Algebraic topology ensemble

Shintaro Fushida-Hardy
381B Sloan Hall, Stanford University, CA

This document contains notes on algebraic topology - during my reading course with Ciprian Manolescu in Spring 2020, he noticed that I didn't really know any algebraic topology/homotopy theory beyond the first couple of chapters of Hatcher! After concluding the reading course (on knot theory and 3-manifolds) we decided that I should learn some more algebraic topology, as it is fundamental to topology as a whole. In the first chapter, we introduce classifying spaces for vector bundles and principal G-bundles. In the second chapter we try to improve our classifications of vector bundles by introducing characteristic classes (and solve several exercises). We continue the trend of delving into vector bundles by studying K-theory in the third chapter, with an emphasis on the Bott periodicity theorem and classification of division algebras. This was my first experience with an extraordinary cohomology theory! Next we investigate another generalised (co)homology theory, namely cobordism theory. Finally we study some additional tools that can help us understand both K-theory and cobordism theory (and extraordinary cohomology theories in general): equivariant cohomology and the Atiyah-Hirzebruch spectral sequence. We give examples of calculations involving each of these. The primary sources are listed at the start of each chapter.

Contents

Chapter 1

Classifying spaces

Fibre bundles (and in particular vector bundles) appear throughout maths, so we want some way of classifying bundles over a given space X. Given a topological group G, we will construct the *classifying space* BG, and show that isomorphism classes of principal G-bundles over X are in natural bijective correspondence with homotopy classes of maps $X \to BG$. In this chapter, we start by recalling general definitions for fibre bundles and fibrations. We then look at the special cases of vector bundles and principal bundles, before constructing classifying spaces and establishing the main result mentioned above. Finally we will study some examples. This chapter is primarily sourced from [Kot12, Mit06, Coh98].

1.1 Fibrations and fibre bundles

The most general notion we consider is a *fibration*. Most diagrams we consider will involve fibrations or cofibrations. Covering spaces and fibre bundles are also examples of fibrations.

Definition 1.1.1. A *fibration* $p : E \to B$ is a map satisfying the *homotopy lifting property*. That is, any homotopy $f : A \times I \to B$ lifts to $\widetilde{f} : A \times I \to E$.

As a consequence of this definition, the fibre of any two points in B are homotopy equivalent. Therefore we speak of the *fibre* F of a fibration, and write

$$F \xrightarrow{i} E \xrightarrow{p} B$$

where p is the fibration and i is an inclusion. Although not all maps are fibrations, we can approximate all maps by fibrations.

Proposition 1.1.2. Let $f : X \to Y$ be any map between topological spaces. There exists a fibration $\widetilde{f} : \widetilde{X} \to Y$ which approximates f in the sense that there is a homotopy equivalence $h : X \to \widetilde{X}$ such that the following diagram commutes:

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\widetilde{f}"] \\ X \arrow[ur, "h"] \arrow[r, "f"] & Y. \end{tikzcd}$$

Proof outline. Define $\widetilde{X} := \{(x, \gamma) \in X \times Y^I$ such that $\gamma(0) = f(x).\}$ Here Y^I is the space of all paths $I = [0, 1] \to Y$ equipped with the compact-open topology. This is called the *mapping path space* of f. Define $\widetilde{f} : \widetilde{X} \to Y$ by

$$\widetilde{f} : (x, \gamma) \mapsto \gamma(1).$$

We claim without proof that this map is a fibration. Next we define $h : X \to \widetilde{X}$ by

$$h : x \mapsto (x, \gamma_x),$$

where γ_x is the constant path $\gamma_x(t) = f(x)$. It is clear that the above diagram commutes, because $(\widetilde{f} \circ h)(x) = \widetilde{f}(x, \gamma_x) = \gamma_x(1) = f(x)$. Finally, to see that h is a homotopy equivalence, define $g : \widetilde{X} \to X$ by

$$g : (x, \gamma) \mapsto x.$$

One can show that $h \circ g$ and $g \circ h$ are homotopic to the identity. □

The *mapping path space* introduced in the above proof is a canonical choice for $\widetilde{X}$, which also gives us a canonical choice of fibre.

