# FREEDMAN'S THEOREM II

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Abstract. We give a proof outline of the topological 4-dimensional Poincaré conjecture. Specifically, we describe how the $h$-cobordism theorem (i.e. proof of Poincaré conjecture) for high dimensions fails in dimension 4, and introduce Casson handles to patch the proof.

## 1. Review of high dimensions

We begin by recalling the proof outline for the *$h$-cobordism theorem*. This was the focus of last week's talk by Judson.

**Theorem 1.1.** *Let $M^n, N^n$ be simply connected closed manifolds, $n \geq 5$. Let $(W, M, N)$ be an $h$-cobordism from $M$ to $N$. Then $W$ is a cylinder; i.e. $W \cong M \times [0,1]$.*

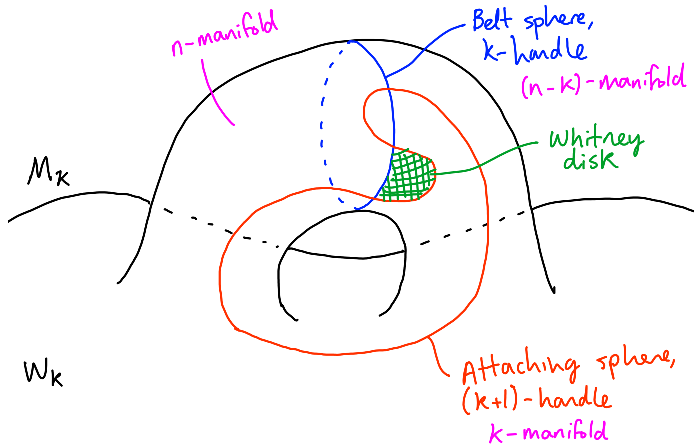
*Proof.* We give a proof outline:

(1) Choose a handle decomposition of $M$.
(2) For algebraic reasons, handles can be rearranged so that they occur in algebraically cancelling pairs. That is, attaching spheres of $k$ handles algebraically intersect belt spheres of $k-1$ handles once.
(3) By the *Whitney trick*, the algebraic intersections can be upgraded to geometric intersections. Now the handles can actually be cancelled!
(4) After cancelling all handles, a cylinder remains.

□

The key ingredient of the proof is the Whitney trick. Unfortunately this only holds in dimension at least 5. The Whitney trick relies on the *Whitney embedding theorem*, that embeddings $A^n \to B^{2n+1}$ are dense in the space of (smooth) maps $A^n \to B^{2n+1}$. In particular, embeddings are dense in the space of maps $D^2 \to B^5$, but not necessarily in the space of maps $D^2 \to B^4$. Indeed, embeddings of disks into 4-manifolds are generally *not* dense.

**Remark.** It's important to be careful about where we're considering the embedding/Whitney disk. The $k$-handles of the cobordism $W$ are attached at height $k$, given a Morse function on $W$. The level set $W_k$ is an $n$-dimensional manifold, even though $W$ is $n+1$ dimensional. The attaching sphere of the $k$-handle and belt sphere of the $(k-1)$-handle both occur at height $k$, and the Whitney disk must embed in this level set as well.




## 2. A four dimensional programme

In dimension 4, we wish to carry out the same programme as above. The biggest difficulty is embedding disks. If we can figure out how to eliminate self-intersections of immersed disks and obtain embedded disks even in the 4-dimensional setting, then the same proof as in higher dimensions will carry through, and we'll obtain an $h$-cobordism theorem for 4-dimensional manifolds.

It turns out that by introducing a topological (but not smooth) notion of handles that we can "easily embed" we can solve the problem. We now state the theorem and describe the proof outline.

**Theorem 2.1.** *Let $M^4, N^4$ be simply connected closed manifolds, and $W$ an $h$-cobordism between them. Then there is a* homeomorphism $W \cong M \times [0, 1]$.

