su}(2)) &= \{\varphi : \pi_1 M \to \mathfrak{su}(2) : \varphi(x) = \operatorname{Ad}_{f(x)} u_\varphi - u_\varphi, u_\varphi \in \mathfrak{su}(2)\}.
\end{aligned}
$$

The reason we are investigating group cohomology is that they turn out to encode the local structure of spaces of representations we're interested in.

Proposition 4.3.6. Let $f \in R(M)$. The *tangent space* of $R(M)$ at f is

$$T_f R(M) \cong Z^1_f(\pi_1 M; \mathfrak{su}(2)).$$

The tangent space of the *character variety* $\chi(M) = R(M)/\operatorname{SO}(3)$ at f is

$$T_f\chi(M) \cong H^1_f(\pi_1 M; \mathfrak{su}(2)).$$

To understand this proposition we must define the *Zariski tangent space*. This is because $R(M)$ is not generally a smooth manifold, so the usual notion of a tangent space does not apply.

Definition 4.3.7. Let X be an affine algebraic set over k, and $\alpha \in X$. Then the Zariski tangent space $T_\alpha X$ is the set of k-valued derivations D of $k[X]$. More explicitly,

$$T_\alpha X = \{D : k[X] \to k \mid D(pq) = p(\alpha)D(q) + D(p)q(\alpha)\}.$$

Each D as above is called a *tangent vector*.

By using the auxiliary ring of *dual numbers* $k[\varepsilon]/(\varepsilon^2)$, one can show that the Zariski tangent space has a more down-to-earth construction:

Proposition 4.3.8. Let $X = \mathcal{Z}(F_1, \ldots, F_m) \subset k^n$. Fix $\alpha \in X$. Then $T_\alpha X$ is the zero set of the linearisations of each F_i at α. More explicitly, write

$$F_i(\alpha + \varepsilon\beta) = F_i(\alpha) + \varepsilon L_i(\beta) + G_i(\varepsilon\beta) = \varepsilon L_i(\beta).$$

$G_i(\varepsilon\beta)$ vanishes since ε^2 factors in it, while $F_i(\alpha) = 0$ by definition of X, leaving only the *linearisation* L_i. Then $T_\alpha X = \mathcal{Z}(L_1, \ldots, L_m) \subset k^n$.

Example. Our space $R(M) = \operatorname{Hom}(\pi_1 M, \operatorname{SU}(2))$ is a real algebraic set, so we can apply the above notion of a tangent space. Explicitly, write

$$\pi_1 M = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle.$$

(We only concern ourselves with compact manifolds of dimension at most 3, which admit finite triangulations, and hence have finite presentations.) Each map $f \in R(M)$ is determined by the image of each x_j:

$$f : x_j \mapsto \begin{pmatrix} a_j + ib_j & -c_j + id_j \\ c_j + id_j & a_j - ib_j \end{pmatrix} \sim (a_j, b_j, c_j, d_j) \in \mathbb{R}^4.$$

This gives an inclusion $R(M) \hookrightarrow \mathbb{R}^{4n}$. The polynomials defining $R(M)$ are exactly the following:

- The constraint on $\operatorname{SU}(2)$: each matrix must have determinant 1, so $a_j^2 + b_j^2 + c_j^2 + d_j^2 = 1$ for every j.
- The relations in $\pi_1 M$: each r_k is a product of x_js and their inverses. These are each represented by an $\operatorname{SU}(2)$ matrix with entries polynomials in the $a_j, \ldots, d_j$. Each such matrix must vanish, giving four polynomial constraints for each r_i.

This gives a description of $R(M)$ as an algebraic set, with which we can compute the tangent space.

We are now ready to prove the earlier proposition which relates the tangent spaces of $R(M)$ and $\chi(M)$ to group cohomology. We'll re-state the results and prove them one at a time.

Proposition 4.3.9. Let $f \in R(M)$. Then $T_f R(M) = Z^1_f(\pi_1 M; \mathfrak{su}(2))$.

Proof. The map $f : \pi_1 M \to \mathrm{SU}(2)$ can be perturbed to obtain $f + \varepsilon\eta$, where $\eta : x \to \eta(x) \in T_{f(x)} \mathrm{SU}(2)$. Since $T_{f(x)} \mathrm{SU}(2)$ is just the right translation of $T_1 \mathrm{SU}(2) = \mathfrak{su}(2)$ by $f(x)$, our perturbation can be written as

$$\widetilde{f} : x \mapsto f(x) + \varepsilon\xi(x)f(x), \quad \xi : \pi_1 M \to \mathfrak{su}(2).$$

Our constraint equation is that $\widetilde{f}$ must be the linearisation of a representation. Therefore we must have

$$\begin{aligned} \widetilde{f}(xy) &= \widetilde{f}(x)\widetilde{f}(y) \\ \Leftrightarrow f(xy) + \varepsilon\xi(xy)f(xy) &= (f(x) + \varepsilon\xi(x)f(x))(f(y) + \varepsilon\xi(y)f(y)) \\ &= f(x)f(y) + \varepsilon\xi(x)f(x)f(y) + \varepsilon f(x)\xi(y)f(y), \end{aligned}$$

Where the second order term has been dropped. Subtracting $f(xy)$ from each side, and multiplying on the right by $f(xy)^{-1}$, we have

$$\varepsilon\xi(xy) = \varepsilon\xi(x) + \varepsilon f(x)\xi(y)f(x)^{-1}.$$

The equation is now independent of ε, and can be expressed as

$$\xi(xy) = \xi(x) + \mathrm{Ad}_{f(x)} \xi(y).$$

This is exactly a 1-cocycle! This proves the claim. ☐

Proposition 4.3.10. Let $f \in R(M)/\mathrm{SO}(3)$. Then $T_f(R(M)/\mathrm{SO}(3)) = H^1_f(\pi_1 M; \mathfrak{su}(2))$.

