TWO RESULTS IN DISCRETE GEOMETRY III

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Abstract. For two quarters now I've given Kiddie talks focussing on two discrete geometry topics each. With no sign of slowing down, here's yet another talk consisting of two topics from discrete geometry.

1. Decomposing shapes into a minimal number of strictly smaller pieces. *Borsuk's conjecture*, dating back to the 30s, asks if n dimensional bounded bodies can be partitioned into $n+1$ pieces with strictly smaller diameter. We'll observe that this is false in sufficiently many dimensions!

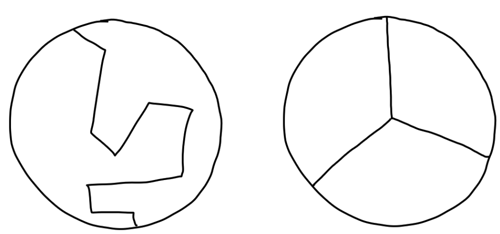
2. Covering shapes with a minimal number of (smaller) scaled copies. *Hadwiger's conjecture*, from the 50s, states that any n dimensional convex body can be covered by 2^n (strictly smaller) homothetic copies of itself. This is still an unsolved problem, but we'll inspect some known results.

1. Borsuk's conjecture

I'll open with some examples.

Example. Suppose we decompose a unit disk in the plane into two pieces. Then at least one piece contains two antipodal boundary points. This means the piece also has diameter 2, which is equal to the diameter of the original disk. *Any partition of a disk into two pieces has at least one piece with diameter equal to that of the disk.*

Example. Suppose we decompose a unit disk into three pieces as follows. Then the diameter of each piece is $\sqrt{3}$, which is strictly less than 2. *The disk has a partition into three pieces, each with diameter strictly less than that of the disk.*



In fact, the following result is true:

Theorem 1.1 (Borsuk, Bonnesen-Fenchel)**.** *Every set $S \subset \mathbb{E}^2$ of diameter d can be partitioned into three pieces, each with diameter at most*

$$\frac{\sqrt{3}}{2}d.$$

Before proving this result, I want to formally define a couple of concepts.

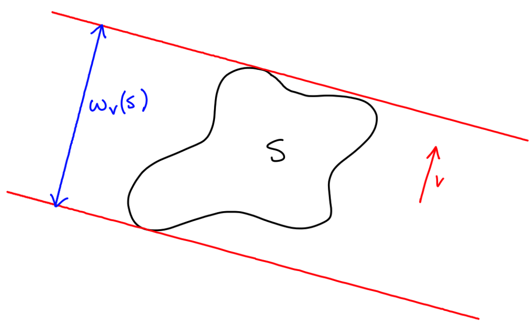
Definition 1.2. Let $S \subset \mathbb{E}^n$ be bounded. For each direction $v \in \mathbb{S}^{n-1} \subset \mathbb{E}^n$, we can consider pairs of hyperplanes perpendicular to v and sandwiching S. We define the *width* $w_v(S)$ of S in the given direction to be the infimum of the distances between such hyperplanes.

The *diameter* of a shape S is defined to be

$$\operatorname{diam}(S) = \sup_{v \in \mathbb{S}^{n-1}} w_v(S).$$

A shape S has *constant width* if for all v,

$$w_v(S) = \operatorname{diam}(S).$$



Proof. The proof outline is as follows:

(1) Embed our set inside a *shape of constant width.*
(2) Embed the shape of constant width inside a regular hexagon of known diameter.
(3) Partition the hexagon into three pieces with known diameter.

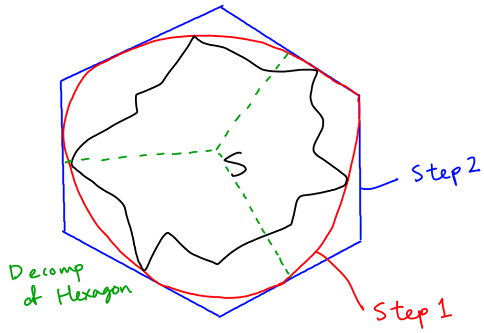
Step 1 is, I believe, due to Lebesgue. Let S be our given shape with diamater d. Consider the family

$$\mathcal{F} = \{S : S \subset A, \operatorname{diam} A = d\}.$$

Observe that $\mathcal{F}$ is non-empty, and there's a partial order given by inclusion. Moreover, chains of the partial order have upper bounds (since the union of sets in the chain is an upper bound). By Zorn's lemma we can find an upper bound. This upper bound can be shown to be a shape of constant width.

Step 2. The result comes from the intermediate value theorem: choose an angle θ. We can deterministically build a hexagon whose "first edge" has gradient θ, and whose internal angles are all $2\pi/3$, but need not be regular. Rotating θ by π results in an identical hexagon as above, but with opposite pairs of edges swapping lengths. For some intermediate θ, opposite lengths were the same - so the hexagon was regular.

