AN APPLICATION OF MODULI SPACES TO LEGENDRIAN LINKS

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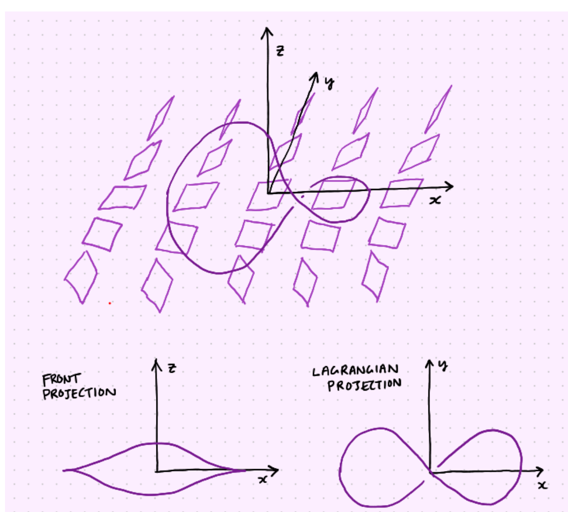
ABSTRACT. This quarter the Stanford student symplectic seminar was all about moduli spaces of J-holomorphic curves. To finish the quarter, here's a talk applying what we've learned about so far. I'll introduce Legendrian contact homology and discuss how it answered an important question in knot theory, as well as some higher dimensional applications. The description of Legendrian contact homology comes from Ekholm, Etnyre, and Sullivan (2005).

1. LEGENDRIAN LINKS AND THE CLASSICAL INVARIANTS

Knots in the usual sense are defined as smooth embeddings $K : \mathbb{S}^1 \to \mathbb{R}^3$, and studied up to smooth isotopy. Knots are interesting in their own right, but also help us understand three dimensional topology. *Legendrian* knots and links are their counterparts in the contact category.

Definition 1.1. Let M be a 3-manifold. A *contact structure* on M is a completely non-integrable plane field ξ in M. This means that there are no surfaces in M which are everywhere tangent to ξ.



Definition 1.2. A link L in (M, ξ) is *Legendrian* if it is tangent to ξ at all points in L.

Example. The standard contact structure on $\mathbb{R}^3$ is given by

$$\xi = \text{span}\left\{\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right\}.$$

An example of a Legendrian (un)knot has been drawn in the contact structure. To better understand how it sits in $\mathbb{R}^3$ (since looking at a picture makes this difficult), see the two projections underneath.

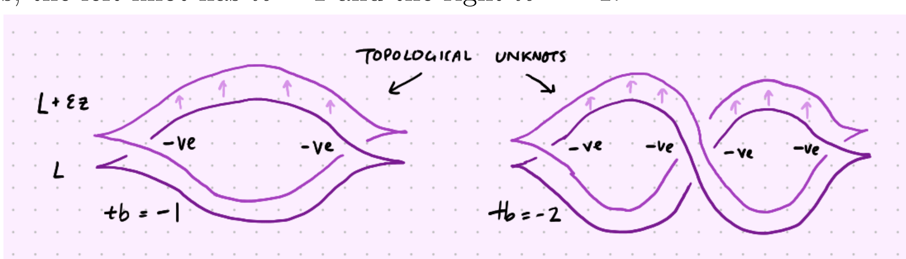
These two projections are very important and have their own names! The left one is called the "front projection", and the right the "Lagrangian projection".

In topology, we consider knots and links up to smooth ambient isotopy. In contact topology, we consider Legendrian knots and links up to isotopy through a family of Legendrian knots and links. Given two Legendrian knots which are topologically isotopic, we shouldn't expect them to be *Legendrian isotopic*. Indeed, there are Legendrian isotopy invariants of Legendrian knots (and links) unrelated to topological type:

Definition 1.3. The *Thurston-Bennequin number* $tb(L)$ of L is the linking number of L with a push-off of L in the z-direction. (A more contact-geometric definition is to compare the framings given by a Seifert surface with that induced by the intersection of the contact distribution with the normal bundle to L.) This is most easily computed using the front projection: simply draw a second copy of the link translated vertically, and make sure that in every crossing the top-left to bottom-right arc crosses over the bottom-left to top-right arc.

