CASH IN ON THE CASSON INVARIANT

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Abstract. We motivate the Casson invariant through its role in proving the existence of non-triangulable topological 4-manifolds. Next we describe a construction of the Casson invariant using representation theory, with a focus on why various choices have to be made. (Why do we use SU(2)? Why only homology spheres?) Finally we mention some generalisations of the Casson invariant, and return to the triangulation conjecture. The bulk of the results are from Saveliev's two texts *Lectures of the Topology of 3-Manifolds* [Sav12] and *Invariants for Homology 3-Spheres* [Sav02].

Contents

1. Motivation: triangulation conjecture

1.1. **Preliminaries: the Rokhlin invariant.** Imagine you're in the world of 1980s low dimensional topology. Not only does the internet not exist yet, you don't even have access to the Poincaré conjecture. To get around this difficulty, you need ways of comparing and contrasting homology and homotopy 3-spheres. The Casson invariant is an example of an invariant of homology 3-spheres that helped topologists in a pre-Poincaré world, and it still has uses today.

Before introducing the Casson invariant, we must introduce the Rokhlin invariant. This is an invariant of homology 3-spheres valued in $\mathbb{Z}/2\mathbb{Z}$. The Casson invariant is a lift of the Rokhlin invariant to $\mathbb{Z}$. To define the Rokhlin invariant, we recall some preliminary definitions and theorems.

Definition 1.1. Let X be a simply connected topological 4-manifold. Then its only trivial cohomology group is $H^2(X;\mathbb{Z}) \cong \mathbb{Z}^n$. The *intersection form* of X is the map

$$Q : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}, \quad Q(\alpha, \beta) = \langle \alpha \smile \beta, [X] \rangle.$$

This is a unimodular symmetric bilinear form

$$Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}.$$

The *signature* of X, denoted by $\sigma(X)$, is the *signature* of Q. That is, $\sigma(X) = b_+ - b_-$, where Q has b_+ positive eigenvalues and b_- negative eigenvalues. Finally the *type* of X is the type of Q, which is *even* if $Q(\alpha, \alpha) = 0 \mod 2$ for all α, and *odd* otherwise.

The intersection form is a very important invariant of simply connected 4-manifolds, but it can be unwieldy to work with. Instead we will proceed by using the signature and type of manifolds. It turns out that every integral homology 3-sphere bounds a simply connected *even* 4-manifold, so we focus our attention on even intersection forms.

Theorem 1.2. *The signature of an even symmetric unimodular bilinear form is divisible by 8. Conversely, there exists an even symmetric unimodular bilinear form; the E_8-form.*

This is purely algebraic, and a consequence of Serre's classification of symmetric unimodular bilinear forms. Next we add some topology into the mix.

Theorem 1.3 (Rokhlin)**.** *The signature of an even simply connected smooth 4-manifold is divisible by 16.*

By a theorem of Freedman, for any symmetric unimodular bilinear form Q, there exists a simply connected closed 4-manifold whose intersection form is Q. In particular, there is an "E_8-manifold". But by Rokhlin's theorem, this manifold doesn't admit a smooth structure. We will soon use the Casson invariant to show that this manifold doesn't even admit a triangulation.

We are now ready to define the Rokhlin invariant.

Definition 1.4. Let Σ be a homology 3-sphere. Then Σ bounds an even smooth simply connected 4-manifold W. We define the *Rokhlin invariant* of Σ to be

$$\mu(\Sigma) = \frac{1}{8}\sigma(W) \mod 2.$$

This definition really gives an element of $\mathbb{Z}/2\mathbb{Z}$ by the algebraic property that even symmetric unimodular forms have signature divisible by 8. But it's also independent of the choice of W due to Rokhlin's theorem. Explicitly, suppose W and W' are both even smooth simply connected 4-manifolds. Gluing W to W' along Σ, one can show that

$$\sigma(W) - \sigma(W') = \sigma(W \sqcup_\Sigma -W') \equiv 0 \mod 16.$$

Therefore

$$\frac{1}{8}\sigma(W') = \frac{1}{8}\sigma(W) - \frac{1}{8}(\sigma(W) - \sigma(W')) = \frac{1}{8}\sigma(W) \mod 2.$$

The Rokhlin invariant is genuinely an invariant of homology 3-spheres!