Step 3. The regular hexagon necessarily has diameter $2d/\sqrt{3}$. A regular decomposition of the hexagon as depicted has diameter $\frac{\sqrt{3}}{2}d$. In particular, this is strictly less than d. $\square$



Corollary 1.3. *Every bounded body $S \subset \mathbb{E}^2$ can be partitioned into three pieces each with strictly smaller diameter than S. Two pieces is generally not enough.*

I won't prove the following theorem, but the same result also holds in one higher dimension:

Theorem 1.4 (Perkal, Eggleston)**.** *Every bounded body $S \in \mathbb{E}^3$ can be partitioned into four pieces each with strictly smaller diameter than S. Three pieces is generally not enough.*

This leads to the general question:

Question. Can every bounded body $S \in \mathbb{E}^n$ be partitioned into $n+1$ pieces, each with diameter strictly less than S?

This problem was open for many decades, and resolved (in the negative) in the 90s by Kahn and Kalai. In the remainder of this section of the talk we'll construct a counter example.

We'll follow a construction from Kalai's blog.

Example. We will construct a counter example inside n^2-dimensional Euclidean space, for $n = 4p$, with p a sufficiently large prime.

(1) Let S' be the set of vectors in $\mathbb{E}^n$ with entries ± 1. Note that S' is a finite subset of $\mathbb{R}^n$ with size 2^n.
(2) Let S be the subset of $\mathbb{E}^{n^2}$ defined by
$$S = \{x \otimes x : x \in S'\} \subset \mathbb{E}^n \otimes \mathbb{E}^n \cong \mathbb{E}^{n^2}.$$
Concretely, we can think of S as consisting of vectors with entries $x_i x_j$.

It turns out that for n sufficiently large, S cannot be decomposed into n^2+1 pieces with strictly smaller diameter than S.

Let's now consider the geometry of S.

(1) All points (vectors) in S have length n:

$$\langle x \otimes x, y \otimes y \rangle = \sum_{i,j} x_i x_j y_i y_j = (\sum_i x_i y_i)^2 = \langle x, y \rangle^2.$$

Since S' consisted of vectors of length $\sqrt{n}$, the above calculation shows that S consists of vectors of length n.

(2) The above calculation also shows that any two points have non-negative dot products!

(3) Given any two vectors, the square of the distance between them is given by

$$\begin{aligned} \langle x \otimes x - y \otimes y, x \otimes x - y \otimes y \rangle &= 2n - 2\langle x \otimes x, y \otimes y \rangle \\ &= 2n - 2\langle x, y \rangle^2 \leq 2n. \end{aligned}$$

Since S' contains orthogonal vectors, the diameter of S is $\sqrt{2n}$.

(4) Now suppose S is partitioned into subsets with diameter strictly less than $\sqrt{2n}$. This is possible only if each subset doesn't contain orthogonal vectors.

To really establish that our set X is the desired counterexample, we'll use a theorem from combinatorics.

Theorem 1.5 (Frankl-Wilson)**.** *(Intuive informal version.) Let $\mathcal{F}$ be a collection of subsets of size r of a given set S. If we have some understanding of the intersections of the subsets, we can bound the size of $\mathcal{F}$. Specifically, it's a non-trivial bound that comes from prime number properties.*

Theorem 1.6 (Frankl-Wilson)**.** *(Specific version.) Let $n = 4p$ for an odd prime p. Let $\mathcal{F}$ be a set of vectors in $\mathbb{E}^n$ with entries ± 1, so that no two vectors in $\mathcal{F}$ are orthogonal. Then*

$$|\mathcal{F}| \leq 4\left(\binom{n}{0} + \cdots + \binom{n}{p-1}\right).$$

Proof. Proof outline:

(1) For each x, y in $\mathcal{F}$ we define the polynomial

$$P_y(x) = \prod_{k=1}^{p-1} (\langle x, y \rangle - k)$$

(considered over $\mathbb{F}_p$.)

(2) It turns out that $P_y(x) = 0$ if and only if $x \neq y$. As a consequence, the set of polynomials $\{P_y\}$ is linearly independent over $\mathbb{F}_p$.

(3) We define new polynomials P'_y by replacing each x_i^{2k} with 1, and each x_i^{2k+1} with x_i. These attain the same values as P_y on $\mathcal{F}$, so they're again linearly indepdendent.

(4) The P'_y are square free, and $|\{P'_y\}| = |\mathcal{F}|$ (because of linear independence). This bounds $|\mathcal{F}|$ by the dimension of the space of square free degree $p-1$ polynomials in n variables, which is

$$4\left(\binom{n}{0} + \cdots + \binom{n}{p-1}\right).$$

$\square$

Remark. Actually, I lied a bit: for the proof to carry through properly in step 2, we want to restrict to vectors in $\mathcal{F}$ which all have first entry ± 1 and an even or odd number of 1s. The dimension of the space of polynomials constructed in the above manner is one 4th of what I claim. For each of the four options (two choices) the bound holds, so covering each quarter of $\mathcal{F}$ one at a time gives the desired result.