Definition 1.4. The *rotation number* $r(L)$ of L is the winding number of L in the Lagrangian projection. (A more contact-geometric definition is to use a Seifert surface to induce a trivialisation on $\xi|_L$, from which the winding number can be computed.)

The following are examples of Legendrian unknots with distinct Thurston-Bennequin invariants; the left knot has $tb = 1$ and the right $tb = -1$.



The *classical invariants* of Legendrian knots are exactly the topological knot type, Thurston-Bennequin number, and rotation number.

Remark. Why are there no more classical invariants? The Thurston-Bennequin number captures the normal direction to L in the contact distribution, and the rotation number the tangent direction.

. Now that we have a collection of invariants, there are two natural questions we can ask:

- Are the invariants *necessary*? That is, is every combination of values for knot type, tb and r satisfied by some Legendrian knot?
- Are the invariants *sufficient*? That is, are two Legendrian knots (or links) Legendrian isotopic if they have the same classical invariants?

(1) Bennequin showed that not all combinations of classical invariants are realised by Legendrian knots: this follows from the *Bennequin inequality*

$$tb(L) + |r(L)| \leq -\chi(L).$$

(2) Eliashberg and Fraser showed that the invariants *are* sufficient in a special case: two Legendrian unknots are Legendrian isotopic if and only if they have the same classical invariants.

(3) This is not true in general! (Chekanov.) We can use *Legendrian contact homology* to find Legendrian knots which share classical invariants but are non-Legendrian isotopic. The rest of this talk will be a proof of this result.

2. Summary of moduli space results

Over the course of the quarter we established various properties of the *moduli space of J-holomorphic curves*. The fundamental object of interest was the following:

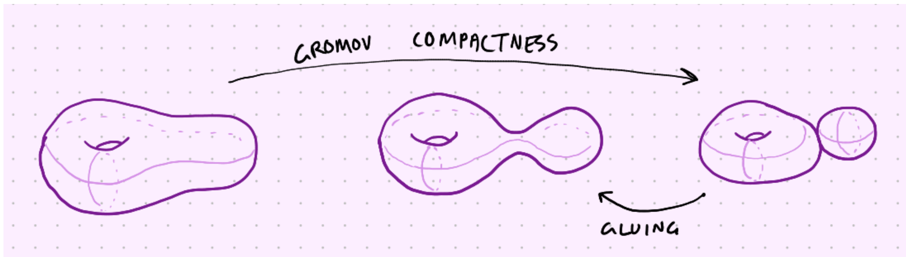
Definition 2.1. Fix a compact symplectic manifold (M, ω), a compatible almost-complex structure J, and a homology class $A \in H_2(M; \mathbb{Z})$. The *moduli space of J-holomorphic curves in M with genus g and m marked points representing A* is

$$\mathcal{M}_{g,m}(A; J) := \{(\Sigma_g, j, u, (z_1, \ldots, z_m))\}/\sim$$

where we're modding out by biholomorphisms preserving the additional structure.

What results do we have?

- $\mathcal{M}_{g,m}(A; J)$ is a finite dimensional manifold.
  1. Express the moduli space as the zero-set of some section of a vector bundle over an infinite dimensional Fréchet manifold.
  2. Upgrade this to a Banach manifold using **Banach theory**.
  3. Use **Fredholm theory** to show that the zero set is finite dimensional.
  4. Use **transversality** together with the above to conclude that the zero set is a finite dimensional *manifold*.
- **Gromov compactness**: The "closure" $\overline{\mathcal{M}}_{g,m}(A; J)$ is compact. $\overline{\mathcal{M}}_{g,m}(A, J)$ is defined by including J-holomorphic bubble trees. Compactness is the statement that sequences of J-holomorphic curves with bounded energy converge to something in $\overline{\mathcal{M}}_{g,m}(A, J)$.
- **Gluing**: given two J-holomorphic curves we can glue them together to form a bubble tree. The gluing theorem says that this is approximated by a J-holomorphic curve. This is a converse to Gromov compactness.



