TWO RESULTS IN DISCRETE GEOMETRY

SHINTARO FUSHIDA-HARDY

Abstract. This talk will concern polyhedra and polygons.

1. It's well known that there are exactly 5 platonic solids, but why are there only five? We'll observe that this follows from a remarkably short topological proof. Next we'll study "convex deltahedra", which is another collection of polyhedra, and observe that there are exactly 8 of them using a combinatorial version of the Gauss-Bonnet theorem.

2. Given an integer n, can we cut a square into n equal area triangles? This turns out to be impossible for all odd n, by application of graph theory and p-adics! We'll marvel at this awesome proof I learned about earlier this year.

1. Polyhedra

1.1. **Classifying platonic solids.** The most famous polyhedra are the five platonic solids; the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Draw the platonic solids. Why are there exactly five? What even is a platonic solid?

Definition 1.1. A platonic solid is a convex regular polyhedron; i.e. all vertices have the same degree, and all faces have the same number of edges.

Proposition 1.2. There are at least five platonic solids.

To prove that there are at most five platonic solids, we use the Euler characteristic:

Definition 1.3. The *Euler characteristic* $\chi(\Gamma)$ of a graph Γ embedded on a surface Σ is $V - E + F$, where V is the number of vertices, E the number of edges, and F the number of faces. The *Euler characteristic* $\chi(\Sigma)$ of a closed surface is $2 - 2g$, where g is the genus (number of holes) in the surface. In fact,

$$V - E + F = \chi(\Gamma) = \chi(\Sigma) = 2 - 2g.$$

Theorem 1.4. *There are exactly 5 platonic solids.*

Proof. We know there are at least 5 of them, so we must show there are at most five. By convexity, a platonic solid must be homeomorphic to a sphere, so $V - E + F = 2$. By regularity, every vertex has the same degree $d \geq 3$. Since every edge meets exactly two vertices, $dV = 2E$. Similarly, by regularity, every face has the same number of edges $n \geq 3$. Every edge meets exactly two faces, so $nF = 2E$. Now

$$V - E + F = \frac{2E}{d} - \frac{E}{1} + \frac{2E}{n} = 2.$$

Dividing through by $2E$, we have

$$\frac{1}{d}+\frac{1}{n}=\frac{1}{2}+\frac{1}{E}>\frac{1}{2}.$$

Since $d, n \geq 3$, we can enumerate the solutions to this equation:

$$(3,3),(3,4),(4,3),(3,5),(5,3).$$

There are only five as desired! For example, the solution $(3,4)$ corresponds to degree 3 vertices and faces with 4 edges - these are cubes! □

1.2. **A Combinatorial Gauss-Bonnet theorem.** We might now hope to classify some more families of polyhedra. The main result of the first half of my talk is a *combinatorial Gauss-Bonnet theorem*, which we'll use to classify convex deltahedra later. First, the traditional Gauss-Bonnet theorem:

Theorem 1.5 (Gauss-Bonnet theorem). *Let Σ be a Riemannian surface (that is, a surface with a metric). Then*

$$\int_{\Sigma}\kappa(x)=2\pi\chi(\Sigma)$$

where $\kappa(x)$ is the curvature of Σ at x.

This is probably the coolest theorem in geometry, and undoubtedly the equation I'd get tattooed on me if I had to choose one equation to tattoo. It turns out that a combinatorial version of this result holds:

Theorem 1.6 (Combinatorial Gauss-Bonnet theorem). *Let Γ be a triangulation of a surface Σ. (That is, a graph embedded in Σ whose faces are all triangles.) Then*

$$\sum_{v\in\Gamma}\kappa(v)=2\cdot 3\chi(\Sigma).$$

The curvature of a vertex v is defined to be

$$\kappa(v)=6-deg(v).$$

Remark. The definition of curvature is "correct". If we have equilateral triangles, a flat tiling has vertices of degree 6, while positively curved tilings have degree at most 5 and so on.

Remark. This seems to be saying that $\pi = 3$ when you change perspectives from smooth surfaces to triangulations. This is actually unironically true: π is defined to be the ratio of perimeter to diameter, given a perimeter-minimising smooth shape. On the other hand, 3 is the ratio of perimeter to diameter for an equilateral triangle, and equilateral triangles are perimeter minimising triangles.

