The first Chern number

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1 Motivation

Our statement of the Riemann-Roch theorem will look something like

$$\mathrm{ind}(D) = \chi(\Sigma) + 2c_1(E).$$

At the moment this is just a bunch of symbols. Before we prove the statement we need to unravel what everything means. The left side of the statement is purely analytic. The D is some differential operator called a Cauchy-Riemann operator. I still don't know anything about it, but we'll prove that it's a Fredholm operator (meaning its kernel and cokernel are finite dimensional) and then define the index to be the difference of these dimensions. On the other hand, the right side of the statement is purely topological. It involves the Euler characteristic of our manifold and something called the first Chern number of a complex vector bundle E over our manifold. In this talk I'll build up definitions to the point of defining the first Chern number.

2 Complex vector bundles

Definition 2.1. Let $E \to \Sigma$ be a real vector bundle. This induces a vector bundle $\mathrm{End}(E) \to \Sigma$, where the fibres above x are the endomorphisms of E_x. A *complex structure* is a smooth section J of $\mathrm{End}(E)$ such that $J^2 = -1$. A *complex vector bundle* is a real vector bundle equipped with a complex structure.

Locally all complex vector bundles look the same: if $E \to \Sigma$ is a complex vector bundle, locally one can find smooth sections $X_1, \dots, X_n, Y_1, \dots, Y_n$ of E which restrict to a basis on each fibre, and satisfy $JX_i = Y_i$. In this basis, for each x, we have

$$J_x = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

By this expression for J it's immediately clear that complex vector bundles have even rank, and are orientable. The standard orientation of a complex vector bundle is the one determined by the ordered basis $X_1, \dots, Y_n$ such that J has the form above.

Definition 2.2. An *almost complex manifold* is a manifold Σ equipped with a complex structure on $T\Sigma$. A *Riemann surface* is an almost complex manifold with real dimension 2. A *complex manifold* is a manifold whose charts map into $\mathbb{C}^n$, with holomorphic transition maps.

Every complex manifold can be realised as an almost complex manifold, but the converse isn't true in general. Examples of such manifolds can be found in real dimension 4, and it's an open problem whether or not such examples exist in real dimension at least 6. Some time in the next few weeks, apparently someone will prove that all Riemann surfaces can be realised as complex manifolds.

Suppose $(E_1, J_1), (E_2, J_2)$ are complex vector bundles over Σ. Let L be any real linear map between them, in the sense that the restriction of L to fibres gives a real linear map $L_x : E_{1x} \to E_{2x}$. Clearly we have

$$L = \frac{1}{2}(L - J_2 L J_1) + \frac{1}{2}(L + J_2 L J_1) = L^{1,0} + L^{0,1}.$$

But now $L^{1,0}$ is *complex linear* in the sense that

$$L^{1,0}(U + V) = L^{1,0}(U) + L^{0,1}(V), \quad L^{1,0}((a + J_1 b)U) = (a + J_2 b)L^{1,0}(U).$$

On the other hand, $L^{0,1}$ is antilinear. This shows that all real linear maps are captured by complex linear and complex antilinear maps. These complex bundles are denoted by

$$\mathrm{Hom}^{1,0}(E_1, E_2), \text{ and } \mathrm{Hom}^{0,1}(E_1, E_2).$$

Definition 2.3. Given a manifold Σ, $\mathbb{C}_\Sigma$ denotes the trivial complex line bundle over Σ.

In a few weeks, someone will introduce *Cauchy Riemann operators* which are certain differential operators $E \mapsto \mathrm{Hom}^{0,1}(T\Sigma, \mathbb{C}_\Sigma) \otimes E$.

One of the ingredients in our statement of the Riemann Roch theorem is the first Chern number. Chern numbers generally arise from Chern classes, but in our setting, it's sufficient to not consider these and instead use the *Euler invariant* which a certain cobordism class.

A cobordism is an equivalence relation on the collection of closed manifolds of a given dimension. Intuitively, two manifolds are *cobordant* if their disjoint union is the boundary of a compact manifold of higher dimension. In our case, the definition of the *Euler invriant* uses cooriented cobordisms and depends on some transversality results.

3 Transversality and Coorientation

Recall that two submanifolds are transverse if their tangent spaces at each point of intersection generate the entire tangent space of the ambient manifold. This definition can be adapted to create a notion of transversality for sections of vector bundles. First we must define the analogue of the "tangent spaces" of the section.