3. Defining Legendrian contact homology

Originally this was carried out in dimension 3, in which the whole theory reduces to combinatorics, and we never have to mention moduli spaces. This is actually sufficient for the purposes of showing that Legendrian knots aren't determined by their classical invariants. However in this section I'll present the more general case.

Definition 3.1. Parametrise $\mathbb{R}^{2n+1}$ by $(x_1, y_1, \ldots, x_n, y_n, z)$. The standard contact structure is the kernel of the 1-form

$$\alpha = dz - \sum_{j=1}^{n} x_j \, dy_j.$$

A *Legendrian submanifold* of $\mathbb{R}^{2n+1}$ is an n-manifold which is everywhere tangent to the contact distribution.

Remark. A Legendrian submanifold of $\mathbb{R}^3$ is exactly a Legendrian knot.

Legendrian submanifolds in general also have analogues of the classical invariants. Legendrian contact homology establishes that these classical invariants do not determine Legendrian submanifolds up to Legendrian isotopy.

Definition 3.2. Let $L \in \mathbb{R}^{2n+1}$ be Legendrian. We will define a corresponding *differentially graded algebra* $(\mathcal{A}, \partial)$. The Legendrian contact homology of L is the homology of $(\mathcal{A}, \partial)$.

First, we must define the algebra itself, $\mathcal{A}$. This is generated by *Reeb chords* of L.

Definition 3.3. Let $(M, \ker \alpha)$ be a contact manifold, and $L \subset M$ Legendrian. The *Reeb vector field* of M is the unique vector field determined by

$$\alpha(X) = 1, \quad d\alpha(X, \cdot) = 0.$$

A *Reeb chord* of L is a segment of a flow line of X starting and ending on L. In $\mathbb{R}^{2n+1}$, the Reeb vector field is exactly $\frac{\partial}{\partial z}$. The Reeb chords are exactly the vertical arcs in $\mathbb{R}^{2n+1}$ which start and end on L.

As in the 1-dimensional case, we have a *Lagrangian projection* $\Pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ which forgets the z component. Assuming L is generic (so that $\Pi(L)$ has at worst double-points of intersection), the Reeb chords of L are in bijective correspondence with double points of its Lagrangian projection. We'll write $\mathcal{C} = \{c_1, \ldots, c_n\}$ to denote the set of Reeb chords.

Now we're ready to define $\mathcal{A}$. Let $\{t_1, \ldots, t_k\}$ be a basis of $H_1(L)$, with its elements written multiplicatively. The *group ring* $\mathbb{Z}/2\mathbb{Z}[H_1(L)]$ is a free $\mathbb{Z}/2\mathbb{Z}$-module, generated with basis the elements of $H_1(L)$, but also with a ring structure induced from group multiplication in $H_1(L)$. Now $\mathcal{A}$ is the free associative unital algebra generated by $\mathcal{C}$ over this group ring. An arbitrary element of $\mathcal{A}$ looks like

$$\sum_i t_1^{n_{1,i}} \cdots t_k^{n_{k,i}} \mathbf{c}_i,$$

where $\mathbf{c}_i = c_{i_1} \cdots c_{i_r}$ is a word in the c_i.

Next, we define the differential. As an equation it looks pretty familiar, it's like all the other differentials we see in Floer-type homology theories:

$$\partial a = \sum_{\substack{\mathbf{b} \in \langle \mathcal{C} \rangle,\ A \in H_1(L) \\ \dim \mathcal{M}(a; A, \mathbf{b}) = 0}} \left(\# \mathcal{M}(a; A, \mathbf{b}) \right) Ab$$

for generators $a \in \mathcal{C}$. I've written $\langle \mathcal{C} \rangle$ to mean the words in $\mathcal{C}$.