Proposition 4.3.11. We must show that if any two vectors are related by the SO(3) conjugation action, then their difference is a 1-coboundary. Concretely, two perturbations $f + \varepsilon\xi_1 f$ and $f + \varepsilon\xi_2 f$ are related by the SO(3) action if there exists $u \in \mathfrak{su}(2)$ such that

$$f(x) + \varepsilon\xi_1(x)f(x) = (1 + \varepsilon u)(f(x) + \varepsilon\xi_2(x)f(x))(1 + \varepsilon u)^{-1}.$$

But we find that

$$(1 + \varepsilon u)(1 - \varepsilon u) = 1 - \varepsilon^2 y^2 = 1,$$

so $(1 + \varepsilon u)^{-1} = 1 - \varepsilon u$. Making this substitution and dropping second order terms gives

$$f(x) + \varepsilon\xi_1(x)f(x) = f(x) + \varepsilon\xi_2(x)f(x) + \varepsilon u f(x) - \varepsilon f(x)u.$$

Rearranging the equation and multiplying on the right by $f(x)^{-1}$ gives

$$\xi_1(x) - \xi_2(x) = u - f(x)uf(x)^{-1} = u - \operatorname{Ad}_{f(x)} u.$$

Therefore $\xi_2 = \xi_1 + \varphi$ where $\varphi \in B^1_f(\pi_1 M, \mathfrak{su}(2))$ is a 1-coboundary. But now

$$T_f(R(M)/\operatorname{SO}(3)) = Z^1_f(\pi_1 M, \mathfrak{su}(2))/B^1_f(\pi_1 M, \mathfrak{su}(2)) = H^1_f(\pi_1 M, \mathfrak{su}(2))$$

as required.

This shows that the local structure of the character variety $\chi(M)$ can be understood via group cohomology. In particular, it sheds light on the representation space $\mathcal{R}(M)$ (which is really a subspace of $\chi(M)$). We will observe that in some cases the group cohomology can be identified with singular cohomology. This completes the general theory, and we now study the specifics which will lead to the Casson invariant.

4.3.2 Step 2: representation spaces of homology spheres

Our goal is to define the Casson invariant, an integral invariant of homology 3-spheres. We will define this by choosing a Heegaard splitting of the homology sphere;

$$\Sigma = M_1 \sqcup_F M_2.$$

In the remainder of this subsection we will develop an understanding of representation spaces of Σ, M_i, and F. First we will investigate the representation space of the total space, Σ.

Proposition 4.3.12. $R(\Sigma)$ has a unique reducible representation, namely the trivial representation.

Proof. Recall that $f : \pi_1\Sigma \to \operatorname{SU}(2)$ is reducible if and only if f factors through $U(1)$. But $U(1)$ is abelian, so the derived subgroup $[\pi_1\Sigma, \pi_1\Sigma]$ must map to $1 \in U(1)$. Therefore f factors as

$$f : \pi_1\Sigma \to \frac{\pi_1\Sigma}{[\pi_1\Sigma, \pi_1\Sigma]} \to U(1) \to \operatorname{SU}(2).$$

This means f factors through $H_1(\Sigma)$, which is trivial since Σ is a homology sphere. Therefore f is the trivial representation. □

This result will later be essential, as it gives control over the reducible representations. There is no reason to believe that $R(\Sigma)$ is a manifold, but we might hope that $R^{\text{irr}}(\Sigma)$ is a manifold. Unfortunately this turns out not to be the case, but the individual pieces of a Heegaard decomposition of Σ indeed give rise to manifolds of representations.

Proposition 4.3.13. Let M be a handlebody of genus $g \geq 1$. Then $R^{\text{irr}}(M)$ is a smooth open manifold of dimension $3g$, and $\mathcal{R}(M)$ is a smooth open manifold of dimension $3g - 3$. When $g = 1$ the manifold is empty.

Proof. Since SO(3) has dimension 3 and acts freely on $R^{\mathrm{irr}}(M)$, the second claim follows immediately provided the first claim is true.

For the first claim, observe that $\pi_1 M$ is the free group on g generators. Therefore $R(M)$ is diffeomorphic to $\prod_g \mathrm{SU}(2)$ (by recalling the description of $R(M)$ as an algebraic set). This is a closed smooth manifold of dimension $3g$. But now $R^{\mathrm{irr}}(M) = R(M) - R^{\mathrm{red}}(M)$ and $R^{\mathrm{red}}(M)$ is a topologically closed subset of $R(M)$. Therefore $R^{\mathrm{irr}}(M)$ is an open smooth manifold of dimension $3g$, as required.

When $g = 1$ the manifold is empty because $\chi(M)$ consists of a single point, the trivial representation. □

Finally, we determine the topological structure of the representation space of a surface.

Proposition 4.3.14. Let F be a closed oriented surface of genus $g \geq 1$. Then $R^{\mathrm{irr}}(F)$ is a smooth open manifold of dimension $6g - 3$, and $\mathcal{R}(F)$ is a smooth open manifold of dimension $6g - 6$.

Proof. We give just a proof outline. Let $D \subset F$ be a disk, $\gamma = \partial D$, and $F_0 = F - D$. Define

$$h : R(F_0) \to \mathrm{SU}(2)$$

by $h(f) = f(\gamma)$. Then

$$h^{-1}(1) = \{f \in R(F_0) : f(\gamma) = 1\} \cong R(F).$$

Therefore by understanding h we can understand $R(F)$.

We consider F_0 since $\pi_1 F_0$ is free of rank $2g$, with standard generators $a_1, \ldots, a_g, b_1, \ldots, b_g$. Then γ is given by

$$\gamma = \prod_{i=1}^{g} [a_i, b_i].$$

Since $\pi_1 F_0$ is free we have an identification

$$R(F_0) \cong \mathrm{SU}(2)^{2g}, \quad f \mapsto (f(a_1), f(b_1), \ldots, f(a_g), f(b_g)).$$

But now h can be expressed as

$$h : \mathrm{SU}(2)^{2g} \to \mathrm{SU}(2), \quad (A_1, B_1, \ldots, A_g, B_g) \mapsto \prod_{i=1}^{g} [A_i, B_i].$$

One can explicitly show that h is surjective. Moreover, by some technical calculations (carried out in Saveliev), one can show that h is regular exactly at irreducible representations.