Definition 3.1. Let s be a smooth section of a vector bundle $E \to \Sigma$. Choose any local frame $X_1, \ldots, X_n$ (i.e. sections so that for each x in a neighbourhood, $X_1(x), \ldots, X_n(x)$ forms a basis of E_x). Then s can be written as $s = a_1 X_1 + \cdots + a_n X_n$ for some smooth functions a_i. At each point x in $\mathcal{Z}(s)$, the *Linearisation* of s at x is

$$L_x(s) : T_x\Sigma \to E_x, \text{ defined by } L_x(s)(v) = \sum \mathrm{d}a_i(v) X_i.$$

This is exactly what the name suggests it is - it's the derivative of s at x. Of course it's not immediately clear that the map is well defined - it might depend on the local frame. For example, it turns out that if we try to extend this definition of the linearisation to points that don't lie in $\mathcal{Z}(s)$, then the choice of frame matters. This is expected because of the Leibniz rule - a true linear approximation to s should look like

$$\nabla s(v) = \sum \mathrm{d}a_i(v) X_i + \sum a_i \nabla X_i(v),$$

so without a connection, whenever we change our frame, we obtain different contributions from the second term. However, if $x \in \mathcal{Z}(s)$, the second term always vanishes independent of the choice of frame. It follows that our definition of the linearisation is well defined.

Definition 3.2. A section s of a vector bundle is *transverse* if $L_x(s)$ is surjective for all $x \in \mathcal{Z}(s)$.

In particular, a section of a vector bundle with rank greater than the dimension of the manifold cannot be transverse unless it has no zeroes. One can gain some visual intuition by considering a smooth section of a real vector bundles over a circle.

Now we have a couple of theorems that I won't prove which agree with geometric intuition:

Theorem 3.3 (Transversality is generic)**.** *Let s be a section of a vector bundle $E \to \Sigma$. If s is transverse outside an open set U, then s can be perturbed in U to make it transverse everywhere.*

Theorem 3.4 (Implicit function theorem)**.** *Let s be a transverse section of a vector bundle $E \to \Sigma$. Then $\mathcal{Z}(s)$ is a submanifold of Σ, and $\ker L_x s = T_x\mathcal{Z}(s)$. Moreover, if Σ has boundary $\partial\Sigma$, and $s|_{\partial\Sigma}$ is transverse, then $\mathcal{Z}(s)$ is a smooth submanifold with boundary $\partial\mathcal{Z}(s) = \mathcal{Z}(s) \cap \partial\Sigma$.*

This is in fact a general form of the implicit function theorem for a smooth function f: transversality corresponds to the premise that $\mathrm{d}f$ has full rank, and the existence of an implicit function gives us a chart for the submanifold.

Definition 3.5. Recall that for a vector space, an *orientation* is a choice of ordered basis, and a vector bundle is oriented if every fibre has a an orientation, and all local frames are orientation preserving. Given a vector bundle $E \to \Sigma$ and a subbundle $P \subset E$, a *coorientation* of P is an orientation of the quotient bundle $E/P \to \Sigma$.

As an example, suppose a vector bundle $E \to \Sigma$ is oriented. Suppose s is a transverse section of E. Let $x \in \mathcal{Z}(s)$ Then by the implicit function theorem and the first isomorphism theorem,

$$T_x\Sigma / T_x\mathcal{Z}(s) = E_x.$$

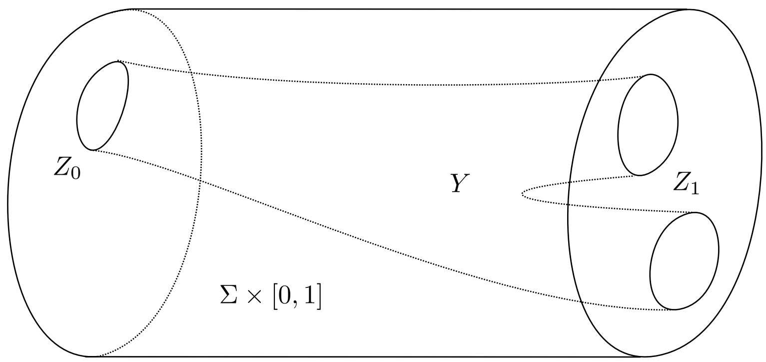
Therefore the tangent space $T_x\mathcal{Z}(s)$ has a coorientation, which extends to a coorientation of the submanifold $\mathcal{Z}(s)$. Recall from earlier that every complex vector bundle is canonically oriented, so the zero set of any transverse section of a complex vector bundle has a canonical coorientation. This is idea is essential for the definition of the Euler invariant.