Definition 1.1.3. Let $f : X \to Y$ be a map. We have the following commutative diagram:

$$\begin{tikzcd} F_f \arrow[r, "i"] \arrow[d, "p"] & E_f \arrow[d, "\widetilde{f}"] \\ X \arrow[ur, "h"] \arrow[r, "f"] & Y. \end{tikzcd}$$

The map $\widetilde{f} : E_f \to Y$ is a *fibration*, where E_f is the *mapping path space* defined by

$$E_f := \{(x, \gamma) \in X \times Y^I : \gamma(0) = f(x)\}.$$

E_f is homotopic to X, so $\widetilde{f}$ can be seen as a fibration that approximates f. F_f denotes the *homotopy fibre* of f, which is the pullback of the above diagram, and the fibre of $\widetilde{f}$. Explicitly, we have

$$F_f := \{(x, \gamma) \in X \times Y^I : \gamma(0) = f(x), \gamma(1) = y_0\},$$

where y_0 is a base point of Y. The map p is the projection onto the first coordinate, and i is the inclusion map.

The most important theorem concerning fibrations is that they extend to a long exact sequence. Recall that given a pointed space X, the *loop space* of X, denoted ΩX, is the space of pointed maps from $\mathbb{S}^1$ into X equipped with the compact-open topology.

Let $F \to E \to B$ be a fibre sequence (i.e. $p : E \to B$ is a fibration, with fibre F). Replacing the fibre sequence with $F_p \to E_p \to B$, one can show that $i : F_p \to E_p$ is itself a fibration, with fibre ΩB. To see why the fibre is ΩB, we simply write it out explicitly:

$$i^{-1}((e_0, \gamma)) = \{(e_0, \gamma) : \gamma(0) = f(e_0) = b_0 = \gamma(1)\} \cong \{\gamma \in B^I : \gamma(0) = \gamma(1) = b_0\} = \Omega B.$$

Given a map $f : X \to Y$, we also have a canonical map $f' : \Omega X \to \Omega Y$ defined by

$$f' : \gamma \to f \circ \gamma.$$

Combining these constructions actually gives a long fibre sequence, as described in the following theorem.

Theorem 1.1.4. *Let $F \to E \to B$ be a fibre sequence. Then there is a long fibre sequence*

$$\cdots \to \Omega^2 B \to \Omega^1 F \to \Omega^1 E \to \Omega^1 B \to F \to E \to B.$$

By a long fibre sequence, we mean that any two consecutive maps define a fibre sequence.

Corollary 1.1.5. *If $F \to E \to B$ is a fibre sequence, there is an associated long exact sequence of homotopy*

$$\cdots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots.$$

Proof outline. We induce a long exact sequence

$$\cdots \to [\mathbb{S}^0, \Omega^{n+1} B]_0 \to [\mathbb{S}^0, \Omega^n F]_0 \to [\mathbb{S}^0, \Omega^n E]_0 \to [\mathbb{S}^0, \Omega^n B]_0 \to \cdots.$$

Recall that $[X, Y]$ denotes the homotopy classes of maps from X to Y, and here we have used $[X, Y]_0$ to denote the classes of based maps. By Eckmann-Hilton duality, we have

$$[\mathbb{S}^0, \Omega^n X]_0 = [\Sigma^n \mathbb{S}^0, X]_0 = [\mathbb{S}^n, X]_0 = \pi_n(X),$$

where ΣY denotes the suspension of Y. The result follows. □

This is a key result which we wish to apply in many scenarios, an in particular in the context of bundles. It turns out that under mild conditions, bundles are indeed fibrations and we have a long exact sequence of homotopy groups.

Definition 1.1.6. A *fibre bundle* is the data (π, E, B, F), where π is a locally trivial surjection $\pi : E \to B$ of topological spaces. E is called the *total space*, and B the *base space*, and F the *fibre space*. *Locally trivial* means that for any $b \in B$ there is a neighbourhood U of b so that $\pi^{-1}(U) \subset E$ is homeomorphic to $F \times U$, and the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{proj}"] \\
B. &
\end{tikzcd}$$

Definition 1.1.7. A *morphism* of fibre bundles $F : (\pi, E, B) \to (\pi', E', B')$ is a pair (f_E, f_B) such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[r, "f_E"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\
B \arrow[r, "f_B"] & B'.
\end{tikzcd}$$

We often write $\pi : E \to B$, or even just E, to denote a fibre bundle. A morphism of fibre bundles is often denoted by $f : E \to E'$.

Remark. The collection of fibre bundles forms a category, with morphisms as defined above. Fixing a base space gives a subcategory.

Next we observe that for both fibrations and fibre bundles, we have pullbacks.

