RELATIVE BRIDGE TRISECTION DIAGRAMS WITH A VIEW TO MINIMAL GENUS PROBLEMS

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Abstract. What is the minimum genus of a surface embedded in a 4-manifold representing a given homology class? Answers to these questions (such as the adjunction inequality) were originally found with gauge theory, but recently Peter Lambert-Cole gave a novel proof of the adjunction inequality using trisections. The gauge theory machinery also applies to manifolds with boundary, provided the boundaries are convex. I'll describe how trisections could enable us to extend some results to the non-convex setting.

1. The landscape

My ambitious goal is to solve problems of the following form:

Given some data, what is the minimum genus of a surface $\mathcal{K}$ embedded in X^4 and satisfying this data?

Traditionally theorems of this form were proven using gauge theory. A couple of years ago, Peter Lambert-Cole gave a combinatorial proof of the following result.

Theorem 1.1 (Adjunction inequality)**.** *(Lambert-Cole) Let X be a closed symplectic 4-manifold, and $\mathcal{K} \subset X$ a smoothly embedded essential surface with $[\mathcal{K}]$ non-torsion, then*

$$-\chi(\mathcal{K}) \geq [\mathcal{K}]^2 - \langle c_1(X), [\mathcal{K}] \rangle.$$

This can be thought of as

$$2g - 2 \geq \text{homological info.}$$

Therefore the homology class of the surface gives a lower bound on the genus of any embedded representative of the surface.

Proof. The outline is as follows:

(1) Show that every closed symplectic 4-manifold admits a *Weinstein trisection.* (We will imminently define these.)
(2) Show that an embedded surface in *transverse bridge position* satisfies the inequality.
(3) Attempt to isotope an arbitrary surface to be in *transverse bridge position.* In general this can't be done, but if we carefully keep track of exactly how it fails we can do surgery to achieve the desired normal form without altering the inequality.

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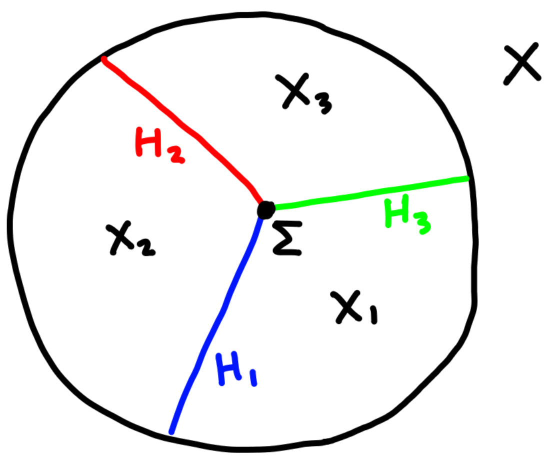
2. What are trisections?

For the next while, I'll be dishing out definitions and examples - so what exactly is a trisection?

Definition 2.1. Let X be a closed 4-manifold. A *trisection* of X is a decomposition of X into three standard pieces which all glue together in a standard way. Specifically:

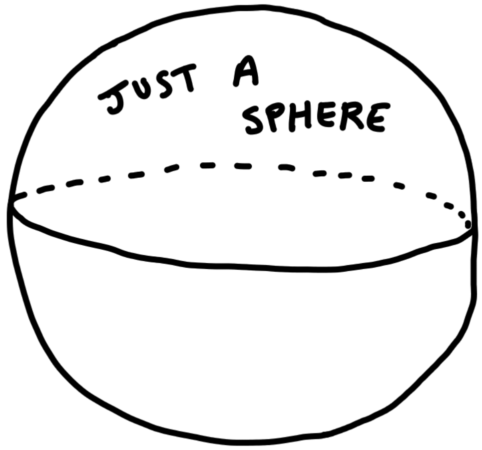
(1) Each $X_i = \natural^k \mathbb{S}^1 \times B^3$ for some k.
(2) Each $H_i = X_i \cap X_{i+1} = \natural^g \mathbb{S}^1 \times B^2$ for some g.
(3) $\Sigma = \cap_i H_1 = \Sigma_g$ for the above g. (Note that any two H_is form a Heegaard splitting.)