Example. The actual 3-sphere bounds the 4-ball. This has trivial homology, and hence trivial intersection form. Its signature is zero, so $\mu(\mathbb{S}^3) = 0$.

Any integral homology 3-sphere Σ that bounds the E_8-manifold W_{E_8} has

$$\mu(\Sigma) = 1 \mod 2,$$

since $\sigma(W_{E_8}) = 8$. An example of such a homology sphere is the Poincaré homology sphere.

1.2. **Disproving the triangulation conjecture in dimension 4.** As mentioned earlier, the Casson invariant is an extension of the Rokhlin invariant from $\mathbb{Z}/2\mathbb{Z}$ to all of $\mathbb{Z}$. We are now ready to describe two essential properties of the Casson invariant, which we use to disprove the triangulation conjecture, before really going into the details of the invariant.

Proposition 1.5. There exists a function

$$\lambda : \{\text{integral homology 3-spheres}\} \to \mathbb{Z}$$

such that

- $\lambda(\Sigma) = 0$ if Σ is simply connected.
- $\lambda(\Sigma) \equiv \mu(\Sigma) \mod 2$.

An example of such a function is the Casson invariant, which will be introduced later in the talk.

Theorem 1.6 (Casson, 1985)**.** *There exist topological 4-manifolds that do not admit triangulations.*

By a *triangulation*, we mean a simplicial complex whose underlying topological space is homeomorphic to our given manifold.

Proof. Consider the 4-manifold $W = W_{E_8}$, with intersection form $Q = E_8$.

Note that Q has signature 8, so in particular $\sigma(Q)$ isn't divisible by 16. Then by Rokhlin's theorem W is not smoothable. In particular, W does not admit a combinatorial structure (a triangulation whose links of vertices are all homeomorphic to $\mathbb{S}^3$).

Now assume for a contradiction that W admits a triangulation K. By taking a refinement if necessary, there is a vertex v in K so that the complement of the open star of v in K is combinatorial. (That is, the only link of a vertex in K which *isn't* homeomorphic to $\mathbb{S}^3$ is the link of v.) Denote the link of v by L, and the star of v by $C(L)$. Then $W - \operatorname{int} C(L)$ is combinatorial, so in particular it is smooth. But $W - \operatorname{int} C(L)$ is an even smooth simply connected 4-manifold with boundary L (a homology sphere), so modulo 2 we have

$$\mu(L) = \frac{1}{8}\sigma(W - C(L)) = 1.$$

On the other hand, one can further show that the link L of v is a homotopy 3-sphere. Therefore the Casson invariant of L must vanish, and hence its Rokhlin invariant vanishes: $\mu(L) = 0$. This is a contradiction. $\square$

Remark. We know from the Poincaré conjecture that a homotopy sphere is homeomorphic to $\mathbb{S}^3$. Since the Rokhlin invariant of $\mathbb{S}^3$ is trivial, the Poincaré conjecture removes the need to use Casson's invariant.

Remark. The power of the Casson invariant is that by construction, it is clear that $\lambda(\Sigma) = 0$ for Σ a simply connected homology 3-sphere. However, the Rokhlin invariant was "too coarse" to prove that $\mu(\Sigma) = 0$ (before the Poincaré conjecture was resolved).

2. The Casson invariant

2.1. **Construction via** $\mathrm{SU}(2)$**-representations.** In this section, we will finally actually construct the Casson invariant. We will begin by briefly outlining the construction and then some properties of the Casson invariant, before explaining the construction in some more detail.

The Casson invariant can be constructed as a certain count of a moduli space of $\mathrm{SU}(2)$-valued representations of the fundamental group of homology spheres. The outline is as follows:

(1) For any manifold M consider the space of representations
$$R(M) = \mathrm{Hom}(\pi_1 M, \mathrm{SU}(2)).$$
This is a topological space (equipped with the compact-open topology). $\mathrm{SO}(3)$ acts on $R(M)$ by conjugation, and $\mathcal{R}(M) \coloneqq R^{\mathrm{irr}}(M)/\mathrm{SO}(3)$ is called the *representation space* of M. (Note that $R(M)/\mathrm{SU}(2)$ is a *character variety.*) Here $R^{\mathrm{irr}}(M) \subset R(M)$ is the subspace of irreducible representations.