*Proof.* We give a proof outline:

(1) Deal with all $k$-handles with small $k$ or large $k$, as in the high dimensional case. This leaves us with 3-handles and 2-handles, which meet along 2-dimensional belt spheres and 2-dimensional attaching spheres.
(2) Choose a pair of intersection points between the attaching and belt spheres. By drawing arcs in each of the two spheres between intersection points, we bound an immersed disk $D$ in $W_2$. We want to apply the Whitney trick to $D$, but we can't. We're instead going to a series of moves to the immersed disk to push all the badness to infinity.
(3) Miraculously, after pushing the badness to infinity, everything actually works out anyway in the topological category. This is essentially the *disk embedding theorem* (1984, combining work of Casson, Freedman, and Quinn). This means we have an analogue of the Whitney trick, and we can topologically cancel the remaining handles to produce a cylinder.

□

## 3. Casson handles

In this section we introduce Casson handles. Historically these were introduced in the early 70s by Casson. He called them *flexible handles* because they're easier to embed than standard handles:

**Theorem 3.1** (Casson, 73)**.** *Let $(M, \partial M)$ be a simply connected 4-manifold with boundary. Let $f_1, \ldots, f_n : D^2 \to M$ be immersions whose boundaries are disjoint embeddings. Moreover, assume that for $i \neq j$, $f_i \cdot f_j = 0$, and that there exist "dual spheres" for each $f_i$. By dual spheres, we mean classes $a_i \in H_2(M; \mathbb{Z})$ such that $a_i \cdot f_i = 1$ but $a_i \cdot f_j = 0$ for $i \neq j$.*

*Then there exist Casson handles $C_1, \ldots, C_n$ such that*

1. *there are proper homotopy equivalences $(C_i, C_i \cap \partial M) \to (D^2 \times \mathbb{R}^2, \mathbb{S}^1 \times \mathbb{R}^2)$,*
2. *each $C_i \cap \partial M$ is an open tubular neighbourhood of the circles $f_i(\partial D^2)$, and*
3. *each $f_i$ is homotopic rel boundary to a map into $C_i$.*

This already looks like some sort of 4-dimensional Whitney trick analogue: we're replacing our immersed disk with something embedded - but unfortunately these objects we're embedding are a little bit elusive.

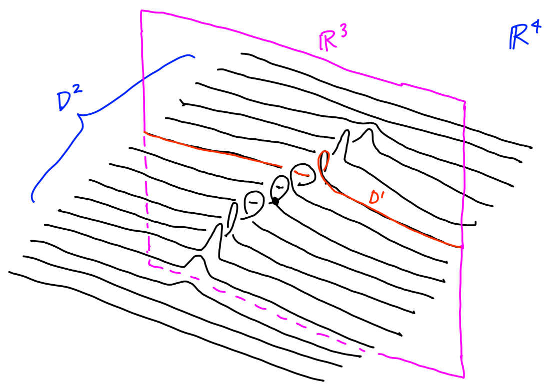
**Theorem 3.2** (Freedman, 82)**.** *Any Casson handle is homeomorphic to a thickened disk $D^2 \times \mathbb{R}^2$. In particular, Casson handles are genuine topological 2-handles with core a topologically embedded 2-disk.*

So what exactly is a Casson handle? We discuss this next.

**Definition 3.3.** *Casson handles* are the union of topological spaces obtained by carrying out certain procedures an infinite number of times! (To understand what Casson handles are, we spend the rest of this section describing certain moves in 4 dimensions.)

3.1. **Creating self intersections.** Warm up: suppose $K$ is an immersed circle in $\mathbb{R}^2$. Locally, away from double points, $K$ looks like $\mathbb{R}$ embedded in $\mathbb{R}^2$. We can add a double point by utilising the direction normal to $\mathbb{R}$ to add a kink.