It follows that $R^{\mathrm{irr}}(F)$ (which is $h^{-1}(1)$ restricted to irreducible representations) is an open submanifold of $\mathrm{SU}(2)^{2g}$ with dimension $3(2g) - 3 = 6g - 3$. By taking a further quotient by SO(3), $\mathcal{R}(F)$ is a smooth open manifold of dimension $6g - 6$. □

Finally we note that if $\Sigma = M_1 \sqcup_F M_2$, then the representation space of Σ is the intersection of the representation spaces of M_1 and M_2. This means that even if $\mathcal{R}(\Sigma)$ is not known to be a manifold, we can completely understand it in terms of the $3g-3$ dimensional manifolds $\mathcal{R}(M_i)$.

Proposition 4.3.15. Let $\Sigma = M_1 \sqcup_F M_2$ be a Heegaard splitting of a closed 3-manifold Σ. Then $R(M_1) \cap R(M_2) = R(\Sigma)$. Moreover, it follows that $R^{\text{irr}}(M_1) \cap R^{\text{irr}}(M_2) = R^{\text{irr}}(\Sigma)$ and $\mathcal{R}(M_1) \cap \mathcal{R}(M_2) = \mathcal{R}(\Sigma)$.

Proof. Suppose the Heegaard splitting is of genus g. Then

$$\pi_1 M_1 = \langle x_1, \ldots, x_g \rangle, \quad \pi_1 M_2 = \langle y_1, \ldots, y_g \rangle.$$

By the Seifert-van Kampen theorem, we have

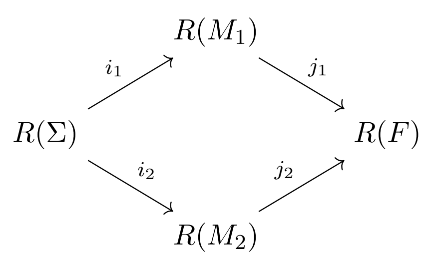
$$\pi_1 \Sigma = \langle x_1, \ldots, x_g \mid r_1, \ldots, r_g \rangle = \langle y_1, \ldots, y_g \mid r'_1, \ldots, r'_g \rangle,$$

where $r_1, \ldots, r_g$ are determined by $y_1, \ldots, y_g$ expressed in the basis $x_1, \ldots, x_g$, and similarly for $r'_1, \ldots, r'_g$. It follows that the inclusions induce quotient maps $\pi_1 M_1, \pi_1 M_2 \to \pi_1 \Sigma$. Now any representation $f : \pi_1 \Sigma \to \mathrm{SU}(2)$ pulls back to representations $\pi_1 M_i \to \mathrm{SU}(2)$.

On the other hand, $\pi_1 F$ is generated by $2g$ generators,

$$\pi_1 F = \langle x_1, \ldots, x_g, x'_1, \ldots, x'_g \mid [x_i, x'_i] \rangle = \langle y_1, \ldots, y_g, y'_1, \ldots, y'_g \mid [y_i, y'_i] \rangle.$$

Therefore we have quotient maps $\pi_1 F \to \pi_1 M_i$ as above, so that any representation $f : \pi_1 M_i \to \mathrm{SU}(2)$ pulls back to a representation $\pi_1 F \to \mathrm{SU}(2)$. In summary we have the following diagram of inclusions:



These inclusions allow us to make sense of $R(M_1) \cap R(M_2)$, by considering the intersection in $R(F)$. It is clear that $R(\Sigma) \subset R(M_1) \cap R(M_2)$. For the converse, we inspect the presentations of the corresponding fundamental groups. If $f \in R(M_1) \cap R(M_2)$, then it is defined on both the x_i and the y_i. Since the y_i determine the relations of $\pi_1 \Sigma$ with generators x_i, f is also well defined as map on $\pi_1 \Sigma$. Therefore $R(\Sigma) = R(M_1) \cap R(M_2)$.

Next we restrict to irreducible representations. The lift of any irreducible representation of F is irreducible on M_i (since it cannot factor through $U(1)$). Conversely, reducibles lift to reducibles. Therefore

$$R^{\text{irr}}(M_1) \cap R^{\text{irr}}(M_2) = R^{\text{irr}}(\Sigma), \quad R^{\text{red}}(M_1) \cap R^{\text{red}}(M_2) = R^{\text{red}}(\Sigma).$$

The last result is immediate. ∎

This concludes our investigations of the global structures of representation spaces of manifolds with a view to defining the Casson invariant. Next we investigate some of the local structure by using tangent spaces. In the degenerate case of the trivial representation, the tangent space is in fact given by singular cohomology.

Proposition 4.3.16. If $\theta \in \chi(X) = \mathrm{Hom}(\pi_1 X, \mathrm{SU}(2))/\mathrm{SO}(3)$ is the trivial representation, where X is any topological space with finitely presented fundamental group, then

$$T_\theta \chi(X) = H^1_\theta(\pi_1 X; \mathfrak{su}(2)) \cong H^1(X; \mathfrak{su}(2)).$$

(Group cohomology on the left, singular cohomology on the right.)

Proof. Recall the universal coefficient theorem: there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_0(X;\mathbb{Z}), \mathfrak{su}(2)) \to H^1(X; \mathfrak{su}(2)) \to \mathrm{Hom}_{\mathbb{Z}}(H_1(X;\mathbb{Z}), \mathfrak{su}(2)) \to 0.$$

But $H_0(X;\mathbb{Z})$ is free, so $H^1(X;\mathfrak{su}(2))$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(H_1(X;\mathbb{Z}), \mathfrak{su}(2))$.

Next we notice that for θ the trivial representation, the 1-cocycles $Z^1_1(\pi_1 X; \mathfrak{su}(2))$ are exactly $\mathfrak{su}(2)$-valued homomorphisms of $H_1(X;\mathbb{Z})$. This because the cocycle condition becomes

$$\xi(xy) = \xi(x) + \mathrm{Ad}_{f(x)} \xi(y) = \xi(y) + \xi(y) = \xi(yx).$$

Therefore cocycles factor through the abelianisation of $\pi_1 X$ (which is exactly $H_1(X)$) and are $\mathbb{Z}$-linear.