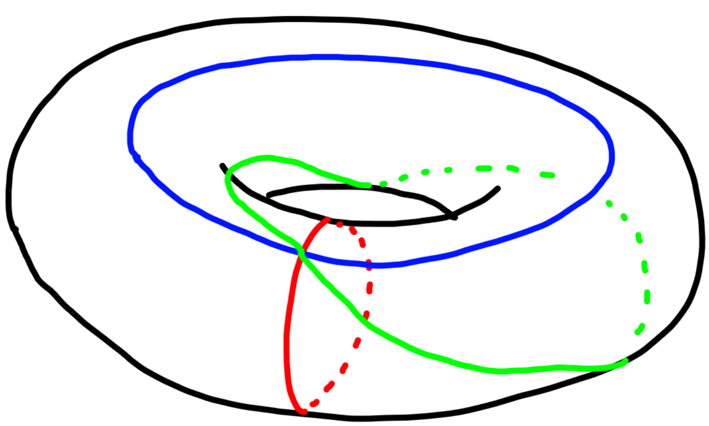
Theorem 2.2. *Basic properties of trisections of 4-manifolds.*

(1) They exist! For any oriented closed 4-manifold.
(2) The 3-dimensional spine ($H_1 \cup H_2 \cup H_3$) determines the 4-manifold.
(3) Since 3-dimensional manifolds are determined by Heegaard diagrams*, and the spine consists of 3-dimensional pieces, the whole 4-manifold is encoded by a* trisection diagram *(three simultaneous Heegaard diagrams).*

Example. The 4-sphere has the following trisection:



Example. The complex projective plane $\mathbb{CP}^2$ has the following trisection:



We can also encode surfaces in trisections.

Theorem 2.3. *Let X be a trisected 4-manifold, and $\mathcal{K} \subset X$ an embedded surface. Then $(\mathcal{K}, X)$ is a* bridge trisection *if $\mathcal{K}$ is partitioned into pieces in a standard way inside X. Specifically:*

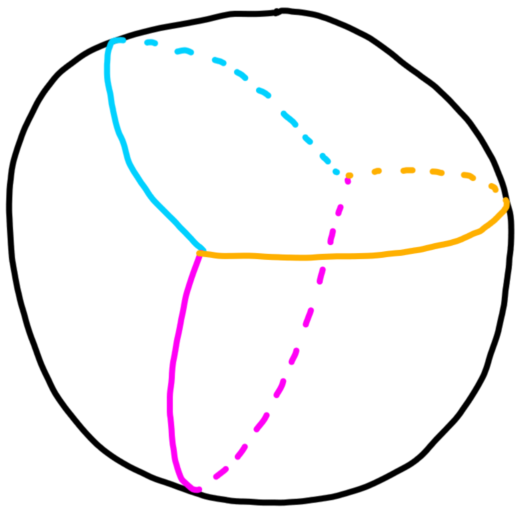
(1) Each $\mathcal{K}_i = \mathcal{K} \cap X_i$ is a disjoint union of disks. Moreover, these disks can be isotoped rel boundary to lie in $\partial X_i = H_i \cup H_{i-1}$. (i.e. a trivial disk tangle.)

(2) Each $\tau_i = K_i \cap H_i$ is a disjoint union of arcs which can be isotoped rel boundary to lie in $\partial H_i = \Sigma$. (i.e. a trivial tangle.)

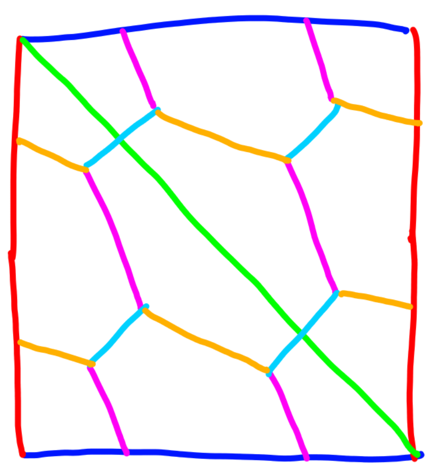
Theorem 2.4. *Basic properties of bridge trisections.*

(1) *They exist! Any embedded surface in a trisected 4-manifold can be isotoped to be in bridge position.*
(2) *The 3-dimensional spine* $(\tau_1 \cup \tau_2 \cup \tau_3,\ H_1 \cup H_2 \cup H_3)$ *determines the pair* $(\mathcal{K}, X)$.
(3) *Since each tangle* τ_i *is determined by a tangle diagram (i.e. a projection to* $\partial H_i = \Sigma$*), the whole bridge trisection is encoded by a* bridge trisection diagram *(i.e. a trisection diagram together with three simultaneous tangle diagrams)*

Example. Here's a diagram for a 2-sphere sitting inside a 4-sphere.