Example. Now we go back to our (counter)example of Borsuk's conjecture. Recall that we constructed a set S in $\mathbb{E}^{n^2}$ (for $n = 4p$) which had the property that

(1) S has diameter $\sqrt{2n}$ (coming from the distance between two orthogonal vectors in S).
(2) Any subset $\mathcal{F}$ of S with strictly smaller diameter cannot contain any orthogonal vectors.

By our version of the Frankl-Wilson theorem, we immediately have the bound

$$|\mathcal{F}| \leq 4\left(\binom{n}{0} + \cdots + \binom{n}{p-1}\right).$$

On the other hand, we know that S contains exactly 2^n points (because S is in bijection with the set of vectors in n-dimensions with entries ± 1). We obtain the following result:

Proposition 1.7. To cover S with subsets of strictly smaller diameter than S, we need at least

$$\frac{2^n}{4\left(\binom{n}{0} + \cdots + \binom{n}{p-1}\right)}$$

subsets.

I didn't want to do algebra so I plugged this equation into Desmos. It turns out that for $n = 4 \times 13$, the above fraction is *larger than* $n^2 + 1$. This means in dimension

$$2704$$

we've obtained a counter example to Borsuk's conjecture!

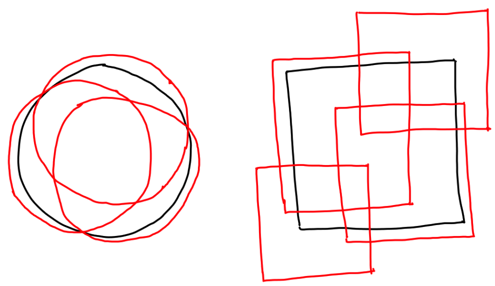
Remark. The original published counterexamples to Borsuk's conjecture found by Kahn and Kalai were in all dimensions at least 2014. As of today the conjecture is known to fail in all dimensions at least 64. The status of the conjecture is unknown in all dimensions between 4 and 63.

2. Hadwiger's conjecture

Again, I'll open with some examples before introducing the conjecture.

Example. Consider the disk in $\mathbb{E}^2$. Three slightly smaller disks can be used to cover the original disk, as shown in the figure.

Example. Consider a square in $\mathbb{E}^2$. Any smaller translation of a square can cover at most one of the vertices of the square. This means at least 4 smaller translated squares are needed to cover the original square. On the other hand, 4 is certainly enough, as shown in the figure.



We now introduce some terminology:

Definition 2.1. Given a set S, a *homothet* of S is a scaled translation of S. If S is scaled by a positive constant, the homothet is called *positive*, and if the scaling factor has magnitude less than 1, it's *smaller*.

Proposition 2.2. An n-dimensional hypercube can be covered by 2^n positive smaller homothets, but no fewer.

Conjecture 2.3. The *Hadwiger conjecture* states that a convex (bounded) subset S of $\mathbb{E}^n$ can be covered by 2^n positive smaller homothets.

Writing $H(S)$ to mean the *minimum number of positive smaller homothets needed to cover* S, the conjecture can be written as

$$H(S) \leq 2^n.$$

It turns out that very little is known about this problem!

Remark. The general conjecture is known to be true in dimension 2. It's unknown in all other dimensions.

Remark. In dimension 3, the best bound that currently exists is $H(S) \leq 16$ - much more than the conjectured value of 8.

Remark. The current best asymptotic upper bound is

$$H(S) \leq 4^n(5n \log n).$$

This is a lot bigger than 2^n.

To tie everything back to the very first part of the talk, we'll finish by considering forms of constant width.

Proposition 2.4. Let $S \subset \mathbb{E}^n$ be a convex bounded set of constant width. Then for n large enough S is covered by 2^n positive smaller homothets.

To the best of my knowledge, the current best upper bound for bodies of constant width is

$$H(S) \leq 5n\sqrt{n}(4 + \log n)(3/2)^{n/2},$$

which was proven using the probabilistic method. On the other hand, we can ask for a lower bound. These actually come from the Borsuk conjecture!

Let S be a bounded set in $\mathbb{E}^n$, and let $P(S)$ be the minimum size of a partition in which each piece has diameter strictly less than S. Recall from the start of the talk that S lies inside a body of constant width C of the same diameter as S. Now suppose C can be covered by $H(C)$ positive smaller homothets h_i. Then it's necessarily the case that

$$P(S) \leq H(C).$$

This is because $\{S \cap h_i\}$ is a covering of S of size $H(C)$ consisting of sets with diameter strictly less than that of C.