Definition 1.1.8. Let $\pi : E \to B$ be a fibre bundle, and $f : A \to B$ continuous. The pullback bundle $\pi' : f^*(E) \to A$ is defined to be the pullback of the diagram $E \to B \leftarrow A$. Explicitly, define

$$f^*(E) = A \times_B E = \{(a, e) \in A \times E : f(a) = \pi(e)\},$$

and consider the diagram

$$\begin{tikzcd}
f^*(E) \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"] & E \arrow[d, "\pi"] \\
A \arrow[r, "f"] & B.
\end{tikzcd}$$

Then the arrow on the left defines the pullback bundle.

Remark. More generally, given a fibration $\pi : E \to B$ and a continuous map $f : A \to B$, the pullback of $E \to B \leftarrow A$ also defines a fibration $f^*(E) \to A$.

In fact, fibre bundles of interest are fibrations!

Theorem 1.1.9. *Let $\pi : E \to B$ be a fibre bundle. If B is paracompact, then π is a fibration. (Recall that a space is paracompact if every open cover has a locally finite subcover.)*

This means that for B paracompact, a fibre bundle gives rise to the long exact sequence of homotopy. In particular, whenever B is a manifold, the long exact sequence of homotopy is induced.

Remark. Because of the above theorem, fibre bundles are sometimes called locally trivial fibrations.

Finally we introduce the sections. In any category, a section of a map $f : A \to B$ is some map $\sigma : B \to A$ such that $f \circ \sigma = \mathrm{id}_B$. This applies in the context of fibre bundles, which are defined to be surjections.

Definition 1.1.10. Let $\pi : E \to B$ a fibre bundle, and $U \subset B$ a subspace. Then $\Gamma(U, E)$ denotes the *local sections* $\sigma : U \to E$ of π. The *global sections* $\sigma : B \to E$ are simply denoted by $\Gamma(E)$.

In the next section we see that not all bundles admit global sections.

1.2 Vector bundle and principal bundle basics

In the previous section, we laid out the most general definitions of relevance. In practice, we are interested in settings with more structure, which generally manifests in the choice of fibre. In this section, we explore the case where F is a vector space (which gives rise to *vector bundles*), and the case where F consists of G-torsors (which gives rise to *principal G-bundles*).

Both of the above scenarios arise by fixing a group of automorphisms for the fibre, called the *structure group*, and requiring bundle morphisms to preserve the structure.

Definition 1.2.1. Let G be a topological group, and (π, E, B, F) a fibre bundle. Suppose G acts continuously and faithfully on F by left multiplication (so that $F \subset \mathrm{Homeo}(F)$). A *$G$-atlas* for (π, E, B, F) is an atlas of local trivialisations $\{(U_k, \varphi_k)\}$ of $E \to B$ so that whenever $U_i \cap U_j$ is nonempty,

$$\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

is given by

$$\varphi_i \circ \varphi_j^{-1} : (x, f) \mapsto (x, g_{ij}(x) f).$$

Here $g_{ij} : U_i \cap U_j \to G$ is a continuous map called the *transition function.*

Two G-atlases are equivalent if their union is a G-atlas, and a *G-bundle* is a fibre bundle as above equipped with an equivalence classes of G-atlases. The group G is called the *structure group* of G.

Using the notion of a G-bundle, we can define vector bundles and principal G-bundles. Namely, a vector bundle is a G-bundle with structure group $\mathrm{GL}(n, k)$ for some field k, while a principal G-bundle is a G-bundle whose G-action is free and transitive. In general a vector bundle is not a principal G-bundle and vice versa. We now explore these two concepts in some more depth.

Definition 1.2.2. A *vector bundle* is a fibre bundle $\pi : E \to B$ with fibre a vector space V and structure group $\mathrm{GL}(V)$.

More concretely, we require the local trivialisations $\pi^{-1}(U) \cong U \times V$ to restrict to vector space isomorphisms $\pi^{-1}(b) \cong V$ (where U is a neighbourhood of b). The *rank* of a vector bundle is the dimension of V. A rank 1 vector bundle is often called a *line bundle.*

A *morphism of vector bundles* is a bundle morphism which restricts to linear maps on fibres.

Example. The cylinder is a trivial (real) line bundle over $\mathbb{S}^1$, while the Möbius strip is a non-trivial line bundle over $\mathbb{S}^1$.