Example. Here's a diagram for a degree 2 complex curve in $\mathbb{CP}^2$.



3. Relative trisections and geometry

Next I'll say a bit about what I'm doing. The theory of trisections is well developed in the closed case, and its useful in geometry has been exhibited by Peter Lambert-Cole's proof of the adjunction inequality. I'm working in the relative case - I want to establish analogous results for manifolds with boundary. This is where we switch from the Adjunction inequality to the related slice Bennequin inequality:

$$-\chi(\mathcal{K}) \geq sl(K),$$

for transverse knots in the boundary of a 4-ball. This can be generalised:

Theorem 3.1 (Slice Bennequin inequality)**.** *(Lisca-Matić.) Let X be a Stein 4-manifold with boundary$Y$. Let $K \subset Y$ be a transverse knot, and suppose $\mathcal{K}$ is an orientable smooth surface in X bound by K. Then*

$$-\chi(\mathcal{K}) \geq sl(K, \mathcal{K}).$$

It's a similar form to the slice Bennequin inequality for knots on the boundary of a 4-ball, but now we need to be a little careful about a few things.

(1) What exactly is a Stein manifold with boundary? You can think of this as a complex 4-manifold with boundary with extra properties, and in particular the boundary is *convex*. Formally, convex means there's an outward pointing vector field ρ such that

$$\iota_\rho \omega = \alpha$$

induces the contact structure on the boundary.

(2) What's $sl(K, \mathcal{K})$? In the 4-ball case we defined the self linking number as an actual linking number of the knot with a certain perturbation of the knot. This is because the self linking number in the 4-ball is independent of the surface bound by the knot. In a more general 4-manifold, we need to keep track of the surface as well. Specifically, we define

$$sl(K, \mathcal{K}) = e(N\mathcal{K}, s) - c_1(\det TX|_{\mathcal{K}}, s),$$

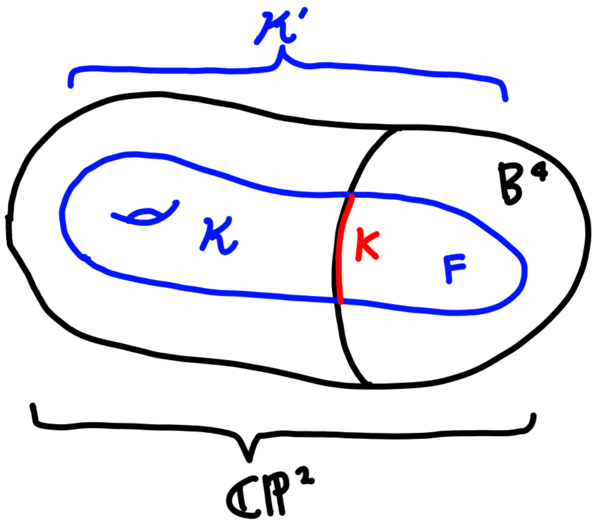
where s is a non-vanishing vector field in the contact distribution restricted to K. (That is, it's defined as the difference of two characteristic classes.)

Goal. I wish to understand genus bounds for surfaces in symplectic 4-manifolds with boundary, even if the boundary isn't convex.

Example. Consider

$$\mathbb{CP}^2 - B^4.$$

This is a symplectic 4-manifold with boundary, but the boundary is concave. The slice Bennequin inequality from above (or any other gauge theoretic versions) don't apply because of this. Unfortunately, the slice Bennequin inequality is false, because I can provide a counter example.



We'll make use of the formula

$$sl(K, \mathcal{K}) = [\mathcal{K}']^2 - \langle c_1(\mathbb{CP}^2), \mathcal{K}' \rangle + sl(K).$$

This comes from expressing both of the self linking number terms as differences of characteristic classes. Each of the characteristic classes can be expressed as an intersection, and the sum of intersections in the two pieces add to the global intersection terms in the formula.