(2) For M a handlebody of genus $g \geq 1$, $\mathcal{R}(M)$ is a smooth open manifold of dimension $3g - 3$ (and empty if $g = 1$). For F a closed oriented surface of genus $g \geq 1$, $\mathcal{R}(F)$ is a smooth open manifold of dimension $6g - 6$.

(3) For Σ a homology sphere, we can find a Heegaard decomposition $\Sigma = M_1 \sqcup_F M_2$. Then $\mathcal{R}(\Sigma) = \mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ is a compact manifold of dimension 0 in $\mathcal{R}(F)$. The above inclusions and equalities and so on are obtained from the following chain of diagrams:
$$\begin{tikzcd} & M_1 \arrow[dr, hook] & \\ F = M_1 \cap M_2 \arrow[ur, hook] \arrow[dr, hook] & & \Sigma = M_1 \cup M_2. \\ & M_2 \arrow[ur, hook] & \end{tikzcd}$$
Now apply the π_1 functor followed by the (contravariant) Hom functor to obtain
$$\begin{tikzcd} & \mathrm{Hom}(\pi_1 M_1, \mathrm{SU}(2)) \arrow[dl, hook] & \\ \mathrm{Hom}(\pi_1 F, \mathrm{SU}(2)) & & \mathrm{Hom}(\pi_1 \Sigma, \mathrm{SU}(2)). \arrow[ul, hook] \arrow[dl, hook] \\ & \mathrm{Hom}(\pi_1 M_2, \mathrm{SU}(2)) \arrow[ul, hook] & \end{tikzcd}$$

Removing irreducibles and modding out by the SO(3) action gives

$$\begin{tikzcd}
& \mathcal{R}(M_1) \arrow[dl, hook] & \\
\mathcal{R}(F) & & \mathcal{R}(\Sigma). \arrow[ul, hook] \arrow[dl, hook] \\
& \mathcal{R}(M_2) \arrow[ul, hook] &
\end{tikzcd}$$

Therefore $\mathcal{R}(M_1) \cap \mathcal{R}(M_2)$ can be understood as lying in $\mathcal{R}(F)$. It is also clear that $\mathcal{R}(\Sigma) \subset \mathcal{R}(M_1) \cap \mathcal{R}(M_2)$, and with some work this can be shown to be an equality.

By orienting $\mathcal{R}(M_1)$ and $\mathcal{R}(M_2)$ we obtain an algebraic count of this intersection, from which we define the *Casson invariant*:

$$\lambda(\Sigma, M_1, M_2) = \frac{(-1)^g}{2} \#(\mathcal{R}(M_1) \cap \mathcal{R}(M_2)).$$

(4) One can show that the invariant is independent of the choice of Heegaard splitting.

Theorem 2.1. *The Casson invariant satisfies the following properties:*

(1) If Σ is simply connected, then $\lambda(\Sigma) = 0$.
(2) $\lambda(-\Sigma) = -\lambda(\Sigma)$, where $-\Sigma$ denotes Σ with reversed orientation.
(3) $\lambda(\Sigma \# \Sigma') = \lambda(\Sigma) + \lambda(\Sigma')$.
(4) $\lambda(\Sigma) \equiv \mu(\Sigma) \mod 2$.

Proof. (1) If Σ is simply connected, then $\pi_1 \Sigma$ is trivial. Therefore $R(\Sigma) = \mathrm{Hom}(\pi_1\Sigma, \mathrm{SU}(2))$ consists of a single point - the trivial representation. This is reducible because any proper subspace of $\mathbb{C}^2$ is preserved by the representation. Therefore $R^{\mathrm{irr}}(\Sigma)$ is empty, so the signed count of points in $\mathcal{R}(M) = R^{\mathrm{irr}}(\Sigma)/\,\mathrm{SO}(3)$ is 0. That is, $\lambda(\Sigma) = 0$. In words, all homotopy 3-spheres have trivial Casson invariant.

(2) By reversing the orientation of the homology sphere, each sign in the signed count is swapped.

(3) Heegaard splittings $M_1 \cup M_2$ of Σ and $M_1' \cup M_2'$ of Σ' induces a Heegaard splitting $(M_1 \natural M_1') \cup (M_2 \natural M_2')$ of $\Sigma \# \Sigma'$. One can count intersections and show that they match up correctly to give the desired connected sum formula.