Now suppose $S$ is a surface immersed in a 4-manifold. Locally, away from double points, $S$ looks like $\mathbb{R}^2$ embedded in $\mathbb{R}^4$. We actually think of this as $(\mathbb{R} \times \mathbb{R}, \mathbb{R}^3 \times \mathbb{R})$, where the second $\mathbb{R}$ term is parametrised by $t$. Now as we increase $t$, we pinch $\mathbb{R}$ at a point and start to twist it. At a certain value of $t$ the twisting copy of $\mathbb{R}$ intersects itself creating a double point. We then keep twisting and eventually remove the pinch, getting back to standard $\mathbb{R}$ in $\mathbb{R}^3$. This procedure adds a single double point, which can have positive or negative sign depending on the direction of twisting.



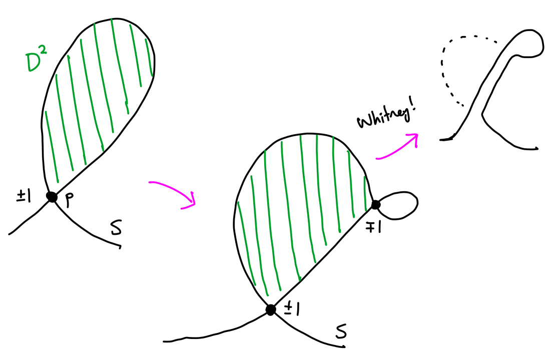

3.2. **Removing self intersections.** Suppose we have a double point $p$ in an immersed surface $S$ that we wish to remove. There exists a loop $\gamma$ in $S$ that starts at $p$ along one branch of the double point and returns back to $p$ along the other.

Warm up: assume our curve $\gamma$ bounds an embedded disk. Then, somewhere along the curve $\gamma$, we can *add a self intersection* as above, and choose the intersection to have opposite sign to the pre-existing one at $p$. Now the Whitney trick applies, and both intersections can be removed.

In reality, our curve $\gamma$ probably won't bound an embedded disk. What we want is the following:

(1) We have a closed 4-manifold $M$ and an immersed surface $S$ in $M$. We also consider a loop $\gamma$ in $M$, lying on the image of $S$.
(2) We want a disk in $M - S$ whose boundary is the loop $\gamma$.
(3) If $M - S$ isn't simply connected, we can find a loop that doesn't bound any disk, and push the loop into the boundary of $M - S$.

In summary, for an arbitrary loop to bound an embedded disk, we certainly need the complement to be simply connected. We now describe how to achieve this:

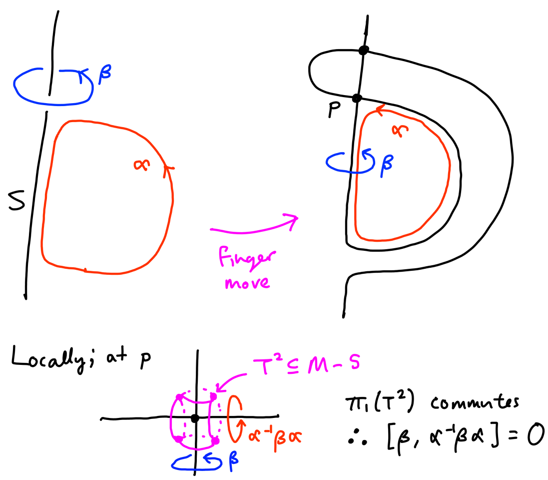


3.3. **Finger moves.** We now describe finger moves which are a method to remove certain commutators from the fundamental group of the complement $M - S$.

(1) Suppose $\beta$ is a loop *around* $S$, and $\alpha$ is any loop in $M - S$. We can push part of $\alpha$ to be parallel to $S$, which decomposes $\alpha$ into $\alpha_1$ and $\alpha_2$. Here $\alpha_1$ is the part of $\alpha$ parallel to $S$ and $\alpha_2$ is an arc from $S$ to itself, through $M - S$.