By taking K to be the right handed trefoil, we can find a surface $\mathcal{K}$ bound by K in $\mathbb{CP}^2 - B^4$ with genus 0 and trivial relative homology class. On the other hand, $\overline{K}$ bounds a genus 1 surface in B^4. Their union is a homologically trivial surface in $\mathbb{CP}^2$. From general theory,

$$sl(K) = 1.$$

This tells us that

$$sl(K, \mathcal{K}) = 1.$$

On the other hand, since $\mathcal{K}$ is a disk, $\chi(\mathcal{K}) = 1$. Clearly it's not true that

$$-\chi(\mathcal{K}) \geq sl(K, \mathcal{K}).$$

Peter Lambert-Cole has described a criterion for the slice Bennequin inequality to hold in a general setting (which doesn't require convexity). This example shows that the criterion doesn't always hold, even though it's guaranteed to hold for the closed case (which is how he proved the adjunction inequality). Namely:

Theorem 3.2 (slice Bennequin inequality)**.** *(Lambert-Cole) Let (X, J) be a compact, symplectic 4-manifold with boundary with a Weinstein trisection. Let $\mathcal{K}$ be an embedded surface in homotopically transverse bridge position, and transverse boundary $K = \partial\mathcal{K}$. Then*

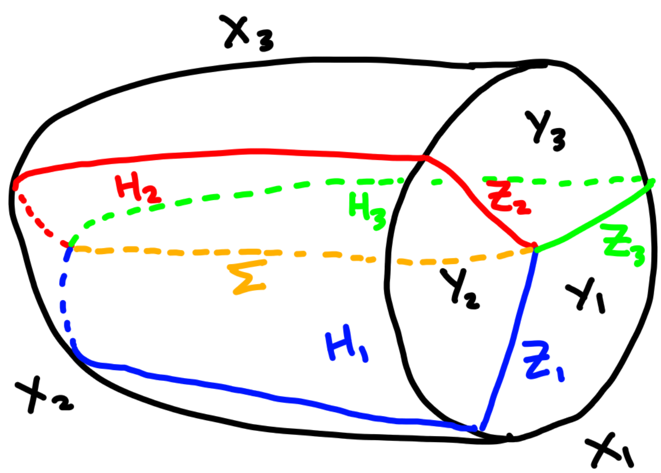
$$-\chi(\mathcal{K}) \geq sl(K, \mathcal{K}).$$

To interpret this theorem, I need to define trisections for manifolds with boundary, as well as Weinstein trisections and the condition of a surface being in homotopic transverse bridge position. The main idea is that the above example fails the conclusion of this theorem, even though $\mathbb{CP}^2 - B^4$ admits a Weinstein trisection, and $\mathcal{K}$ can be put in bridge position. This means there's some obstruction to putting $\mathcal{K}$ into *homotopically transverse* bridge position.

Goal. I wish to understand when surfaces in trisections can be put into homotopic transverse bridge position. (This is being pursued by first developing an understanding of *diagrams* of relative bridge trisections.)

Definition 3.3. Let X be a 4-manifold with boundary $\partial X = Y$. (This is a slightly informal definition, but) a trisection of (X, Y) is a decomposition $X = X_1 \cup X_2 \cup X_3$ and $Y = Y_1 \cup Y_2 \cup Y_3$ such that

1. Each $X_i = \natural^k \mathbb{S}^1 \times B^3$ for some k.
2. Each $\partial X_i = \#^k \mathbb{S}^1 \times \mathbb{S}^2 = Y_i \cup H_i \cup H_{i-1}$.
3. $X_1 \cap X_2 \cap X_3$ is a surface Σ of some genus g with b boundary components.
4. Each $X_i \cap X_{i+1}$ is a 3-dimensional compression body H_i from Σ to $Y_i \cap Y_{i+1}$.



A *relative bridge trisection* of $(X, \mathcal{K})$ is a surface $\mathcal{K}$ embedded in a trisection 4-manifold X with boundary, such that

1. Each $\mathcal{K}_i = \mathcal{K} \cap X_i$ is a trivial disk-tangle.
2. Each $\tau_i = \mathcal{K} \cap H_i$ is a trivial tangle.