Finally we observe that the coboundaries are trivial, since the coboundary condition reduces to

$$\xi(x) = \mathrm{Ad}_{\theta(x)} u - u = u - u = 0.$$

Therefore we have

$$T_\theta \chi(X) = H^1_\theta(\pi_1 X; \mathfrak{su}(2)) = Z^1_\theta(\pi_1 X; \mathfrak{su}(2)) = \mathrm{Hom}(H_1(X;\mathbb{Z}), \mathfrak{su}(2)) = H^1(X; \mathfrak{su}(2)).$$

□

In some non-degenerate cases (that is, when we restrict to irreducible representations), we have shown that the representation space is sometimes a smooth manifold. In this case the Zariski tangent space agrees with the usual tangent space.

Example. We have established that $\mathcal{R}(F)$ is a smooth manifold of dimension $6g - 6$. Therefore for any $f \in \mathcal{R}(F)$,

$$\mathbb{R}^{6g-6} \cong T_f \mathcal{R}(F) = T_f \chi(F) = H^1_f(\pi_1 F; \mathfrak{su}(2)).$$

If f is the trivial representation, then

$$T_f \chi(F) = H^1_f(\pi_1 F; \mathfrak{su}(2)) = H^1(F; \mathfrak{su}(2)) \cong \mathbb{R}^{6g}.$$

Similarly for M a handlebody of genus g, we have

$$T_f\chi(M) = \begin{cases} \mathbb{R}^{3g-3} & f \text{ irreducible} \\ \mathbb{R}^{3g} & f \text{ trivial} \\ ? & \text{other reducibles.} \end{cases}$$

In these two examples, the change in dimension reflects that the reducible representations are singular points of the character variety.

4.3.3 Step 3: the Casson invariant as a signed count

We now have enough understanding of the global and local structures of representation spaces to define the Casson invariant. The definition is as follows:

Definition 4.3.17. Let $M_1 \sqcup_F M_2$ be a genus g Heegaard splitting of a homology sphere Σ. Then the *Casson invariant* λ is defined by

$$\lambda(\Sigma, M_1, M_2) = \frac{(-1)^g}{2}\#(\mathcal{R}(M_1) \cap \mathcal{R}(M_2))$$

where # denotes a signed count, given orientations on $\mathcal{R}(M_i)$.

It is not yet immediate that this definition makes sense. We must verify the following facts:

1. $\mathcal{R}(M_2)$ can be perturbed to $\widetilde{\mathcal{R}}(M_2)$ so that $\mathcal{R}(M_1) \cap \widetilde{\mathcal{R}}(M_2)$ is a finite collection points.
    - $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ is compact.
    - $\dim \mathcal{R}(M_1) + \dim \mathcal{R}(M_2) = \dim \mathcal{R}(F)$.
2. The signed count of points is independent of the perturbation.
3. The invariant is independent of choices of orientation for the Heegaard splitting.

In the next subsection we will also show that the Casson invariant does not depend on the choice of Heegaard splitting. We now prove the first of the two bullet points above.

1. By derivations in the previous subsection, we know that $\dim \mathcal{R}(M_1) + \dim \mathcal{R}(M_2) = (3g-3)+(3g-3) = 6g-6 = \dim \mathcal{R}(F)$. Therefore by perturbing $\mathcal{R}(M_2)$ to achieve transversality in $\mathcal{R}(F)$, the two submanifolds intersect along a 0-manifold. For a 0-manifold, finiteness is equivalent to compactness. Therefore it remains to show that $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ is compact.

By the previous subsection, we know that $\mathcal{R}(M_1) \cap \mathcal{R}(M_2) = \mathcal{R}(\Sigma) \subset \mathcal{R}(F)$. Moreover, we know that $R(\Sigma)$ is a closed subset of $\mathrm{SU}(2)^{\pi_1\Sigma}$ and hence compact, so $\chi(\Sigma)$ is compact. It remains to show that

$$\mathcal{R}(\Sigma) = \chi(\Sigma) - R^{\mathrm{red}}(\Sigma)/\,\mathrm{SO}(3)$$

is closed, or equivalently that $R^{\mathrm{red}}(\Sigma)$ is open in $R(\Sigma)$. Since Σ is a homology sphere, we have established that it has a unique reducible representation, namely the trivial representation. We will show that the trivial representation is an isolated point in $R(\Sigma)$.

Recall the following result from transversality: if $U, V \subset X$ are submanifolds of codimension n and m, and $U \pitchfork V$ in X, then $U \cap V$ is an embedded submanifold of codimension $n+m$. In particular, if $n+m$ is the dimension of X, then $U \cap V$ intersect along isolated points. Therefore to show that the trivial representation θ is isolated in $R(\Sigma)$, it suffices to show that $R(M_1)$ and $R(M_2)$ intersect transversely at θ. This means that

$$T_\theta R(M_1) + T_\theta R(M_2) = T_\theta R(F).$$

But recall that the trivial representation satisfies

$$T_\theta \chi(X) = T_\theta R(X) = H^1(X; \mathfrak{su}(2))$$

for any X (with finitely presented fundamental group). Therefore we must establish that the induced map

$$H^1(M_1; \mathfrak{su}(2)) \oplus H^1(M_2; \mathfrak{su}(2)) \to H^1(F; \mathfrak{su}(2))$$

is an isomorphism, to verify transversality as above. This comes immediately from the Mayer-Vietoris sequence, since Σ is a homology sphere.

Therefore $R(M_1)$ and $R(M_2)$ intersect transversely at θ, so θ is an isolated point of $R(\Sigma)$ in $R(F)$. But then $R(\Sigma) - \{\theta\} = R^{\mathrm{irr}}(\Sigma)$ is topologically closed, and in particular compact. The result follows.

