CHOPPING UP 4-MANIFOLDS TO STUDY EMBEDDED SURFACES

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Abstract. This expository talk will start by defining bridge trisections of surfaces in 4-manifolds. We'll then introduce a more rigid version of bridge trisections for surfaces in closed symplectic 4-manifolds and run through a proof outline of the adjunction inequality in this setting. Finally we'll investigate generalisations of this to not-necessarily-closed manifolds. Almost everything in this talk will come from various papers of Peter Lambert-Cole.

1. Overview of the talk

First I'll introduce trisections, and provide one example - the standard trisection of $\mathbb{CP}^2$. I'll then introduce bridge trisections, which are trisections of a pair $(X, \mathcal{K})$ where $\mathcal{K}$ is a surface in X.

Next I'll make things more geometric: if X is a symplectic manifold, we can define a more rigid type of trisection which is compatible with the added geometry. Similarly the notion of a bridge trisection can be made more rigid as well. This geometric version of a trisection was used in a recent proof of the adjunction inequality:

$$2g(\mathcal{K}) - 2 \geq [\mathcal{K}] \cdot [\mathcal{K}] - \langle c_1(\omega), [\mathcal{K}] \rangle$$

(This inequality states that for certain embedded surfaces in symplectic manifolds, the genus is bound below by homological information.) We'll work through a proof outline of this result using geometric trisections.

Finally we'll discuss generalisations to manifolds with boundary.

2. Bridge trisections

In 3 dimensions, a classical result is that every oriented 3-manifold can be cut into two handlebodies glued along their common boundary (a closed surface of genus g). A more recent result is that the same holds for 4-manifolds, provided we use three pieces:

Definition 2.1. Let X be a closed oriented 4-manifold. A *trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$, where

- each X_i is a 4-dimensional 1-handlebody,
- each $H_i = X_i \cap X_{i-1}$ is a 3-dimensional 1-handlebody,
- and $\Sigma = X_1 \cap X_2 \cap X_3$ is a closed surface and the boundary of each H_i.

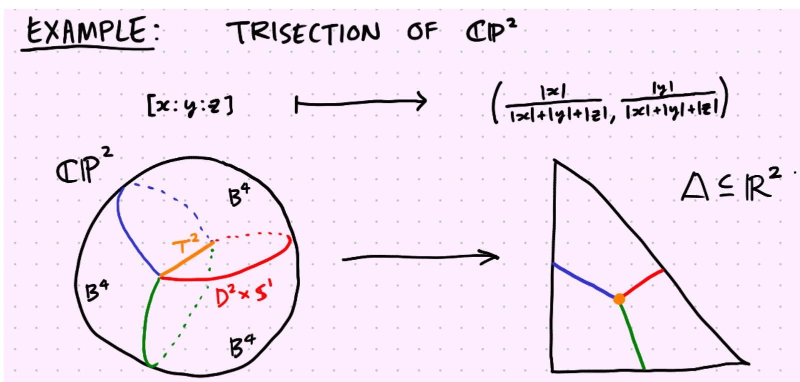
A motivation for studying trisections is that the *spine* (that is, the 3-dimensional part of the trisection) determines the 4-manifold. Therefore trisections may reduce certain

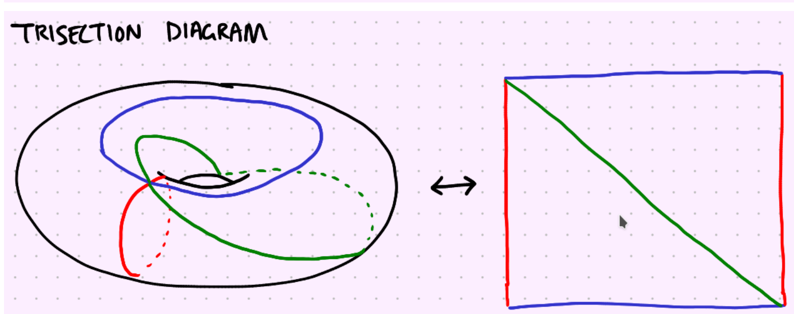
4-dimensional problems to 3-dimensional problems, where algebraic topology is more powerful. (This is exactly why trisections were useful for reproving the adjunction inequality - most of the proof is carried out within the spine.)