The proof of (4) is left as an exercise! This is generally done by means of establishing some surgery formulae for the Casson invariant in terms of the Alexander polynomial (of a knot along which surgery is carried out). The Alexander polynomial contains the information of the Arf invariant, which in turn determines the Rokhlin invariant. Details are given in *Lectures on the topology of 3-manifolds* by Saveliev. ☐

2.2. **Explaining the choices in the Casson invariant.** We next explain the construction of the Casson invariant by illuminating some of the choices that were used in the definition.

(1) Why must we remove reducible representations?

(2) We consider representations $\pi_1 M \to \mathrm{SU}(2)$. Why are they valued in SU(2)? Why not some other Lie group?
(3) This is supposedly an invariant of homology 3-spheres. Why does it not apply to other n-manifolds? Why does it not apply to other 3-manifolds?

(1) Reducible represents are singular points in the character variety. To use transversality arguments, we want $\mathcal{R}(M)$ to be a smooth manifold given our space M. We take for granted the fact that

$$T_f(R(M)/\operatorname{SO}(3)) = T_f R(M) = H^1_f(\pi_1 M; \mathfrak{su}(2))$$

where $H^1_f(\pi_1 M; \mathfrak{su}(2))$ is group cohomology, with $\mathfrak{su}(2)$ realised as a $\pi_1 M$ module by pulling back the adjoint action of SU(2) by f. When f is reducible, the dimension of $H^1_f(\pi_1 M; \mathfrak{su}(2))$ is generally higher than that of irreducible cases.

For example, take M to be a solid figure 8, so that $\pi_1 M$ is the free group $\mathbb{Z} * \mathbb{Z}$. Then $\operatorname{Hom}(\pi_1 M, \mathrm{SU}(2)) = \mathrm{SU}(2)^2$, which is a 6 dimensional topological manifold (since $\mathrm{SU}(2) \cong \mathbb{S}^3$). Then modding out by the action of SO(3) gives a 3 dimensional topological manifold.

On the other hand, by $H^1_f(\pi_1 M; \mathfrak{su}(2)) = T_f \operatorname{Hom}(\pi_1 M, \mathrm{SU}(2))/\operatorname{SO}(3)$, take f to be the trivial representation. Then $\mathfrak{su}(2)$ is a trivial $\pi_1 M$-module, and it can be shown that $H^1_f(\pi_1 M; \mathfrak{su}(2)) = H^1(M; \mathfrak{su}(2))$ where the latter is singular cohomology. But

$$H^1(M; \mathfrak{su}(2)) = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathfrak{su}(2) \cong \mathbb{R}^6$$

which is 6 dimensional.

(2) Now that we have established the need to remove reducible representations, we want an easy check that representations are indeed reducible. It turns out that a representation $f : G \to \mathrm{SU}(2)$ is reducible if and only if it factors through $U(1) \subset \mathrm{SU}(2)$. This can be routinely checked by writing

$$A = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1$$

for $A \in \mathrm{SU}(2)$. Then requiring f to be reducible, i.e. requiring each $f(g) = A_g$ to fix a proper subspace of $\mathbb{C}^2$, is equivalent to the requirement that there exists a basis in which $b = 0$. Then we obtain

$$A = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}, \quad |a|^2 = 1.$$

The collection of all such A is parametrised by $\{a \in \mathbb{C} : |a|^2 = 1\} = U(1)$. The main benefit of $U(1)$ is that it is abelian: all irreducible SU(2)-valued representations must factor through an abelian group.

(3) Next suppose Σ was an arbitrary n-manifold. Of course we don't have access to the notion of a Heegaard splitting anymore, so it is difficult to define an analogous invariant. For 4-manifolds, we have a notion of trisections! However, with three pieces as opposed to two, the notion of a signed count makes little sense.

Finally we wish to observe the relevance of choosing Σ to be a homology 3-sphere rather than any other 3-manifold. The primary reason to control reducible representations. For the signed count of $\mathcal{R}(M_1)\cap\mathcal{R}(M_2)$ to make sense, we require the spaces to be 0 dimensional and compact. Since $\mathcal{R}(M_1)\cap\mathcal{R}(M_2) = \mathcal{R}(\Sigma)$, we must show that $\mathcal{R}(\Sigma)$ is compact. Note that $R(\Sigma)$ can be shown to be closed subset of $\mathrm{SU}(2)^{\pi_1\Sigma}$, and is hence compact. It remains to show that when reducibles are removed, the remaining space is still closed. Thus we want to know:

- what the reducibles are,
- and show that they are *isolated*.