2. This is a result of the general theory of differential topology.

3. Formally the signed count $\#(\mathcal{R}(M_1) \cap \mathcal{R}(M_2))$ is really $\#(\mathcal{R}(M_1) \cap \widetilde{\mathcal{R}}(M_2))$, where $\widetilde{\mathcal{R}}(M_2)$ is a perturbation of $\mathcal{R}(M_2)$ so that the intersection is transverse. The signed count is then defined to be

$$\#(\mathcal{R}(M_1) \cap \mathcal{R}(M_2)) = \sum_{f \in \mathcal{R}(M_1) \cap \widetilde{\mathcal{R}}(M_2)} \mathrm{sgn}(f),$$

where $\mathrm{sgn}(f)$ is defined to be ± 1 depending on whether or not the following orientations agree:

$$T_f \mathcal{R}(M_1) \oplus T_f \mathcal{R}(M_2), \quad T_f \mathcal{R}(F).$$

We show that the signed count is independent of choices of orientation as follows:

(a) Write $\Sigma = M_1 \sqcup_F M_2$. Choose an orientation for F, and observe how it determines an orientation of $H^1(F; \mathbb{R})$ and hence an orientation of $\mathcal{R}(F)$.

(b) Use the orientation of $H^1(F; \mathbb{R})$ to induce orientations on $H^1(M_i, \mathbb{R})$, and hence on $\mathcal{R}(M_i)$.

(c) Study how changing the orientation of F affects the signed count (we will see that it changes sign).

(d) Study how changing the labels M_1 and M_2 (to reverse the order) changes the signed count (it also changes sign).

(e) Conclude that the signed count is independent of choices of orientation for a fixed Heegaard splitting, since the orientation of F and indexing of M_i determine the orientation of Σ, which is fixed.

We now work through the above outline in a bit more detail.

(a) Let F be oriented, with genus g. Then F is equipped with an intersection form,

$$I : H^1(F;\mathbb{R}) \times H^1(F;\mathbb{R}) \to \mathbb{R}, \quad I(a,b) = (a \smile b)[F].$$

In some basis of $H^1(F;\mathbb{R})$, I is described by the block diagonal matrix

$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is called a *symplectic basis*, and any two symplectic bases for which I has the same form have the same orientation. This is the canonical orientation induced on $H^1(F;\mathbb{R}) = \mathbb{R}^{2g}$.

Now we describe the orientation of $\mathcal{R}(F)$. Choose a basis for $H^1(F;\mathbb{R})$ which is consistent with the orientation. Define $F_0 = F - D$ for some disk in F, and recall that F_0 has fundamental group free of rank $2g$. Therefore $\mathrm{Hom}(\pi_1 F_0, \mathrm{SU}(2)) \cong \mathrm{SU}(2)^{2g}$. But now the basis for $H^1(F;\mathbb{R})$ fixes an ordering for the generators of $\pi_1 F_0$, and hence an identification $\mathrm{Hom}(\pi_1 F_0, \mathrm{SU}(2)) = \mathrm{SU}(2)^{2g}$.

We have now induced an orientation on $R(F_0) = \mathrm{Hom}(\pi_1 F_0, \mathrm{SU}(2))$. But if $h : R(F_0) \to \mathrm{SU}(2)$ is defined by $h(f) = f(\partial D)$, then $R(F) = h^{-1}(1)$ is oriented by the orientation of $\mathrm{SU}(2)$. Finally by requiring that the identification $\mathfrak{su}(2) \oplus T\mathcal{R}(F) = T\mathcal{R}(F_0)$ is orientation preserving, we can further induce an orientation on $\mathcal{R}(F)$ as required.

All choices above were canonical, except for the choice of orientation of Σ. Suppose we change the orientation of Σ. Then the orientation of $H^1(F,\mathbb{R})$ changes by $(-1)^g$, and the orientation of $\mathcal{R}(F)$ changes by $(-1)^{g+1}$.

(b) The orientations on $H^1(M_i;\mathbb{R})$ come from the Mayer-Vietoris sequence. Specifically since Σ is a homology sphere we have

$$H^1(M_1;\mathbb{R}) \oplus H^1(M_2;\mathbb{R}) = H^1(F;\mathbb{R}).$$

Therefore any choice of orientation of M_1 together with the induced orientation of $H^1(F;\mathbb{R})$ canonically orients $H^1(M_2;\mathbb{R})$.

Choose a basis for $H^1(M_1;\mathbb{R})$ which is compatible with the orientation. This defines an identification of $R(M_1)$ with $\mathrm{SU}(2)^g$, and hence an orientation of $\mathcal{R}(M_1) \subset \mathrm{SU}(2)^g/\mathrm{SO}(3)$. Similarly there is an induced orientation on $\mathcal{R}(M_2)$.

(c) The orientation of $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ both change when the orientation of $H^1(M_1;\mathbb{R})$ is changed, so the orientation of

$$T_f\mathcal{R}(M_1) \oplus T_f\mathcal{R}(M_2)$$

does not depend on the orientation of $H^1(M_1;\mathbb{R})$.

However, changing the orientation of F *only* changes the orientation of $\mathcal{R}(M_2)$ and not $\mathcal{R}(M_1)$, so overall the orientation of $T_f\mathcal{R}(M_1) \oplus T_f\mathcal{R}(M_2)$ changes by $(-1)^g$. We noted that changing the orientation of F affects the orientation of $\mathcal{R}(F)$ by $(-1)^{g+1}$. Therefore, the sign of a given point always changes by -1.

(d) Now suppose we change the roles of M_1 and M_2, so that $H_1(M_2;\mathbb{R})$ is the one which is originally oriented. Then the orientation of $T_f\mathcal{R}(M_1) \oplus T_f\mathcal{R}(M_2)$ changes by $(-1)^g$. On the other hand, since $\mathcal{R}(M_1)$ has dimension $3g-3$ (which is odd exactly when g is even), changing from $T_f\mathcal{R}(M_1) \oplus T_f\mathcal{R}(M_2)$ to $T_f\mathcal{R}(M_2) \oplus T_f\mathcal{R}(M_1)$ changes the orientation exactly when g is even, i.e. by $(-1)^{g+1}$. Therefore overall a change in labelling between M_1 and M_2 changes the sign of a point by -1.

(e) It follows that the signed count is independent of the choices of orientation for a Heegaard splitting, since it depends only on the orientation of F and the labelling of M_1 and M_2. But given a choice of orientation for F, the labelling of M_1 and M_2 are determined by the choice of normal vector, which is determined by the orientation of Σ.