Example. Consider $\mathbb{CP}^n$, the space of one dimensional subspaces of $\mathbb{C}^{n+1}$. The *tautological line bundle* is the projection map $\pi : S \to \mathbb{CP}^n$ onto the second coordinate, where S is the subspace of $\mathbb{C}^{n+1} \times \mathbb{CP}^n$ consisting of pairs $(x, [x])$. Morally this is like the quotient map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$. This applies to $\mathbb{RP}^n$, and more generally to any Grassmannian.

Many operations on vector spaces are functorial, giving rise to analogous constructions on vector bundles:

- Each vector bundle E has a corresponding dual bundle E^*, whose fibre is the dual vector space. Formally, if $E \to X$ is a vector bundle, define $E^* \to X$ to be the collection of pairs (x, φ) where $x \in X$ and $\varphi \in (E_x)^*$. The transition functions of E^* are defined to be the dual functions of the inverses of transition functions of E.

- Given a vector bundle $E \to B$ and continuous map $A \to B$, the pullback bundle $f^*(E) \to A$ is a vector bundle.

- Given vector bundles $E_1 \to B_1$ and $E_2 \to B_2$, the product map $E_1 \times E_2 \to B_1 \times B_2$ defines a vector bundle.

- Fix vector bundles $E_1 \to B$ and $E_2 \to B$, and consider their product bundle as above. The *Whitney sum bundle* denoted $E_1 \oplus E_2 \to B$ is the pullback bundle of $E_1 \times E_2$ over the diagonal map $B \to B \times B$. Alternatively, the Whitney sum bundle can be constructed explicitly by declaring the transition functions of $E_1 \oplus E_2$ to be direct sums of the individual transition functions.

- Fix vector bundles $E_1 \to B$ and $E_2 \to B$. The *tensor product bundle* $E_1 \otimes E_2$ is defined similarly, with fibre the tensor products of fibres of E_1 and E_2.

- Fix vector bundles $E_1, E_2 \to B$. The *Hom bundle* $\mathrm{Hom}(E_1, E_2) \to B$ is the vector bundle whose fibre at x is the space of linear maps from E_{1x} to E_{2x}. The sections of $\mathrm{Hom}(E_1, E_2)$ are in bijective correspondence with vector bundle homomorphisms from E_1 to E_2 over X.

Observe that the space of sections $\Gamma(E)$ of a vector bundle is naturally a vector space, by pointwise addition. Therefore every vector bundle admits sections, since they all admit the zero-section. On the other hand, we also know that there exist non-trivial vector bundles (such as the Möbius strip). We will soon see that principal G-bundles admit sections if and only if they are trivial bundles, so this shows that a vector bundle is not generally a principal G-bundle. Without further ado, we discuss principal G-bundles.

Definition 1.2.3. Let G be a topological group, and $\pi : P \to B$ a fibre bundle. Suppose G acts continuously on P (on the right). Then π is said to be a *principal G-bundle* if G acts freely and transitively on the fibres of π, and if π has an atlas of G-equivariant local trivialisations.

Explicitly, the G-equivariant condition is saying that for any g, if $h(p) = (u, f)$, then $h(pg) = (u, fg) = (u, f)g = h(p)g$, where h is shown in the diagram below:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "h"] \arrow[d] & U \times F \arrow[dl] \\ B & \end{tikzcd}$$

Remark. Since G acts continuously, freely, and transitively on the fibres of π, each fibre must be homeomorphic to G. However, there is no canonical choice of identity for the fibres, so the fibres are equipped with a *G-torsor* structure rather than a group structure. (Analogously, consider affine space instead of a vector space.)

Remark. A principal G-bundle is in particular a G-bundle. By identifying F with G, the principal G-bundle structure determines a left action on fibres as required.

We required that local trivialisations of a principal G-bundle are G-equivariant. This applies to morphisms of principal G-bundles as well.

Definition 1.2.4. Let $P, Q \to B$ be principal G-bundles. A *morphism* of principal G-bundles is a G-equivariant morphism of fibre bundles $P \to Q$. More explicitly, we require $f(pg) = f(p)g$, where $f : P \to P'$ is a morphism of fibre bundles P, P'.

The property of being a morphism of principal G-bundles is actually highly restrictive: every morphism is an isomorphism. We then use this result to show that a principal G-bundle is trivial if and only if it has a section.

Proposition 1.2.5. Every morphism of principal G-bundles $P, Q \to B$ is an isomorphism.