Suppose M is an arbitrary 3-manifold, and let $f : \pi_1 M \to \mathrm{SU}(2)$ be reducible. We observed that such a representation must factor through $U(1)$, which is abelian. But then the derived subgroup $[\pi_1 M, \pi_1 M]$ must lie in the kernel of f. The map f factors through $\pi_1 M/\ker f$ and hence through $\pi_1 M/[\pi_1 M, \pi_1 M] = H_1(M;\mathbb{Z})$. For M an arbitrary 3-manifold, $H_1(M;\mathbb{Z})$ can be arbitrary, so we have little control over reducible representations. However, if M is a homology sphere, then $H_1(M;\mathbb{Z})$ is trivial. Therefore requiring f to be reducible forces it to be trivial! That is,

$$R(\Sigma) = R^{\mathrm{irr}}(\Sigma) \cup R^{\mathrm{red}}(\Sigma) = R^{\mathrm{irr}}(\Sigma) \cup \{\text{trivial rep.}\}$$

We mentioned that $R(\Sigma)$ is (topologically) closed. We want $R^{\mathrm{irr}}(\Sigma)$ to be closed as well - for which we need the trivial representation to be an isolated point. The fact that Σ is a homology sphere is also useful here!

Recall that if $M_1, M_2 \subset X$ are embedded submanifolds that meet transversely, then their intersection is also an embedded submanifold, with codimension the sum of the codimensions of M_1 and M_2. In particular, if M_1 and M_2 have codimensions summing to the dimension of X, then $M_1 \cap M_2$ is an embedded 0-manifold, which is topologically a discrete space (and hence consists of isolated points). We use the same reasoning here:

We show that if f is the trivial representation, then

$$T_f R(M_1) + T_f R(M_2) = T_f R(F).$$

Since the codimensions match up, this is equivalent to f being an isolated point. But now we recall an earlier result: for f the trivial representation,

$$T_f R(M) = H^1(M;\mathfrak{su}(2)).$$

By the Mayer-Vietoris sequence, we have

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
H^1(\Sigma;\mathfrak{su}(2)) \arrow[r] & H^1(M_1;\mathfrak{su}(2)) \oplus H^1(M_2;\mathfrak{su}(2)) \arrow[r] & H^1(F;\mathfrak{su}(2)) \arrow[dll] \\
H^2(\Sigma;\mathfrak{su}(2)) \arrow[r] & \cdots &
\end{tikzcd}$$

Since Σ is a homology sphere, we have isomorphisms

$$T_f R(M_1) + T_f R(M_2) \cong H^1(M_1;\mathfrak{su}(2)) \oplus H^1(M_2;\mathfrak{su}(2)) \cong H^1(F;\mathfrak{su}(2)) \cong T_f R(F)$$

as required.

3. Generalisations and related areas

3.1. **Generalising the Casson invariant.** The Casson invariant seems very specific - it only applies to integral homology spheres, and is only defined with respect to SU(2)-valued representations. It is natural to attempt to extend to it a wider class of manifolds, and to a wider class of representations. This has been done as follows:

- Walker generalised the Casson invariant to rational homology spheres.
- Lescop further generalised the invariant to all oriented compact 3-manifolds.
- Taubes gave another description of the (usual) Casson invariant using gauge theory, which we discuss in some more detail.