This is actually very non-trivial: the reason the spine determines the trisection is because up to diffeomorphism there's a unique way to fill $\natural^k \mathbb{S}^1 \times \mathbb{S}^2$ with $\#^k \mathbb{S}^1 \times B^3$. For example, this isn't true for Heegaard splittings: the ‘ ‘spine” of a Heegaard splitting would correspond to the central surface, but $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{S}^2$ both admit Heegaard splittings with the same central surface. This is why Heegaard diagrams require curves on the surface: the curves describe how to fill the surface with the 3-dimensional handlebody. For a trisection, the fact that the 4-dimensional part is determined by the spine means we still only need curves on a surface (to describe the Heegaard splittings of parts of the spine).

Example. This is an example of a trisection - namely the standard trisection of $\mathbb{CP}^2$. Given a trisection of a triangle in $\mathbb{R}^2$, pulling back by the given map induces a trisection of $\mathbb{CP}^2$. The sectors X_i are all 4-balls, the H_i are solid tori, and the central surface is a torus.

The trisection can be described by a *trisection diagram*. This is essentially three Heegaard diagrams, describing $H_1 \cup H_2, H_2 \cup H_3, H_1 \cup H_3$. Since these determine the spine, they then determine the whole 4-manifold.





Next we'll put a surface inside the 4-manifold. In a trisected 4-manifold, we can ask the surface to be isotoped to be in a nice position with respect to the trisection:

Definition 2.2. Let $\mathcal{K} \subset X$ be embedded in a trisection 4-manifold X. $\mathcal{K}$ is said to be in *bridge position* if

- each $\mathcal{K} \cap X_i$ is a disjoint union of disks that can be simultaneously isotoped to lie in ∂X_i.
- each $\mathcal{K} \cap H_i$ is a disjoint union of arcs that can be simultaneously isotoped to lie in ∂H_i.

The main benefit of putting surfaces in bridge position is that they're determined by their intersection with the spine of the trisection.

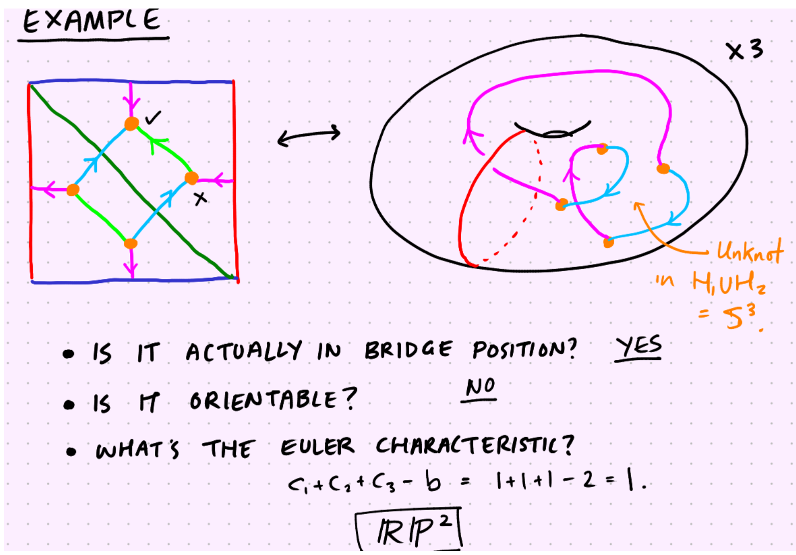
Proposition 2.3. Every surface in a trisected manifold can be isotoped into bridge position.

Another observation is that for a surface in bridge position,

$$\chi(\mathcal{K}) = c_1 + c_2 + c_3 - b$$

where $b = \frac{1}{2}\#\{\mathcal{K} \cap \Sigma\}$ is the bridge number, and each $c_i = \#\{\mathcal{K} \cap X_i\}$ is the number of components of each disk tangle in the sectors. This formula comes from an application of the classic formula $\chi = V - E + F$.