Proof. Let $f : P \to Q$ be a morphism of principal G-bundles. Locally (by choosing a G-equivariant local trivialisation) we can write $f(b, g) = (b, f(b)g)$ where $f(g) \in G$. Therefore, locally $f^{-1} : Q \to P$ is given by $f^{-1}(b, g) = (b, f(b)^{-1}g)$. Gluing together a G-atlas, the local isomorphisms stitch together to a global isomorphism. $\square$

Corollary 1.2.6. *A principal G-bundle is a trivial bundle if and only if it admits a global section.*

Proof. Clearly a trivial bundle admits a global section (by just choosing a constant section). Conversely, suppose $P \to B$ admits a section σ. Define $f : B \times G \to P$ by $f(b, g) = \sigma(b)g$. This is a morphism of principal G-bundles, so by the previous proposition, it is an isomorphism. $\square$

This result establishes that neither vector bundles nor principal bundles are generalisations of the other. However, they are related through the associated bundle construction. Namely, we establish the following:

- Given a principal G-bundle $\pi : P \to B$, and any space F on which G acts on the left by automorphisms, there is an *associated bundle*
$$P \times_G F \to B$$
which is a G-bundle with fibre F. In particular, if F is a vector space and G acts linearly, the associated bundle is a vector bundle.

- Conversely, any fibre bundle E with structure group G arises as the associated bundle of a principal G-bundle. In the context of E a vector bundle, the corresponding principal G-bundle is called the *frame bundle* of E.

We now explore these in more detail. If $\rho : G \to \mathrm{Aut}(F)$ is a left action, and $\pi : P \to B$ is a principal G-bundle, the product $P \times F$ has a canonical right action defined by

$$G \to \mathrm{Aut}(P \times F), \quad (p, f)g = (pg, \rho(g^{-1})f).$$

This defines a fibre bundle over B in the obvious way, but it has fibre $G \times F$. Therefore to obtain a fibre bundle with fibre F, we mod out by the G action:

$$P \times_G F = (P \times F)/G \xrightarrow{\pi_F} B, \quad \pi_F([p, f]) = \pi(p).$$

Definition 1.2.7. Given a principal G-bundle $\pi : P \to B$ and a left action $\rho : G \to \mathrm{Aut}(F)$ on some space F, the *associated bundle* to P is

$$P \times_G F \to B$$

constructed above. The associated bundle has structure group G, and is trivial if P is trivial.

Example. Let $P \to B$ be a principal $\mathrm{GL}(n, \mathbb{R})$-bundle. Let $F = \mathbb{R}^n$. Then F has a canonical left action by $\mathrm{GL}(n, \mathbb{R})$. The associated bundle construction gives a $\mathrm{GL}(n, \mathbb{R})$-bundle (i.e. a vector bundle)

$$E = P \times_{\mathrm{GL}(n,\mathbb{R})} F \to B$$

with fibre F. Then $P \to B$ is called the *frame bundle* of E, since a local section corresponds to a local trivialisation (i.e. a frame) of E. In particular, a global trivialisation of P determines a global frame for E.

Finally, we formalise the converse statement (but do not give a proof).

Proposition 1.2.8. Suppose $\pi : E \to B$ is a fibre bundle with fibre F and structure group G. Then there exists a principal G-bundle $P \to B$ such that $E = P \times_G F$.

1.3 Classifying spaces

Next we want to classify bundles in some way. We do this using *classifying spaces*. Given a group G, we construct a space BG so that homotopy classes of maps $X \to BG$ are in natural bijective correspondence with isomorphism classes of principal G-bundles over X. This gives a corresponding classification for vector bundles.

We take for granted the following non-trivial result:

Theorem 1.3.1. *Let $p : E \to B$ be a fibre bundle with fibre F. Let $f_0, f_1 : X \to B$ be homotopic maps. Suppose B is normal and paracompact. Then the corresponding pullback bundles are isomorphic.*

This is non-trivial because of the weak conditions on B. Often in the literature the result is proven for B a CW complex.

Remark. Hereafter, we assume all spaces are CW complexes.

As a corollary of the previous theorem, we obtain a well defined map as follows:

Let $\pi : P \to B$ be a principal G-bundle, with B connected. Then every homotopy class of maps $X \to B$ determines a unique principal G-bundle by pullback. Therefore, there is a well defined map

$$[X, B] \to \{\text{Principle } G\text{-bundles over } X\}.$$

The hope is to find some "universal" $\pi : P \to B$ so that the above map is a bijection for any X. It turns out that these exist!