4 Cobordism

It was mentioned earlier that the *Euler invariant* is a certain class of cobordisms. We're now ready to introduce the notion of cobordisms.

Definition 4.1. Let Z_0, Z_1 be compact submanifolds of Σ without boundary. A *cobordism* between Z_0 and Z_1 is a submanifold Y of $\Sigma \times [0,1]$ such that $\partial Y = Y \cap (\Sigma \times \{0,1\})$ and $Y \cap (\Sigma \times \{0\}) = Z_0 \times \{0\}$, $Y \cap (\Sigma \times \{1\}) = Z_1 \times \{1\}$, with Y transverse to $\Sigma \times \{0\}, \Sigma \times \{1\}$.

The canonical example is a pair of pants, which shows that a circle is cobordant to the disjoint union of two circles.



Usually the definition of a cobordism is stated without reference to an ambient space. In our case, the ambient space is essential for the cobordism to inherit a coorientation.

Proposition 4.2. *Cobordism is an equivalence relation.*

Proof. Every compact submanifold Z is cobordant to itself by considering $Y = Z \times [0,1]$. Symmetry is even more obvious, and transitivity follows by "gluing" cobordisms together. □

In the definition of a cobordism, we require Y to be transverse to $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. This ensures that we have isomorphisms

$$T_{(p,0)}(\Sigma \times [0,1])/T_{(p,0)}Y \cong T_p\Sigma/T_pZ_0, \quad T_{(p,1)}(\Sigma \times [0,1])/T_{(p,1)}Y \cong T_p\Sigma/T_pZ_1.$$

This means that if Z_0 is cooriented, the cobordism Y inherits a coorientation at $Y \cap (\Sigma \times \{0\})$. If Z_0 and Z_1 are both cooriented, we say that Y is a *cooriented cobordism* if Y is cooriented, and the the inherited coorientations on the two ends agree with this coorientation. It's easy to see that cooriented cobordism is also an equivalence relation, so we can speak of cooriented cobordism classes.

Finally it's time to prove a theorem!

Theorem 4.3. *Let $E \to \Sigma$ be a vector bundle over a compact base Σ. If s_0 and s_1 are two transverse sections, then $s_0^{-1}(0)$ and $s_1^{-1}(0)$ are cobordant.*

Proof. At first this looks tricky because if you're given two arbitrary transverse sections, how can you have any chance of constructing a cobordism between their zero sets? Well somehow it ends up being not very hard at all.

Define $F : \Sigma \times [0,1] \to E \times [0,1]$ by

$$F : (x,t) \mapsto ((1-t)s_0(x) + ts_1(x), t).$$

This map is a section of the vector bundle $E \times [0,1] \mapsto \Sigma \times [0,1]$, and it's well defined because vector spaces are convex. What does it look like on the endpoints? $F_0(x) = (s_0(x), 0)$, moreover, the linearisation of F_0 at each $(x,0) \in \mathcal{Z}(s_0) \times \{0\}$ is $L_x(F_0) = L_x(s_0) + \mathrm{d}t$. In particular, it's surjective, so F_0 is transverse. Similarly F_1 is transverse. It follows that F is transverse on $\Sigma \times [0,1]$

outside of the open set $\Sigma \times (0,1)$. But by the transversality theorem stated earlier, we can now perturb F to ensure that it's transverse everywhere on $\Sigma \times [0,1]$.

Now consider $Y = \mathcal{Z}(F)$. By the implicit function theorem, this is a submanifold of $\Sigma \times [0,1]$. Moreover, $Y \cap (\Sigma \times \{0\}) = \mathcal{Z}(s_0) \times \{0\}$ and $Y \cap (\Sigma \times \{1\}) = \mathcal{Z}(s_1) \times \{1\}$. Transversality of F ensures that Y is transverse to $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. Therefore Y is our desired cobordism. $\square$

This theorem applies to the cooriented cobordism case as well. If E is oriented, then $\mathcal{Z}(s_0)$ and $\mathcal{Z}(s_1)$ inherit coorientations, and are guaranteed to be coorientation cobordant. Finally we can define the Euler invariant.

Definition 4.4. Given any oriented vector bundle $E \to \Sigma$, the *Euler invariant* of E is

$$e(E) \coloneqq \text{ cooriented cobordism class of } \mathcal{Z}(s) \text{ for any transverse section } s \text{ of } E.$$

Usually this is a strange object, it's an equivalence class of manifolds with a bunch of additional information. However, if the rank of the vector bundle is the same as the dimension of the manifold, then the zero set of any transverse section will be a zero-dimensional manifold. Recalling that zero dimensional manifolds are characterised by the number of connected components there's a possibility that we can simply count things and obtain a "number" as an invariant of vector bundles.