Example. This is an example of a bridge trisection, represented as a diagram inside the trisection diagram of $\mathbb{CP}^2$. The coloured arcs inside the trisection diagram are the tangles $\mathcal{K} \cap H_i$, and these tangles all meet at vertices which are where $\mathcal{K}$ meets the central surface Σ.



To verify that this diagram really represents a surface, we need to check that given any two colours, the resulting unlink bounds disks in $H_i \cup H_j = \mathbb{S}^3$. In this case this is indeed true - we see that we have an unknot. We can try computing

$$\chi = c_1 + c_2 + c_3 - b$$

from the diagram: we have $1 + 1 + 1 - 2 = 1$. This is a bit strange! If the surface was oriented, we'd have an even Euler characteristic.

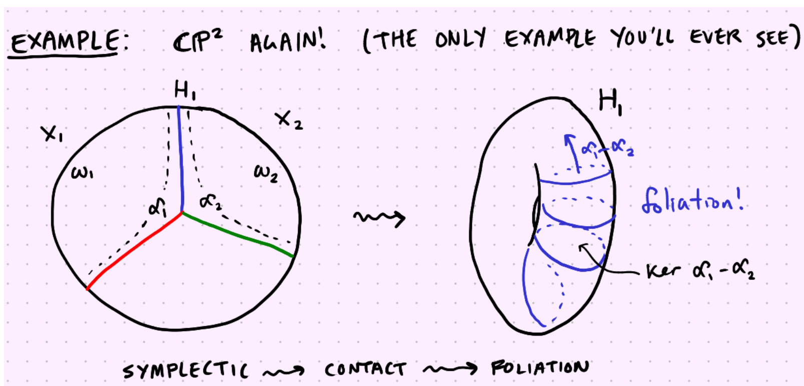
We can also check orientability from the diagram. If we try to orient each arc, we eventually bump into a problem - the surface isn't orientable. (This is a diagram of $\mathbb{RP}^2$ inside $\mathbb{CP}^2$.)

3. Adding geometry to trisections

Now it's time to make things geometric. $\mathbb{CP}^2$ has a symplectic structure, so we ought be able to induce some geometry on the trisection.

Definition 3.1. A *Weinstein trisection* is a trisection whose sectors are all Weinstein domains. That is, each X_i is symplectic, inducing a contact form $\alpha; d\alpha = \omega$ on the boundary ∂X_i for which X_i is a strong filling. (Technically we also need a Morse function for which the Liouville vector field defining the strong filling is gradient-like.)

Example. The standard trisection of $\mathbb{CP}^2$ is a Weinstein trisection. As a consequence, each H_i in the trisection admits two contact structures, induced from the two sectors adjacent to it. Their difference is a closed non-vanishing form β which defines disk foliation $\ker \beta$ in H_i.



We've geometrised trisections, but we have yet to geometrise *bridge trisections*. What does it mean for a surface in bridge position to also be compatible with the extra geometry of a Weinstein trisection?

Definition 3.2. Let $\mathcal{K} \subset X$ be in bridge position in a Weinstein trisection. $\mathcal{K}$ is in *transverse bridge position* if

- $\mathcal{K}$ has complex bridge points; i.e. $\mathcal{K}$ is J-holomorphic in a neighbourhood of its intersections with Σ,
- each $\mathcal{K} \cap H_i$ is a positively transverse tangle in H_i; i.e. $\beta_i(H_i) > 0$ where β_i is the difference of the contact forms on H_i.