Theorem 1.3.2. *Let G be a topological group. There exists a principle G-bundle $EG \to BG$ such that the pullback map*

$$[X, BG] \to \{\textit{Principle } G\textit{-bundles over } X\}$$

is a bijection for any space X. Moreover, BG is unique up to homotopy type. Such a bundle is called a universal bundle*, and BG a* classifying space*.*

Before outlining how the theorem works, we describe a classification of vector bundles.

Fix a space X, and $k = \mathbb{R}, \mathbb{C}$. We write $\mathrm{Vect}_k^n(X)$ to denote the set of all isomorphism classes of k-vector bundles over X, of rank n. The Whitney sum defines an operation

$$\mathrm{Vect}_k^n(X) \times \mathrm{Vect}_k^m(X) \to \mathrm{Vect}^{n+m}(X),$$

which turns $\mathrm{Vect}_k(X) = \bigoplus_n \mathrm{Vect}_k^n(X)$ into an abelian monoid. The *K-theory* $K_k(X)$ of X is defined to be the completion of $\mathrm{Vect}_k(X)$ into an abelian group. The standard notation is to write

$$K(X) \coloneqq K_{\mathbb{C}}(X), \quad KO(X) \coloneqq K_{\mathbb{R}}(X).$$

Here the K, introduced by Grothendieck, is short for *Klasse*, as we think of $K(X)$ as the isomorphism classes of vector bundles over X. The O (from the real case) denotes the orthogonal group.

We now formalise the notion of a K-theory by describing the completion.

Proposition 1.3.3. Let M be an abelian monoid. There exists an abelian group $G(M)$, unique up to isomorphism, and a monoid homomorphism $\iota : M \to G(M)$ such that any morphism $M \to N$ of monoids extends uniquely to a morphism $G(M) \to N$ of monoids. In other words, for any f as in the diagram below, there exists a unique $\widetilde{f}$ making the diagram commute.

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "f"] & G(M) \arrow[dl, dotted, "\widetilde{f}"] \\ N & \end{tikzcd}$$

The abelian group $G(M)$ is called the *Grothendieck completion* of M, and can be thought of as the smallest group containing M.

Proof outline. As with most universal properties, the proof of the result is a construction. We outline the objects in the construction but do not prove that they work.

Let $F(M)$ denote the free group generated by elements of M, and define

$$G(M) = \frac{F(M)}{(a \oplus b - (a+b))}.$$

Here $\oplus$ denotes the group operation of $F(M)$, and $+$ the monoid operation of M. There is a canonical inclusion $M \to G(M)$. This satisfies the universal property. $\square$

Definition 1.3.4. Let X be a space, and $\mathrm{Vect}_k(X)$ the space of classes of k-vector bundles over X as above. The (complex and real) *K-theories* of X are defined by

$$K(X) \coloneqq G(\mathrm{Vect}_{\mathbb{C}}(X)), \quad KO(X) \coloneqq G(\mathrm{Vect}_{\mathbb{R}}(X)).$$

The K-theory of a space classifies the vector bundles. We compute some examples.

Example. Let X be a point. Then every vector bundle of rank n is isomorphic, and is the trivial bundle. Therefore $\mathrm{Vect}_k(X) = \mathbb{N}$. The K-theory of X is now $\mathbb{Z}$.

Next suppose X is a circle. Then a real vector bundle of rank n over X is determined up to isomorphism by orientability. Therefore $\mathrm{Vect}^n_{\mathbb{R}}(X) = \mathbb{Z}/2\mathbb{Z}$ for each n. Inspecting the Whitney sum, we find that $\mathrm{Vect}_{\mathbb{R}}(X)$ has monoid presentation $\langle a, b \mid a^2 = b^2 \rangle$. The Grothendieck completion is the corresponding group presentation, so that

$$KO(\mathbb{S}^1) = \mathbb{Z} *_{2\mathbb{Z}} \mathbb{Z}.$$

In fact, by classifying vector bundles over a given space, we simultaneously classify vector bundles over all homotopic spaces! Applying the first theorem of this section to vector bundles, we have the following:

Proposition 1.3.5. If $f, g : A \to B$ are homotopic maps, and $p : E \to B$ is a vector bundle, the pullback bundles $f^*(E)$ and $g^*(E)$ are isomorphic as vector bundles.