Let $E \to M$ be a principal SU(2)-bundle. Let $\mathcal{A}$ denote the space of SU(2)-connections of E. (These are 1-forms valued in $\mathfrak{su}(2)$ which are SU(2)-equivariant and reproduce the generators of fundamental vector fields.) Each connection $A \in \mathcal{A}$ has a *curvature form* $F_A = dA + A \wedge A$, and is called *flat* if $F_A = 0$. The *gauge group* $\mathcal{G}$ is the collection of automorphisms of $E \to M$. There is an isomorphim

$$\operatorname{Hom}(\pi_1 M; \mathrm{SU}(2))/\operatorname{SO}(3) \cong \{\text{flat SU(2) connections on } M\}/\mathcal{G}.$$

We can further make sense of reducible and irreducible connections via holonomy. We denote by $\mathcal{R}^*_h(\Sigma)$ the space $\{\text{flat SU(2) connections on } M\}/\mathcal{G}$ with reducibles removed, and the flatness condition perturbed by a general function h. Then a theorem of Taubes states that

$$\lambda(\Sigma) = \sum_{A \in \mathcal{R}^*_h(\Sigma)} (-1)^{\mu(A)}$$

where $\mu(A)$ is the *Floer index.*

This perspective can be expanded further - Floer and Taubes developed instanton Floer homology $I_*(\Sigma)$ for homology 3-spheres Σ using SU(2)-gauge theory. Then the Casson invariant arises as the Euler characteristic of instanton Floer homology:

$$\lambda(\Sigma) = \frac{1}{2} \sum_n (-1)^n \operatorname{rank} I_n(\Sigma).$$

This is closely related to instant knot Floer homology developed by Kronheimer and Mrowka, which in turn is closely related to Khovanov homology (whose Euler characteristic is the Jones polynomial). This is getting outside of the scope of the seminar, as all of these invariants depend on difficult compactness results!

3.2. **Disproving the triangulation conjecture in high dimensions.** Finally we get back to the triangulation conjecture. We motivated the Casson invariant by using it to prove that there exist 4-manifolds that do not admit triangulations. Unfortunately this does not generalise to higher dimensions, since the Casson invariant is only defined for homology 3-spheres. However, the following result by Galewski, Stern, and Matumoto reduces the triangulation conjecture in high dimenions to a 3 dimensional problem:

Theorem 3.1. *Every closed topological manifold of dimension at least 5 admits a triangulation if and only if there is an integral homology 3-sphere Σ such that $\mu(\Sigma) = 1$, and Σ*

is homology cobordant to $-\Sigma$. Conversely, if no such Σ exists, then in each dimension at least 5 there exist non-triangulable topological manifolds.

This result is essentially homotopy theoretic. See [FH20] for a detailing of my understanding of the result.

Recall that the Casson invariant reduces mod 2 to the Rokhlin invariant, and detects orientation. If, in addition, it was invariant under homology cobordisms, then we could compute

$$\lambda(\Sigma) = \lambda(-\Sigma) = -\lambda(\Sigma) \quad \Longrightarrow \quad \lambda(\Sigma) = 0$$

for any homology sphere cobordant to its reverse. Therefore reducing mod 2 gives $\mu(\Sigma) = 0$, showing that no homology sphere Σ as in the theorem exists!

Unfortunately, an the Casson invariant is not invariant under homology cobordisms. We must find some other integer-valued invariant β of integral homology spheres such that

- $\beta(\Sigma) \equiv \mu(\Sigma) \bmod 2$.
- $\beta(-\Sigma) = -\beta(\Sigma)$.
- β is invariant under homology cobordism.

While the Casson invariant fails the third condition, using Seiberg-Witten Floer homology, Manolescu constructed an invariant satisfying all three conditions and hence disproved the triangulation conjecture for every dimension at least 5 [Man13].

4. Exercises

Exercise 4.1. Show that for M a handlebody of genus g, $\mathcal{R}(M)$ is a smooth manifold of dimension $3g-3$. Give a heuristic argument to show that for Σ a surface of genus g, $\mathcal{R}(\Sigma)$ is a smooth manifold of dimension $6g-6$.

Exercise 4.2. Attempt to reconstruct an analogous invariant with $\mathrm{SU}(n)$-valued representations. What goes wrong? Reattempt the construction with $\mathrm{SL}(2,\mathbb{C})$-valued representaions. What goes wrong? What doesn't go wrong?

Exercise 4.3. Show that the Poincaré homology sphere has non-trivial Casson invariant. (In fact, depending on orientation, it has Casson invariant 1 or -1.)

Exercise 4.4. Work through the details of inducing orientations on $\mathcal{R}(M_i), \mathcal{R}(F)$, given an orientation on Σ. Verify that $\lambda(-\Sigma) = -\lambda(\Sigma)$. (This is a bit messy, so details can be found in Saveliev.)

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