We'll now prove the combinatorial Gauss-Bonnet theorem.

Proof. The left side of the equation is

$$\sum_{v\in\Gamma}\kappa(v)=\sum_{v\in\Gamma}6-\deg(v)=6V-\sum_{v\in V}\deg(v).$$

The handshaking-lemma says that $\sum\deg(v)=2E$ (because every edge meets two vertices). This gives

$$6V-\sum\deg(v)=6V-2E.$$

Next, notice that in a triangulation, every face meets three edges and every edge meets two faces. This means $3F=2E$, so

$$6V-2E=6V-6E+6F=6\chi(\Gamma).$$

□

1.3. Classifying strictly convex deltahedra.

Theorem 1.7. *There are exactly 8 strictly convex deltahedra: the tetrahedron, octahedron, icosahedron, triangular bipyramid, pentagonal bipyramid, gyroelongated square bipyramid, triaugmented triangular prism, and snub disphenoid.*

Proof. We won't actually prove this, but we'll find an upper bound on the number of strictly convex deltahedra. Then the problem reduces to checking finitely many possibilities rather than a potentially infinite list.

Any polyhedron must have $\deg(v)\geq 3$ for all vertices v, so $\kappa(v)\leq 3$ for all v. On the other hand, strict convexity forces $\kappa(v)\geq 1$ for all v. We can then enumerate solutions to the equation

$$\sum\kappa(v)=2\cdot 3\cdot 2=12;\quad 1\leq\kappa(v)\leq 3.$$

There are exactly 19 solutions. □

Describe the 8 strictly convex deltahedra.

2. Polygons

2.1. Odd equidissections of squares don't exist.

Now we'll change our topic completely, and turn to polygons instead of polyhedra. One question that people have thought about is *equidissections.*

Definition 2.1. An *n-equidissection* is a dissection of a polygon into n triangles all with equal area.

Example. For every integer n, any triangle T admits an n-equidissection.

Example. For every even integer $2n$, a square admits a $2n$-equidissection.

Proposition 2.2 (Monsky)**.** If n is an odd numbder, there are no n-equidissections of squares.

The rest of this talk will be a proof of this cool fact. We need to first introduce some tools from graph theory and algebra.

The proof outline is as follows:

(1) Colour every point in our square with one of three colours. (Algebra.)
(2) Show that in any dissection of our square into triangles, at least one triangle has vertices consisting of all three colours. (Graph theory.)
(3) Compue the area of the colourful triangle and show that it cannot be $1/n$ for n odd.

2.2. **Proof ingredients: algebra.** First we'll introduce the algebraic tool: the 2-adic norm. We'll use this to colour the points in a square. Let $a \in \mathbb{Q}$. Then there is a unique way to write it as

$$a = 2^n \frac{p}{q},$$

where 2 doesn't divide p or q, and p/q is reduced. We now define the 2-adic norm of a by

$$|a|_2 = 2^{-n}.$$

Proposition 2.3. The 2-adic norm extends from $\mathbb{Q}$ to $\mathbb{R}$.

Remark. The extension looks nothing like the usual topology on $\mathbb{R}$. The point is that *there is a norm on $\mathbb{R}$ which restricts to the 2-adic norm on $\mathbb{Q}$.*

We'll now apply this to colour every point on the unit square $[0,1]^2$.

(1) Every point on a square is an ordered pair (a, b). If $|a|_2, |b|_2 < 1$, then (a, b) is red.
(2) If $|a| \geq \max\{1, |b|_2\}$, then (a, b) is blue.
(3) Otherwise (a, b) is green.

To understand this a bit better, let's colour some points on the square.

Example. The vertex $(0, 0)$ has norms $|a|_2 = 0, |b|_2 = 0$. This is because

$$0 = 2^N \frac{0}{1}$$

for all N, and the limit of 2^{-N} is 0. Since these are both less than 1, $(0, 0)$ is coloured red.