This is probably the weirdest definition so far. It's not entirely clear what the properties achieve. One way to motivate the definition is that the conditions guarantee that a certain *adjunction formula* holds (similar to the adjunction formula for algebraic curves):

$$\mathrm{sl}(K) = [\mathcal{K}] \cdot [\mathcal{K}] - \langle c_1(\omega), [\mathcal{K}] \rangle - b.$$

Here $\mathrm{sl}(K)$ is the self linking number of $K = \sqcup K_i$, $K_i = \mathcal{K} \cap \partial X_i$. (Each K_i is a transverse link in $\mathbb{S}^3 = H_i \cup H_{i+1}$ understood as a contact manifold, so the self linking number is well defined.) Combining this with the slice-Bennequin inequality and earlier bridge-position-Euler-characteristic formula gives us the adjunction inequality

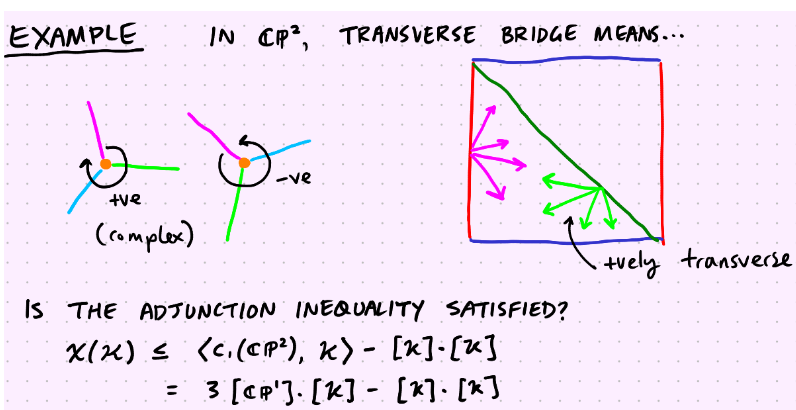
$$\chi(\mathcal{K}) \leq \langle c_1(\omega), [\mathcal{K}] \rangle - [\mathcal{K}] \cdot [\mathcal{K}].$$

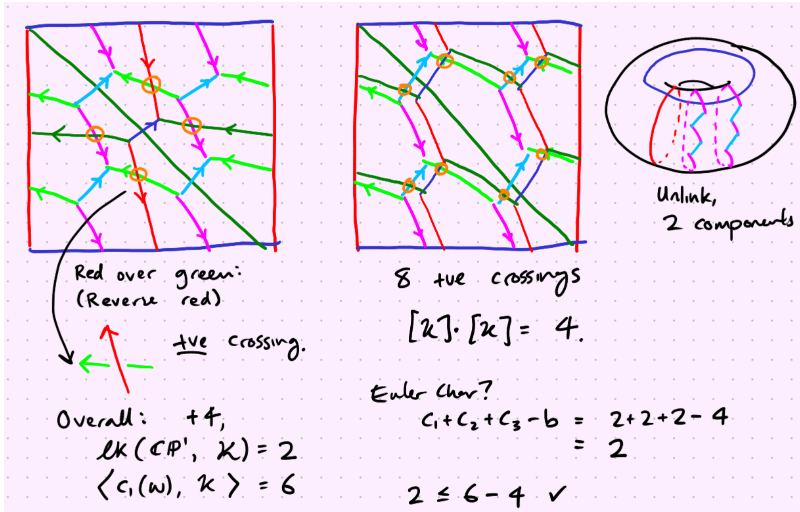
We'll observe this result in an example:

Example. To verify the adjunction formula, we need to compute the self intersection of the surface and the intersection of the surface with the canonical generator ($\mathbb{CP}^1$) of $H_2(\mathbb{CP}^2)$. These are computed by drawing two bridge diagrams on the same trisection diagram and studying the crossings of the arcs. In the given diagram, all the crossings have the same sign! This makes the calculation pretty easy. The diagram has the following properties:

- The arcs all meet the vertex with the same orientation, corresponding to complex bridge points.
- The arcs are all pointing transversely (positively) away from their respective Heegaard diagram curve, corresponding to positive transversality.

These conditions guarantee that signs work out nicely, allowing the adjunction formula/inequality to be derived algebraically.