(2) A *finger move* consists of pushing $S$ along $\alpha_2$ until it eventually gets pushed back into $S$, creating two intersection points.

- This has the effect of killing the commutator $[\beta, \alpha^{-1}\beta\alpha]$. This is because we can embed a torus as follows:
  (1) The manifold $M$ near one of the new double points looks like $D^2 \times D^2$, with branches of $S$ given by $D^2 \times 0$ and $0 \times D^2$. In these coordinates, the complement $D^2 \times D^2 - S$ contains a torus $\mathbb{S}^1 \times \mathbb{S}^1$. (This is immediate because the only points we're removing from $D^2 \times D^2$ have 0 in one of their coordinates.)
  (2) By construction, we actually have that $\beta$ is $\mathbb{S}^1 \times \{-1\}$. On the other hand, conjugating by $\alpha$, we get the other circle $1 \times \mathbb{S}^1$. These are the meridian and longitude of the torus embedded in $M - S$. Since the fundamental group of the torus is commutative, $\beta$ and $\alpha^{-1}\beta\alpha$ must commute in $\pi_1(T^2)$ and thus in $\pi_1(M - S)$.
- This unfortunately increases the number of self intersections by two. It's a tradeoff!




3.4. **Casson handles.** We're now ready to introduce Casson handles. The idea is to replace a failed Whitney disk with a slightly less bad failed Whitney disk and continue to infinity.

Warm up: working with disks, rather than handles. (I.e. the non-thickened version.)

(1) Consider an immersed disk $(D, \partial D) \subset (M, \partial M)$. If we start out with $M$ simply connected, then $M - D$ is actually a perfect group. (This means the group has no non-trivial abelian quotients.) (Why is this true? Does it follow from the Wirtinger

presentation?) In a perfect group, every element *is* a commutator, and taking enough finger moves, we can assume the complement $M - D$ is simply connected.

(2) Consider a self intersection $p$ of $D$. We want to eliminate this, so we consider a loop $\gamma$ from $p$ to itself, eminating from one branch and returning along the other. This loop bounds an immersed disk: if the self intersections of this disk can be removed, then so can the original self intersection!

(3) Effectively we've "pushed the problem away" a little bit. We now perform a sequence of finger moves until the complement of the new surface unioned with disks has simply connected complement.

(4) Now we repeat this infinitely many times. We keep adding new immersed disks, and then carrying out finger moves to purge the fundamental group.

Now we actually define Casson handles by working in the thickened setting.

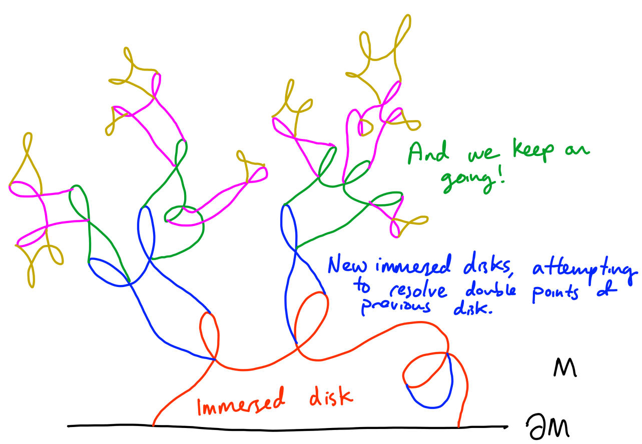
(1) Suppose we have an immersed thickened disk $(D^2 \times D^2, \mathbb{S}^1 \times D^2) \subset (M, \partial M)$. Essentially, this is a 2-handle that's allowed to intersect itself.

(2) The self intersections are *plumbings*. Locally two points in the thickened disk have neighbourhoods $D_1 \times D^2$ and $D_2 \times D^2$ inside $D^2 \times D^2$. These neighbourhoods glue together by swapping factors; $D_1 \times D^2 \sim D^2 \times D_2$.