This completes the proof that the Casson invariant is well defined, given a choice of Heegaard splitting. Next we show that even the Heegaard splitting need not be specified.

4.3.4 Step 4: Heegaard splitting invariance

Let Σ be a homology sphere. Then given a Heegaard splitting $\Sigma = M_1 \sqcup_F M_2$, we have shown that the Casson invariant $\lambda(\Sigma, M_1, M_2)$ is a well defined half-integer. (It has not yet been shown that this quantity is in fact an integer.) We now show that the choice of Heegaard splitting does not affect the Casson invariant. Recall the following classification of Heegaard splittings:

Proposition 4.3.18. Let M be a closed 3-manifold. Any two Heegaard splittings of M are stably equivalent.

This result was discussed at the start of the notes. Therefore to show that the Casson invariant is well defined independent of the Heegaard splitting, it remains to show that it is invariant under stabilisation. We give a proof outline of the following result:

Proposition 4.3.19. Let $\Sigma = M_1 \sqcup_F M_2$ be a homology sphere. Let $M_1' \sqcup_{F'} M_2'$ be the stabilisation of $M_1 \sqcup_F M_2$. Then

$$\lambda(\Sigma, M_1, M_2) = \lambda(\Sigma, M_1', M_2').$$

Proof. We give a proof outline. Let $F_0 = F - D$ and $F_0' = F' - D$, where D is some disk. We have the following fundamental groups:

$$\begin{aligned}\pi_1(F_0) &= \langle a_1, b_1, \ldots, a_g, b_g \rangle \\ \pi_1(M_1) &= \langle a_1, \ldots, a_g \rangle \\ \pi_1(M_2) &= \langle b_1, \ldots, b_g \rangle.\end{aligned}$$

Then in the stabilisation, we have

$$\begin{aligned}\pi_1(F_0') &= \langle a_0, b_0, a_1, b_1, \ldots, a_g, b_g \mid [a_i, b_i] \rangle \\ \pi_1(M_1') &= \langle a_0, a_1, \ldots, a_g \rangle \\ \pi_1(M_2') &= \langle b_0, b_1, \ldots, b_g \rangle.\end{aligned}$$

These further give identifications

$$R(M_i') = \mathrm{SU}(2) \times R(M_i), \quad R(F_0') = \mathrm{SU}(2) \times \mathrm{SU}(2) \times R(F_0).$$

But now we find that

$$\begin{aligned}R(M_1') \cap R(M_2') &= (\mathrm{SU}(2) \times 1 \times R(M_1)) \cap (1 \times \mathrm{SU}(2) \times R(M_2)) \\ &= 1 \times 1 \times (R(M_1) \times R(M_2)).\end{aligned}$$

Making appropriate quotients and perturbations, we can achieve

$$\mathcal{R}(M_1') \cap \widetilde{\mathcal{R}}(M_2') = 1 \times 1 \times \mathcal{R}(M_1) \cap \widetilde{\mathcal{R}}(M_2).$$

The Casson invariants are given by

$$\lambda(\Sigma, M_1, M_2) = \frac{(-1)^g}{2} \sum \varepsilon_\alpha, \quad \lambda(\Sigma, M_1', M_2') = \frac{(-1)^{g+1}}{2} \sum \varepsilon_\alpha'.$$

It suffices to verify that $\varepsilon_\alpha = -\varepsilon_\alpha'$. This is not covered here, but one can show using similar arguments to the orientation invariance of the Casson invariant given a Heegaard splitting that this indeed holds. □

4.4 What if we change the choices?

In this section we explore three modifications to the Casson invariant:

- What if we try to build a Casson invariant for homology 4-spheres?
- What if we try to build a Casson invariant for 3-manifolds which aren't homology spheres?
- What if we change the gauge group to something else? E.g. $\mathrm{SU}(n)$ or $\mathrm{SL}(2, \mathbb{C})$.

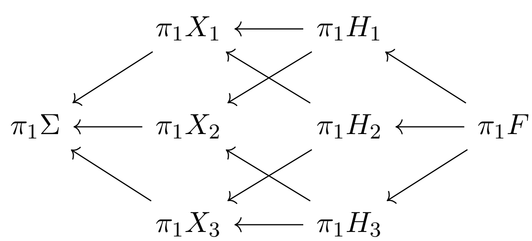
4.4.1 Attempted generalisation to homology 4-spheres

We now attempt to repeat the construction for homology 4-spheres. Recall that closed 4-manifolds have an analogue of Heegaard splittings, called *trisections*.

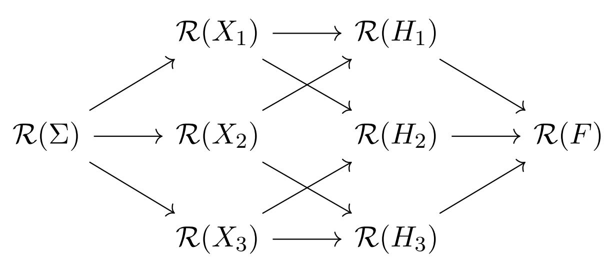
Definition 4.4.1. Let X be a closed smooth connected 4-manifold. Then for $0 \leq k \leq g$, a (g, k)-*trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that

- For each i, there is a diffeomorphism $\varphi_i : X_i \to \natural^k(\mathbb{S}^1 \times B^3)$.
- The boundary of each X_i is $\#^k(\mathbb{S}^1 \times \mathbb{S}^2)$. Each of these has a Heegaard splitting
$$\partial X_i = \#^k(\mathbb{S}^1 \times \mathbb{S}^2) = Y^-_{k,g} \sqcup_{\Sigma_g} Y^+_{k,g}.$$
- Given any i, $\varphi_i(X_i \cap X_{i+1}) = Y^-_{k,g}$, and $\varphi_i(X_i \cap X_{i-1}) = Y^+_{k,g}$.