Is this well defined? the target space is indeed $\mathcal{A}$, provided that the count $\#\mathcal{M}(a; A, \mathbf{b})$ is well defined and finite, with at most finitely many terms in the sum non-zero. (This is not immediately clear, because we could be summing over infinitely many terms.) At this point - we haven't even defined the relevant moduli space!

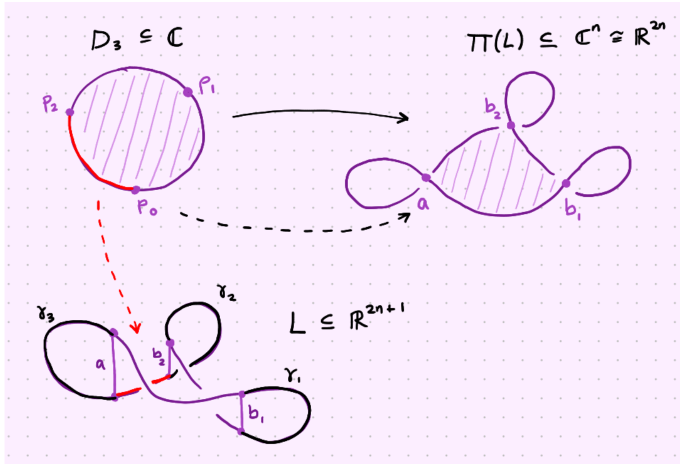
Definition 3.4. Write D_{m+1} to denote the unit disk in $\mathbb{C}$ with $m+1$-marked points $\{p_0, \ldots, p_{n-1}\}$ arranged anticlockwise on the boundary. Then $\mathcal{M}(a; A, \mathbf{b})$ consists of *holomorphic disks with boundary on* L. Formally, this is

$$\mathcal{M}(a; A, \mathbf{b}) = \{u : (D_{m+1}, \partial D_{m+1}) \to (\mathbb{R}^{2n}, \Pi(L))\}/\sim$$

where a is a Reeb chord, $A \in H_1(L)$, and $\mathbf{b} = b_1 \ldots b_m$ is a word of Reeb chords, and the maps u have the following properties:

- They are continuous and holomorphic on the interior.
- $u(p_0)$ is *positive*, while the other $u(p_i)$ are *negative*. Roughly speaking this means that u increase anticlockwise near $u(p_0)$, and conversely at the other p_i. (See the figure below).
- $u(p_0) = a$ and $u(p_i) = b_i$.
- Restricting u to $\partial D_{m+1} - \{p_i\}$, there is a continuous lift $\widetilde{u}$ of the map into L. (Again, see image below.)
- The image of $\widetilde{u}$ glued to *capping paths* of each Reeb chord gives a loop in L representing A. (See image below.)

Here the capping paths are chosen in advance and are used to define the grading on $\mathcal{A}$.



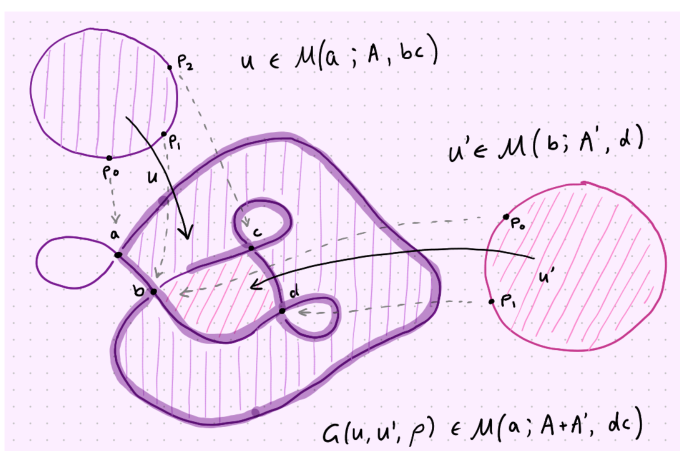
By Fredholm theory etc, the moduli spaces are finite dimensional manifolds! It follows that whenever the moduli space is finite dimensional, it's a discrete space. It turns out that Gromov compactness holds for these moduli spaces, so the moduli spaces are moreover finite. The differential is now well defined.