Theorem 4.5. *The signed count of points of a compact cooriented zero-dimensional submanifold of an oriented ambient manifold is a cooriented cobordism invariant.*

Proof. First we must explain what is meant by *signed count*. Let Σ be an oriented manifold, and $Z \subset \Sigma$ a compact zero dimensional submanifold. Then Z is just a finite collection of points, $\{z_1, \ldots, z_n\}$. At each z_i, $T_{z_i}\Sigma \cong T_{z_i}\Sigma / T_{z_i}Z$ inherits two orientations: the orientation of Σ, and the coorientation of Z. The point z_i is assigned a signed value ± 1 depending on whether or not the two orientations agree. The sum of all of these values is the signed count of Z.

Now suppose Z_0 is coorientation cobordant to Z_1. Let Y be the cooriented cobordism between Z_0 and Z_1. Both Z_0 and Z_1 is a finite collection of signed points, and by the classification of compact 1 dimensional manifolds, Y is a finite disjoint union of circles and arcs. The circles don't have boundary components so they don't intersect $\Sigma \times \{0\}$ or $\Sigma \times \{1\}$, so we can assume without loss of generality that Σ is a finite disjoint union of arcs. Without loss of generality, each arc either has both end points on Z_0, or one on Z_0 and one on Z_1. First consider the latter case. By restriction this single arc gives a cooriented cobordism between two points, so by definition, the orientation at both ends must agree. This means that each arc between Z_0 and Z_1 leaves the signed count invariant. Now consider an arc (say $\gamma : [0,1] \to \Sigma \times [0,1]$) which has endpoints on $z, w \in Z_0$. By transversality, we have

$$T_p\Sigma / T_p z \cong T_{(p,0)}(\Sigma \times [0,1]) / T_0\gamma \cong T_{(p,0)}(\Sigma \times [0,1]) / T_1\gamma \cong T_p\Sigma / T_p w.$$

But now the left most and right most isomorphisms must preserve orientation by definition, but the middle isomorphism swaps orientation. Therefore the signed count of z and w sums to 0. $\square$

Since signed counts of points is a cooriented cobordism invariant, but cooriented cobordism is itself an invariant of oriented vector bundles, if E is an oriented rank n vector bundle over an n-manifold, we can define the *first Chern number* to be

$$c_1(E) \coloneqq \text{ signed count of points in } e(E).$$

In particular, this defines the first Chern number for complex line bundles over Riemann surfaces. This definition relies on the fact that the vector bundle and base manifold have the same real dimension. However, we can extend the definition of the first Chern number of complex line bundles over Riemann surfaces to arbitrary complex vector bundles as follows:

Definition 4.6. Suppose $E \to \Sigma$ is a complex vector bundle of rank k of over a Riemann surface Σ. We then have a complex line bundle $\det E \to \Sigma$ where the fibre above each x is defined by $\Lambda^k E_x$. The *first Chern number* of $E \to \Sigma$ is

$$c_1(E) \coloneqq c_1(\det E).$$

Definition 4.7. The Euler characteristic of a closed Riemann surface Σ can be defined by

$$\chi(\Sigma) = c_1(T\Sigma).$$

We've now defined all of the terms on the topological side of our formulation of the Riemann-Roch theorem.

5 Examples

Example. The torus admits a vector field with no zeroes. This is automatically transverse, so $\chi(T) = 0$. In fact, any surface admitting a non-vanishing vector field has zero Euler characteristic.

Example. $\chi(\mathbb{S}^2) = 2$. Consider coordinates on the sphere given by longitude θ and latitude ϕ. Then $V = \sin(\phi)\frac{\partial}{\partial\theta}$ extends to a global transverse vector field. The zero set of V is the North and South pole. Being careful with orientations, one can show that the signed count is 2.

Example. Transverse hairy ball theorem. Any transverse vector field on $\mathbb{S}^2$ has at least two zeroes, since the signed count of zeroes is an invariant of $T\mathbb{S}^2$.

We've probably all come across the Poincare-Hopf index theorem: it states that a certain signed count of zeroes of a vector field on a surface is the Euler characteristic of the surface in the usual sense. By the Poincare-Hopf index theorem, our definition of the Euler characteristic using the first chern number agrees with the usual definition.