Moreover, the diagram of inclusions of a trisection induces a diagram of surjections of fundamental groups:



In this diagram, each H_i is one of the handlebodies $Y_{k,g}$. The above diagram induces a diagram of inclusions



We would like to define some sort of Casson invariant. Each of the above spaces (except possibly $\mathcal{R}(\Sigma)$) is a manifold. Recall that trisections satisfy

$$2 = \chi(\Sigma) = 2 + g - 3k,$$

where the X_i are $\natural^k(\mathbb{S}^1 \times B^3)$, and F has genus g. Then

$$\begin{aligned} \dim \mathcal{R}(X_i) &= 3k - 3 = g - 3 \\ \dim \mathcal{R}(H_i) &= 3g - 3 \\ \dim \mathcal{R}(F) &= 6g - 6. \end{aligned}$$

This means that, in general, $\mathcal{R}(X_i)$ will not be useful in an attempted transverse signed count construction. We could attempt to count intersections of $\mathcal{R}(H_i)$ in $\mathcal{R}(F)$. However, we then come across a problem concerning induced orientations. Consider the exact sequence

$$\cdots \to H^1(\partial X_1;\mathbb{R}) \to H^1(H_1;\mathbb{R}) \oplus H^1(H_2;\mathbb{R}) \to H^1(F;\mathbb{R}) \to H^2(X_1;\mathbb{R}) \to \cdots .$$

Since ∂X_i has non-trivial homology in general, we cannot identify $H^1(H_1;\mathbb{R}) \oplus H^1(H_2;\mathbb{R})$ with $H^1(F;\mathbb{R}) \to H^2(X_1;\mathbb{R})$ to induce orientations. Therefore the foremost issue is to attempt to understand the Casson invariant for closed 3-manifolds which *aren't* homology spheres (to have any hope of generalising to homology 4-spheres).

4.4.2 Attempted generalisation to more 3-manifolds

In the definition of the Casson invariant, we consistently made use of the fact that our underlying manifold was a homology sphere. This manifested in the following ways: let Σ be a homology sphere, and $\Sigma = M_1 \sqcup_F M_2$ a Heegaard splitting.

- $R(\Sigma)$ has a unique reducible representation (the trivial representation). In general there may be more irreducible representations.
- The trivial representation is isolated. This follows from the fact that

  $$H^1(M_1;\mathfrak{su}(2)) \oplus H^1(M_2;\mathfrak{su}(2)) \to H^1(F;\mathfrak{su}(2))$$

  is an isomorphism (by the Mayer-Vietoris sequence, using that Σ is a homology sphere). But for the trivial representation of $\pi_1 X$ (for any X), we have

  $$H^1(X;\mathfrak{su}(2)) \cong T_\theta \chi(X).$$

  Since $R(\Sigma) \subset \mathrm{SU}(2)^{\pi_1\Sigma}$ is closed and hence compact, $\chi(\Sigma)$ is compact. Since the trivial representation is the unique reducible, $\mathcal{R}(\Sigma)$ is compact. In general the trivial representation may not be isolated, so $\mathcal{R}(\Sigma)$ may not be compact. Moreover, reducibles which are not the trivial representation may be harder to understand.
- To compute the signed count of intersections we must orient F, M_1, and M_2. However, we relied on the identity

  $$H^1(M_1;\mathbb{R}) \oplus H^1(M_2;\mathbb{R}) = H^1(F;\mathbb{R})$$

  to orient $H^1(M_2;\mathbb{R})$ given an orientation of $H^1(M_1;\mathbb{R})$ and F. (The above equality again comes from Mayer-Vietoris using that Σ is a homology sphere). In general it should be more difficult to keep track of orientations. Heegaard splitting independence cannot be considered unless orientations are made sense of.

To understand how things might go wrong, we consider the example $\Sigma = \mathbb{S}^1 \times \mathbb{S}^2$ (rather than a general non-homology 3-sphere).

Example. $\Sigma = \mathbb{S}^1 \times \mathbb{S}^2$ has a canonical Heegaard splitting of genus 1, $M_1 \sqcup_F M_2$. These have fundamental groups as follows:

$$\pi_1\Sigma = \langle a \rangle, \quad \pi_1 M_i = \langle a \rangle, \quad \pi_1 F = \langle a, b \mid [a, b] \rangle.$$

But then $\mathrm{Hom}(\pi_1\Sigma, \mathrm{SU}(2)) \cong \mathrm{SU}(2)$, so $\chi(\Sigma)$ consists of a single point (corresponding to the trivial representation). Therefore $\mathcal{R}(\Sigma)$ is empty - we can make sense of the Casson invariant! it must vanish. Similarly, the "Casson invariant" of Lens spaces must vanish.

These turn out to be trivial examples, so really we want to consider a three manifold whose fundamental group is not generated by one element. To this end, we now consider the three torus.

Example. Let $\Sigma = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. This has a Heegaard splitting defined as follows: let

$$\Gamma = (\mathbb{S}^1 \times \{x_0\} \times \{x_0\}) \cup (\{x_0\} \times \mathbb{S}^1 \times \{x_0\}) \cup (\{x_0\} \times \{x_0\} \times \mathbb{S}^1).$$

Let V be a regular neighbourhood of Γ. Then V and $\Sigma - V$ are handlebodies of genus 3, defining a Heegaard splitting

$$\Sigma = V \sqcup_{\partial V} (\Sigma - V) = M_1 \sqcup_F M_2.$$

Next we describe the relevant fundamental groups. If a_1 is the curve $\mathbb{S}^1 \times \{x_0\} \times \{x_0\}$ and a_2, a_3 are defined analogously, then we have the following diagram of surjections:

$$\begin{array}{ccccc} & & \pi_1 M_1 = \langle a_1, a_2, a_3 \rangle & & \\ & \nearrow & & \searrow & \\ \pi_1 F = \langle a_i, b_i \mid [a_i, b_i] \rangle & & & & \pi_1\Sigma = \langle a_1, a_2, a_3 \mid [a_i, a_j] \rangle \\ & \searrow & & \nearrow & \\ & & \pi_1 M_2 = \langle a_1', a_2', a_3' \rangle & & \end{array}$$

Suppose $\varphi : \pi_1\Sigma \to \mathrm{SU}(2)$ is a reducible representation. Then φ factors through $U(1)$, so we have

$$\begin{aligned} R^{\mathrm{red}}(\pi_1\Sigma) &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, U(1)) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, U(1))^3 \\ &= U(1)^3 = T^3. \end{aligned}$$

The reducible representations are homeomorphic to the 3-torus! What about the irreducibles? What does the space of representations look like in general?