Note that the same result holds when we replace vector bundles with principal$G$-bundles. As an application, we have the following:

Corollary 1.3.6. *If $f : A \to B$ is a homotopy equivalence, then $K_k(B) = K_k(A)$ for any field k.*

Proof. Let $g : B \to A$ be such that $f \circ g \sim \mathrm{id}_B$ and $g \circ f \sim \mathrm{id}_A$. Fix $n \in \mathbb{N}$. There are induced maps

$$f^* : \mathrm{Vect}^n_k(B) \to \mathrm{Vect}^n_k(A), \quad g^* : \mathrm{Vect}^n_k(A) \to \mathrm{Vect}^n_k(B).$$

But $f \circ g$ is homotopic to id_B, so by the previous proposition,

$$E = \mathrm{id}^*_B(E) = (f \circ g)^*(E) = (g^* \circ f^*)(E)$$

for any vector bundle $E \in \mathrm{Vect}^n_k(B)$. Similarly $f^* \circ g^*$ is the identity map. It follows that f^* is an isomorphism. By the universal properties of the direct sum and Grothendieck completion, this extends to an isomorphism on K-theories. □

Rather than going into K-theory any further, by using classifying spaces, we will classify real or complex vector bundles of a fixed rank.

Remark. We hereafter right $\mathrm{Prin}_G(X)$ to denote the isomorphism classes of principle G-bundles over X.

By the associated bundle construction, there is a natural bijection

$$\mathrm{Vect}^n_k(X) \longleftrightarrow \mathrm{Prin}_{\mathrm{GL}(n,k)}(X).$$

Therefore $\mathrm{Vect}^n_k(X)$ is classified by the classifying space $B\,\mathrm{GL}(n,k)$. In the next section we will explore examples such as the above.

The most important theorem concerning classifying spaces and universal bundles is that they are determined by homotopy type. The following result gives existence:

Theorem 1.3.7. *Let $E \to B$ be a principal G-bundle, with E contractible. Then $E \to B$ is a universal bundle, and B is a classifying space.*

We do not prove the theorem, but notice that it is analogous to universal covers. We recall that a universal cover is any covering space which is simply connected. The proof that $E \to B$ is a universal bundle follows from the same obstruction theory results.

Alternatively, existence of classifying spaces can be proven using Brown's representability theorem:

Proposition 1.3.8. Let $F : \mathbf{HoCW} \to \mathbf{Set}$ be a contravariant functor. ($\mathbf{HoCW}$ denotes the homotopy category of pointed connected CW complexes.) Then if F satisfies the following two properties, F is representable.

1. F maps coproducts to products. That is, wedge sums get mapped to cartesian products.

2. F maps homotopy pushouts to weak pullbacks. Equivalently, if $W \in \mathbf{HoCW}$ is covered by subcomplexes U, V, and $u \in F(U), v \in F(V)$ restrict to the same element in $F(U \cap V)$, then there exists $w \in F(W)$ which restricts to each of u and v.

Note that F is said to be *representable* if there exists $B \in \mathbf{HoCW}$ such that $F(-) = [-, B]$.

This immediately implies the existence of classifying spaces. The first condition is satisfied because any two principal G-bundles over a bases B_1 and B_2 glue to give a principal G-bundle over their wedge sum. The second condition is even more intuitive, as it is exactly a gluing condition for sheaves, and sections of bundles are sheaves.

This shows existence, but we also note that universal bundles are unique up to homotopy.

Proposition 1.3.9. Let $E_1 \to B_1, E_2 \to B_2$ be universal principal G-bundles. Then there is a bundle map as in the following diagram, where h is a homotopy equivalence.

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E_2 \arrow[d] \\ B_1 \arrow[r, "h"] & B_2. \end{tikzcd}$$

Proof. This result follows essentially from the definition of a universal bundle (and is analogous to all uniqueness results of objects defined via universal properties). The map h above, and a map g in the opposite direction can be constructed, and the pullbacks of the compositions are necessarily the identity map on bundles. Therefore f is a homotopy equivalence. $\square$

The previous results were not particularly constructive, so we give an outline for how one can generally construct a universal bundle.

Proof outline of the construction of the universal bundle and classifying space. Let G be a topological group. For each n, define

$$EG^n = \star^n G,$$

where $A \star B$ denotes the *join*, $A \star B := A \times B \times I / \sim$, where we identify $(a, b_1, 0) \sim (a, b_2, 0)$ and $(a_1, b, 1) \sim (a_2, b, 1)$. (This can be thought of as taking the disjoint union of A and B and drawing a line between any point in A and any point in B.) Each EG^n is $(n-1)$-connected, as well as naturally being equipped with the diagonal G action. The direct limit

$$EG := \lim_{\rightarrow} EG^n$$

is then aspherical, and is still equipped with a G action. Therefore $EG \to BG = EG/G$ is a universal bundle. $\square$

The classifying space "map" $B : G \mapsto BG$ is actually a functor. That is, given $f \in [G, H]$, we can functorially define a homotopy class $Bf \in [BG, BH]$.