(3) We now follow the the infinite procedure of adding additional immersed disks to try to resolve double points - but instead of gluing in disks, we'll add thickened disks. Technically we should think about framings here!

(4) As we build the iterated chaos handle, we want to remove boundaries so that the end result is an open set in $M$. This means with every immersed disk we glue in, we delete the $D^2 \times \mathbb{S}^1$ from the boundary, and only keep the part of the boundary that gets glued to the preeixsting disk.

The result is an open set in set in $M$, anchored to $\partial M$, which we call a *Casson handle*.



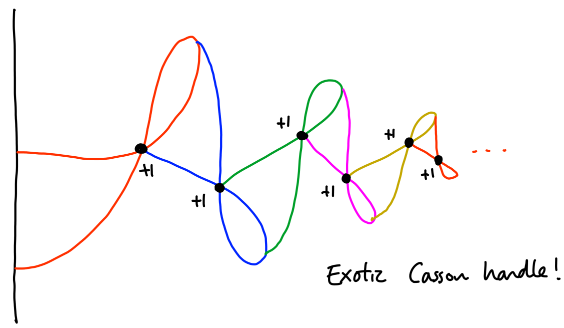

**Theorem 3.4** (Casson, 73)**.** *Casson handles are homotopic to $D^2 \times \mathbb{R}^2$ rel boundary.*

**Theorem 3.5** (Freedman, 81)**.** *Casson handles are homeomorphic to $D^2 \times \mathbb{R}^2$.*

In particular, topologically, Casson handles are real handles.

**Example.** We won't prove any of this, but I'll provide some evidence as to why this might be true. Consider a loop $\gamma$ in a Casson handle. Any loop exists in the handle after a finite number of iterations, and the next iteration kills its homotopy class. This means Casson handles are simply connected.

**Remark.** Casson handles need not be diffeomorphic to real handles. We draw an example. In particular, we can't prove the smooth four dimensional Poincaré conjecture.




## 4. Four dimensional $h$-cobordism theorem

**Theorem 4.1.** *Let $W$ be an $h$-cobordism between $M^4$ and $N^4$, all pieces simply connected. Then there is a homeomorphism $W \cong M \times [0,1]$, and in particular $M$ and $N$ are homeomorphic.*

*Proof.*

(1) We first must find a topological handle decomposition of $W$. This is highly non-trivial, proved by Quinn in 1982. (Recall that Morse theory is inherently a smooth theory.)

(2) Given our handle decomposition, we eliminate 0 and 5-handles, and trade 1-handles for 3-handles, and trade 4-handles for 2-handles. The result consists of a handle decomposition of $W$ consisting exactly of 2-handles and 3-handles. By using handle-slides and handle-creations, all of the 2 and 3-handles are pairwise paired to have their belt and attaching spheres algebraically intersect exactly once.

(3) Let $W_2$ be the union of all the 2-handles. The 3-handles are glued to $W_2$ along the boundary component $M_2$. In $M_2$, we generate Casson handles along immersed disks (which are created when we want to use the Whitney trick). Since Casson handles are actually genuine topological 2-handles, the Whitney trick applies topologically, so all the 2-handles and 3-handles in $W$ cancel each other.

(4) Since all handles have been eliminated, the cobordism is a cylinder.

□

## 5. Four dimensional Poincaré conjecture

**Theorem 5.1.** *Topological homotopy 4-spheres are topological 4-spheres*

*Proof.* We must first argue that the cone of a topological homotopy 4-sphere is a topological manifold. I'm not sure why this is true!

If we start with a smooth (or piecewise linear) homotopy 4-sphere, then the cone is a piecewise linear manifold (and therefore a topological manifold). Next, we remove a 5-ball from around the vertex of the cone. This creates a topological $h$-cobordism from a 4-sphere to the homotopy 4-sphere, so by the $h$-cobordism theorem, homotopy 4-spheres are spheres. □