Example. The vertex $(1, 0)$ has norms $|a|_2 = 1, |b|_2 = 0$. This is because

$$1 = 2^0 \frac{1}{1},$$

and 2^0 is 1. Since $|a|_2 \geq 1$ and $|a|_2 \geq |b|_2$, this vertex is blue.

Example. The vertex $(1, 1)$ is blue, and $(0, 1)$ is green.

FInally, some properties of the norm:

Proposition 2.4 (Ultrametric inequality)**.** For any a, b, we have $|a+b|_2 \leq \max\{|a|_2, |b|_2\}$. If $|a|_2 \neq |b|_2$, then $|a + b|_2 = \max\{|a|_2, |b|_2\}$.

2.3. **Proof ingredients: graph theory.** Next, we introduce a proof ingredient from graph theory.

Lemma 2.5 (Sperner)**.** Colour the vertices of a polygon red, blue, and green. Dissect the polygon into triangles and colour the new vertices. If the number of red-blue edges in the polygon was *odd*, then at least one triangle in the dissection has vertices of all three colours.

I won't prove this, but it's an elementary result that uses things like the handshaking lemma that we saw in the first part of the talk. Why do we care about this? If we inspect our square from earlier, we'll notice that it has one red-blue edge! But now any equidissection is forced to contain a triangle with vertices of all three colours.

2.4. **Completing the proof.** What have we established so far? We've started with a square, and coloured every point on the square using the 2-adic norm. Next, we use Sperner's lemma to see that at least one triangle in any dissection of the square has vertices consisting of all three colours.

To complete the proof, we'll compute the area of this colourful triangle and show that it *cannot* be of the form $1/n$ for n odd. This will prove that there are no n-equidissections of squares for n odd.

Lemma 2.6. The colouring on $\mathbb{R}^2$ induced by the 2-adics is invariant under translations by red points.

We'll use this to translate out colourful triangle so that the red vertex is at $(0, 0)$.

Proof. Case 1: suppose (a, b) is a red point in $\mathbb{R}^2$. Let (s, t) be another red point. Then

$$|a + s|_2 \leq \max\{|a|_2, |s|_2\} < 1, \quad |b + t|_2 \leq \max\{|b|_2, |t|_2\} < 1.$$

Therefore $(a + s, b + t)$ is red.

Cases 2 and 3 are similar, also following from the ultrametric inequality. □

Proposition 2.7. Any colourful triangle in our dissection of the square has area A with $|A|_2 \geq 2$.

To compute the area of the colourful triangle, we'll translate it so the red vertex is at $(0, 0)$. Now it's a triangle with vertices

$$(0, 0), (a, b), (c, d)$$

where (a, b) is blue and (c, d) is green. By linear algebra, we know that the area of the parallelogram with sides given by the vectors (a, b) and (c, d) is the determinant. Therefore the triangle has area

$$A = \frac{|ad - bc|}{2}.$$

Computing A is obviously difficult since all we have is bounds on the 2-adic norms of (a, b) and (c, d), but we can actually bound the 2-adic norm of A:

$$\begin{aligned} |A|_2 &= \left| \frac{1}{2} |ad - bc| \right|_2 \\ &= 2|ad - bc|_2 \\ \text{(a)} \qquad &= 2\max\{|ad|_2, |bc|_2\} \\ \text{(c)} \qquad &\geq 2|a|_2|d|_2 \geq 2. \end{aligned}$$

In line (a), we use that $|ad|_2 \neq |bc|_2$. Essentially this is because (c, d) is green, which forces $|c|_2 < |d|_2$, and (a, b) is blue, so $|b|_2 \leq |a|_2$.

In line (b), we use that $|a|_2 \geq 1$ and $|d|_2 \geq 1$, which again follows from the colouring.

Proposition 2.8. If a triangle has area A with $|A|_2 \geq 2$, then A is not of the form $1/n$ for n odd.

Proof. We compute the 2-adic norm of $1/n$. Notice that $1/n = 2^0 \cdot 1/n$ in 2-adic form, so $|1/n|_2 = 1$. This is less than 2! $\square$

In summary, we've shown that any dissection of a square contains a colourful triangle, and that this triangle cannot have area $1/n$ for n odd. It follows that there are no odd equidissections of squares, as required.