4. Proving the adjunction inequality in general

We have the adjunction inequality in the case where the surface lies in transverse bridge position. To prove the result in general, it would be nice if all surfaces could be isotoped to lie in transverse bridge position. Unfortunately we don't have such a theorem. We also don't necessarily have Weinstein trisections for general symplectic manifolds. To prove the adjunction inequality, we need to do the following:

(1) Find (nice) Weinstein trisections in arbitrary symplectic 4-manifolds.
(2) Isotope surfaces so they're as close to being in transverse bridge position as we can. (*Homotopic transverse bridge position.*)
(3) Modify the surface in a controlled way (via homotopy) to obtain a new surface which is in transverse bridge position. Compute how invariants (such as the self linking number) change during this process.
(4) Apply the adjunction inequality for surfaces in transverse bridge position. Carefully account for the homotopy step (3). (It ends up all cancelling and we just get the usual adjunction inequality anyway!)

I'll describe steps 1 and 3, but not 2 or 4.

Step 1. By a result of Auroux, every symplectic 4-manifold X is a branched cover over $\mathbb{CP}^2$. This meanswe can define a trisection on X by pulling back each of the pieces of the trisection of $\mathbb{CP}^2$ via the covering map. Typically just pulling back a Weinstein trisection by a branched covering map won't ensure that the decomposition of the covering space is itself a Weinstein trisection. However, in this case, the branch locus of Auroux's covering maps can be isotoped to lie in transverse bridge position in $\mathbb{CP}^2$! This is enough to guarantee that the pullback is also a Weinstein trisection.

Step 3. In step 2, we isotope the surface to lie in *homotopic transverse bridge position.* This means the tangles $\tau_i = \mathcal{K} \cap H_i$ can be homotoped rel boundary in H_i so they're positively transverse, and that the resulting surface described by the new tangles is in transverse bridge position. (By homotopy, I really mean the tangle can pass through itself!)

This means everything is again captured by 3-dimensional topology. The homotopy in step 3 all happens in a bridge diagram - the only non-trivial moves are crossing changes. Whenever we make a crossing change, we can track how the self linking number changes.

5. Further generalisations

The main difficulties in the proof are:

- Finding a Weinstein trisection for X.
- Showing that any surface can be isotoped to be in homotopically transverse bridge position.

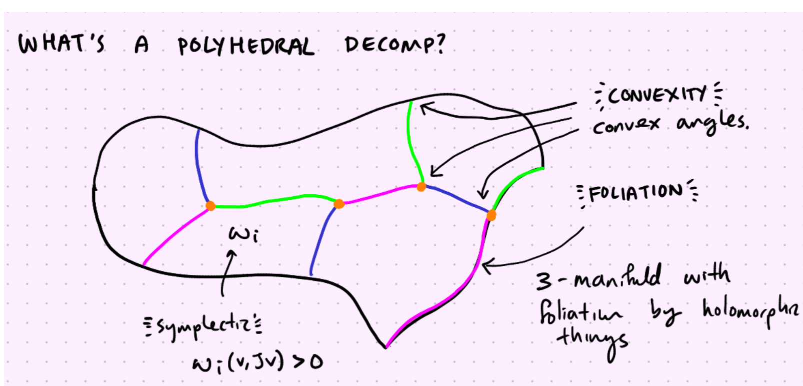
In particular, Auroux's result in which every closed symplectic manifold is a branched cover over $\mathbb{CP}^2$ doesn't generalise well: if we wish to consider symplectic manifolds with boundary, the result falls apart.