This is difficult! Naively calculating the equations which describe $R(\mathbb{Z}^3, \mathrm{SU}(2))$ as a variety in $\mathbb{R}^{12}$, we obtain 27 equations. 24 of these arise as the real and imaginary parts of the equations describing the relations in $\mathbb{S}^3$, and are homogenous of degree 4. The remaining 3 equations specify that each matrix has determinant 1.

Overall this feels wildly intractable.

4.4.3 Attempted generalisation to other gauge groups

At this point we give up on generalising the manifold, and turn to the gauge group. In the construction of the Casson invariant, key facts of SU(2) were the following:

- SU(2) is compact. Therefore if we can identify $R(M)$ as sitting inside $\mathrm{SU}(2)^k$ for some k, then any closed subspace in $R(M)$ is compact. This is used to show that $R^{\mathrm{irr}}(\Sigma)$ is compact for Σ a homology sphere.
- To show that $R^{\mathrm{irr}}(\Sigma)$ is closed, we use that $R(\Sigma)$ has a unique reducible representation, namely the trivial representation. This result uses the fact that all reducible representations of SU(2) factor through an abelian group ($U(1)$).

We now look at some explicit alternatives to SU(2) and see where things go wrong. First we explore $\mathrm{SL}(2, \mathbb{C})$.

Example. What does $\mathrm{Hom}(G; \mathrm{SL}(2, \mathbb{C}))$ look like? This is not necessarily compact, since $\mathrm{SL}(2, \mathbb{C})$ is not compact. This already makes things a bit difficult.

What about reducible representations? As mentioned earlier, these can be classified as having image upper-triangular matrices. In $\mathrm{SL}(2, \mathbb{C})$, these matrices are isomorphic to the group

$$(a, b) \in \mathbb{C}^* \times \mathbb{C}, \quad (a, b) \cdot (a', b') = (aa', ab' + b/a').$$

This is not an abelian group, so reducible representations need not factor through $G/[G, G]$. In particular, reducible representations of $\pi_1\Sigma$ need not factor through $H_1(\Sigma) = 0$. Therefore there may be non-trivial reducible representations. However, in the character variety, this is not an issue! All representations are determined by their traces in the character variety, so the upper triangular matrices reduce to the diagonal matrices. These are abelian as required. Therefore the main difficulty with $\mathrm{SL}(2, \mathbb{C})$ is compactness.

Example. What about SU(3) (instead of $\mathrm{SL}(2, \mathbb{C})$ or SU(2))? This is compact, which is a good start. However, reducible representations may space 1 or 2 dimensional subspaces of $\mathbb{C}^3$, so there are more possibilities. If a reducible representation spans a two dimensional subspace, then in some basis, the image consists of matrices of the form

$$\begin{pmatrix} \lambda_1 & 0 & x_1 \\ 0 & \lambda_2 & x_2 \\ 0 & 0 & x_3 \end{pmatrix}.$$

The special unitary condition forces the following constraints:

$$\lambda_1\lambda_2 x_3 = 1, \begin{pmatrix} \overline{\lambda_1} & 0 & 0 \\ 0 & \overline{\lambda_2} & 0 \\ \overline{x_1} & \overline{x_2} & \overline{x_3} \end{pmatrix} = \begin{pmatrix} \lambda_2 x_3 & 0 & -\lambda_2 x_1 \\ 0 & \lambda_1 x_3 & -\lambda_1 x_2 \\ 0 & 0 & \lambda_1\lambda_2 \end{pmatrix}.$$

Therefore we have $x_1 = x_2 = 0$, and $\lambda_1, \lambda_2, x_3$ are complex numbers with norm 1 which all multiply to the identity. Therefore reducibles factor through the group of matrices of the form

$$\begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\tau} & 0 \\ 0 & 0 & e^{-i\theta-i\tau} \end{pmatrix}.$$

Therefore any reducible with two-dimensional subspace must factor through $U(1) \times U(1)$. It follows that any such reducible representation of $\pi_1\Sigma$ for Σ a homology sphere must be the trivial representation!

What about reducibles which restrict to subspaces of dimension 1? These are matrices with determine 1 satisfying

$$\begin{pmatrix} \overline{\lambda_1} & 0 & 0 \\ \overline{y_1} & \overline{y_2} & \overline{y_3} \\ \overline{x_1} & \overline{x_2} & \overline{x_3} \end{pmatrix} = \begin{pmatrix} x_3y_2 - x_2y_3 & x_1y_3 - x_3y_1 & x_2y_1 - x_1y_2 \\ 0 & \lambda_1 x_3 & -\lambda_1 x_2 \\ 0 & -\lambda_1 y_3 & \lambda_1 y_2 \end{pmatrix}.$$

The constraints can be re-written as $(a, b, c, d) \in \mathbb{C}^4$ such that

$$|ad - bc| = 1, \quad |a|^2 = |d|^2, \quad |b|^2 = |c|^2, \quad |a|^2 - |c|^2 = 1.$$

This is compact, since each parameter must surely have norm bounded by 2. (This is not a tight bound). This is a group, but not an abelian group. Therefore in general there may be reducibles that do not factor through an abelian group, so homology spheres may have non-trivial reducibles.

Finally we note that generalisations of the Casson invariant do in fact exist!

- The *Casson-Walker* invariant is a generalisation of the Casson invariant to a surjective map λ_{CW} from rational homology 3-spheres to $\mathbb{Q}$.
- The *Casson-Walker-Lescop* invariant is a further generalisation to oriented compact 3-manifolds.
- By equating the Casson invariant with a gauge-theoretic invariant, one can define an SU(3)-Casson invariant.
- Finally we note that a Casson-type invariant can in fact be defined for 4-manifolds, namely homology $\mathbb{S}^1 \times \mathbb{S}^3$s. This is also accomplished using gauge-theoretic techniques (and does not use trisections etc.) See [Zen14] or [RS05] for more details.

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