Next, we introduce some geometry:

Definition 3.4. A *Weinstein trisection* is essentially a trisection in which each sector X_i is a *Weinstein domain.* This essentially means there's a Liouville vector field on each sector inducing a contact structure on the boundary of the sector, (and the boundary is convex). Each H_i then inherits a foliation $\ker(\alpha_i - \alpha_{i-1})$.

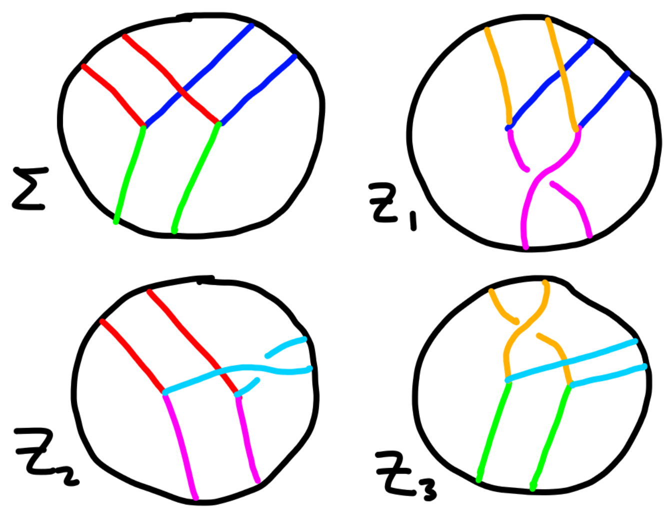
A surface in a Weinstein trisection is in *homotopically transverse bridge position* if it's in bridge position, has complex bridge points, and the tangles $\tau_i = \mathcal{K} \cap H_i$ are homotopically transverse. That is,

(1) $\mathcal{K}$ is J-holomorphic in a neighbourhood of each point in $\mathcal{K} \cap \Sigma$.
(2) Each tangle τ_i can be homotoped rel boundary (in H_i) so that $\beta_i(\tau_i) > 0$. (That is, the tangles are positively transverse to the induced foliation on each H_i.)

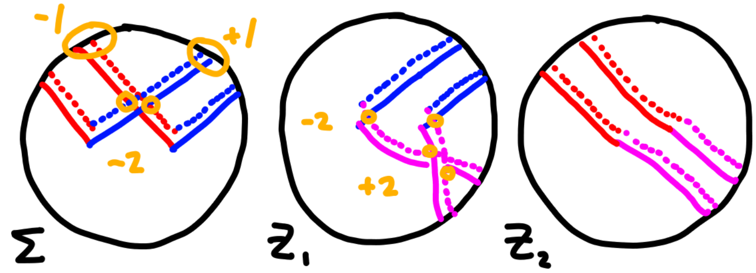
4. Calculations with relative trisections

I'm now going into introduce relative bridge trisection diagrams and do a few calculations with one.

Example. We'll consider the right handed trefoil in $\mathbb{S}^3$. The diagram consists of 4 disks: each one corresponds to the four surfaces lying in the spine of the relative trisection. Each surface also has arcs on it - these come from projecting the tangles $\mathcal{K} \cap H_i, \mathcal{K} \cap Y_i$ onto the boundary of each H_i or Y_i. Each boundary is a 2-sphere decomposed into two disks along $\partial\Sigma = \mathbb{S}^1$. Each component of the diagram is one of these disks together with the projected arcs.



Using this diagram I can compute things like the genus, orientability, and homological data. As an example, I'll compute $e(N\mathcal{K}, s)$ for a section s of the contact distribution on the sphere. (Recall that this is part of the definition of the self linking number.) We can take s to be a perturbation "in the upwards direction" in the diagram. Then I draw another copy of $\mathcal{K}$ perturbed upwards and count intersection points.



$8 - 4 - 4 + 2$
$= 2$

$\therefore \ell k = 2/2 = 1.$

3 triples;

$3 \times 1 = 3.$
$e = 3.$

Goal. I wish to expand my understanding of these diagrams. Specifically, I don't currently have a method of determining a surface corresponding to Chern classes (which I would then intersect with a given surface to compute $c_1(\det(TX)|_{\mathcal{K}}, s)$ and ultimately the self linking number).