That being said, Peter Lambert-Cole recently posted this result to the Arxiv, which shows that *if homotopic transverse bridge position can be achieved*, then the above proof of the adjunction inequality carries through:

Theorem 5.1. *Let (X, J) be a compact, almost-complex 4-manifold with an aspherical* polyhedral decomposition. *Let $\mathcal{K}$ be a homotopically transverse closed embedded surface in X. Then*

$$\chi(\mathcal{K}) \leq \langle c_1(J), \mathcal{K} \rangle - [\mathcal{K}] \cdot [\mathcal{K}].$$

We've replaced the notion of a Weinstein trisection with that of a polyhedral decomposition. This essentially lets us cut our space into many more than just three pieces, while ensuring that each piece is equipped with the correct geometry. I won't give a technical definition, but here's a picture which gives the idea:



Now that we're considering manifolds with boundary, it's natural to want to consider embedded surfaces $\mathcal{D}$ with boundary, where $\partial\mathcal{D} \subset \partial X$. The adjunction inequality for closed surfaces corresponds to the slice-Bennequin inequality for surfaces with boundary:

Theorem 5.2. *Let (X, J) be a compact almost-complex 4-manifold with an aspherical polyhedral decomposition. Let $\mathcal{K}$ be a properly embedded surface with boundary K in ∂X, homotopically transverse with transverse boundary. Then*

$$sl(K, \mathcal{K}) \leq -\chi(\mathcal{K}).$$

To see why this is related to the adjunction inequality, we use the following facts:

- $\mathrm{sl}(K, \mathcal{K}) = e(N\mathcal{K}, s) - c_1(\det_{\mathbb{C}}(TX)|_{\mathcal{K}}, s)$ where s is a non-vanishing section of the field of complex tangencies along K. (Usually the self linking number is defined for knots in contact $\mathbb{S}^3$ bounding B^4. In that setting, the definition doesn't depend on a Seifert surface of the knot, but for a knot in the boundary of a general 4-manifold we need to carry the information of the surface.) This identity should be considered a definition rather than a result.
- The Chern class term in the above formula corresponds to the $\langle c_1(J), \mathcal{K}\rangle$ term for the closed surface case.
- The Euler class term in the above formula corresponds to the $[\mathcal{K}] \cdot [\mathcal{K}]$ term.

Earlier I mentioned that generalising the result is hard because finding trisections (decompositions) is hard, as well as the fact that isotoping surfaces to be in homotopically transverse bridge position is hard. I now provide an example of the latter truly failing.

Example. Trefoil knots are slice in $\mathbb{CP}^2 - B^4$, in other words they bound disks (which are surfaces of genus 0). For the left handed trefoil the simplest disk represents $2H$ in $H_2(\mathbb{CP}^2) \cong H_2(\mathbb{CP}^2 - B^4; \mathbb{S}^3)$, while the right handed trefoil represents 0 in $H_2(\mathbb{CP}^2)$.

For the right handed trefoil, $\mathrm{sl}(RHT, \mathcal{K})$ is trivially 0. On the other hand, $\chi(\mathcal{K}) = 1$. Since 0 is not less that or equal to -1, the right handed trefoil doesn't satisfy the slice Bennequin inequality, so it can't satisfy the premises of the slice Bennequin inequality.

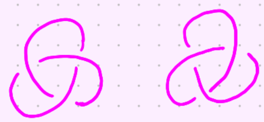
This means the disk bound by the right handed trefoil *cannot* be isotoped to be in homotopic tranverse bridge position in any polyhedral decomposition (or relative trisection) of $\mathbb{CP}^2 - B^4$. On the other hand, this is probably possible for the disk bound by the left handed trefoil, for which we have

$$\mathrm{sl}(LHT, \mathcal{K}) = 4 - 6 = -2.$$

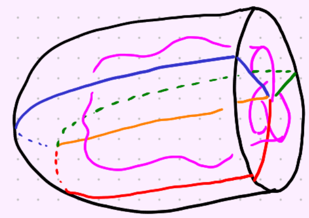
That is, the left handed trefoil does satisfy the slice Bennequin inequality.

EXAMPLE TREFOILS IN $\mathbb{CP}^2 - B^4$

LHT RHT



slice H-slice



$sl(RHT, D) = 0$ (since $D \in H_2$ is trivial)

But $\chi(D) = 2 - 0 - 1 = 1$.

$sl \not\leq -1$.

$sl(LHT, D') = -2$ (since $D' \in H_2$ is $2\mathbb{CP}^1$)

$sl \leq -1$ ✓