Theorem 3.5. *The moduli spaces $\mathcal{M}(a; A, \mathbf{b})$ satisfy the gluing theorem. More formally, suppose $\mathcal{M}(a'; A', \mathbf{b}')$ and $\mathcal{M}(a; A, \mathbf{b})$ are 0-dimensional, with the jth Reeb chord in $\mathbf{b}$ in a'. Then there is an embedding*

$$G : \mathcal{M}(a; A, \mathbf{b}) \times \mathcal{M}(a'; A', \mathbf{b}') \times (\rho, \infty) \to \mathcal{M}(a; A + A', \mathbf{b}_{\{j\}}(\mathbf{b}')).$$

If $u \in \mathcal{M}(a; A, \mathbf{b})$ and $u' \in \mathcal{M}(a'; A', \mathbf{b}')$, then $G(u, u', \rho)$ converges to the broken disk (u, u') as $\rho \to \infty$. Here the notation $\mathbf{b}_{\{j\}}(\mathbf{b}')$ means the Reeb chord b_j is removed from $\mathbf{b}$ and replaced with the word $\mathbf{b}'$.

The following picture shows what the gluing theorem looks like in the context of Legendrian knots:



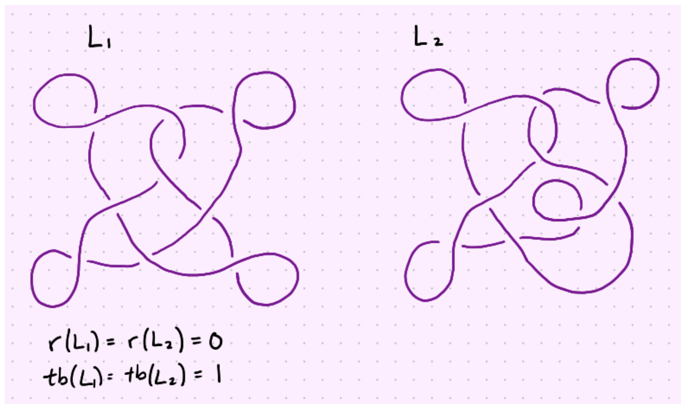
A corollary of the gluing theorem is that the differential defined above squares to zero.

Remark. I've been ignoring gradings entirely, but if they're defined, one can show that the differential has degree -1.

Theorem 3.6. *Legendrian contact homology is an invariant of Legendrian isotopy.*

4. Proving the classical invariants aren't enough

Chekanov showed that the following Legendrian knots have distinct Legendrian contact homologies:



However, from the Lagrangian projection we can compute the classical invariants: $r(L_i) = 0$ and $tb(L_i) = 1$. Moreover, both knots are clearly smoothly isotopic since they only differ by two type 1 Reidemeister moves in the bottom right.

Since these two knots have different Legendrian contact homologies, it follows that the classical invariants are not "sufficient"; they do not determine Legendrian knots up to Legendrian isotopy.

This can be upgraded to higher dimensions.

Theorem 4.1. *For any $n > 1$, there are infinitely many non-Legendrian isotopic Legendrian embeddings of the n-sphere in $\mathbb{R}^{2n+1}$, despite having the same classical invariants.*

Theorem 4.2. *For $n > 1$, there are infinitely many non-Legendrian isotopic Legendrian embeddings of n-tori in $\mathbb{R}^{2n+1}$.*

Moreover, for any $N > 0$, there exist Legendrian isotopy classes of spheres and tori with fixed classical invariants with no representative whose Lagrangian projection has at most N double points. All of these last few results are due to Ekholm, Etnyre, and Sullivan.