Proposition 1.3.10. The classifying space map is a functor.

Proof. Let G and H be topological groups. We must show that there is a map $B : \mathrm{Hom}(G, H) \to [BG, BH]$ such that

- $B(\varphi \circ \psi) = B\varphi \circ B\psi$,
- $B(\mathrm{id}_G) = \mathrm{id}_{BG}$.

Let $\varphi : G \to H$ be a homomorphism. This induces a map $\varphi : G \to \mathrm{Homeo}(H)$ by left multiplication. This is not a group automorphism, but it preserves the H-torsor structure. Therefore the associated bundle $EG \times_\varphi H$ is a principal H-bundle over BG. This is in correspondence with a unique map $B\varphi$ in $[BG, BH]$. This is how we define our map $\mathrm{Hom}(G, H) \to [BG, BH]$.

Now we verify that our map satisfies the desired properties. Let $G \xrightarrow{\varphi} H \xrightarrow{\psi} K$ be group homomorphisms. Then

$$(EG \times_\varphi H) \times_\psi K \cong ((EG \times H)/\sim_\varphi) \times K)/\sim_\psi .$$

The equivalence relations on the right are

$$(x, y) \sim (xg, \varphi(g^{-1})y), \quad ([x, y], z) \sim ([x, y]h, \psi(h^{-1})y).$$

Rewriting the above as a single equivalence relation, we have

$$(x, y, z) \sim (xg, \varphi(g^{-1})yh, \psi(h^{-1})z).$$

Equivalently, we can write $(xg, 1, \psi(\varphi(g^{-1})y)z)$. This establishes an isomorphism

$$(EG \times_\varphi H) \times_\psi K \cong EG \times_{\psi \circ \varphi} K.$$

Therefore $B\psi \circ B\varphi = B(\psi \circ \varphi)$ as required. On the other hand, we have that $EG \times_G G \cong EG$, so $B\mathrm{id}_G = \mathrm{id}_{BG}$ as required. □

One last general result we investigate is how classifying spaces relate to products. Given two topological groups, their product admits the product topology, but also the product group structure, giving rise to a new topological group.

Proposition 1.3.11. Let G, H be topological groups. Then

$$B(G \times H) = BG \times BH.$$

Proof. $B(G \times H)$ can be constructed from an aspherical space $E(G \times H)$ with a $G \times H$ action, by

$$B(G \times H) = E(G \times H)/G \times H.$$

On the other hand, we have universal bundles $EG \to EG/G = BG$ and $EH \to EH/H = BH$. Since EG and EH are aspherical, and the homotopy groups of products are the products of homotopy groups, $EG \times EH$ is aspherical. Moreover, it is equipped by a canonical $G \times H$ action, under which

$$EG \times EH/G \times H = BG \times BH.$$

Since this is a universal bundle, $B(G \times H) \sim BG \times BH$ as required. □

So far everything has been rather abstract, so in the next section we look at some concrete examples.

1.4 Examples of classifying spaces

So far everything has been abstract, but now we give examples. In particular, we will investigate principal circle bundles (i.e. principal $U(1)$-bundles) and more generally $U(n)$-bundles. These will classify complex vector bundles. We will also study principal $O(n)$-bundles, which classify real vector bundles. On the other hand, we can restrict the base space and vary the gauge group: by studying principal G-bundles over $\mathbb{S}^n$, we will obtain some results concerning the homotopy of classifying spaces.

Proposition 1.4.1. The classifying space for the circle group is

$$BU(1) = \mathbb{CP}^\infty.$$

To understand this proposition, we begin with a brief detour into direct limits. Abstractly, these are defined using universal properties in any arbitrary category. However, we describe direct limits more concretely as we concern ourselves with topological spaces.

Definition 1.4.2. Let $X_1 \to X_2 \to X_3 \to \cdots$ be a collection of topological spaces, with $f_i : X_i \to X_{i+1}$. The *direct limit* is defined to be

$$\lim_{\to} X_i \coloneqq \coprod_i X_i / \sim$$

where $\sim$ is the equivalence relation of "eventually equal", i.e. $x_j \sim x_i$ if $(f_{j-1} \circ \cdots \circ f_i)(x_i) = x_j$. This set is equipped with the final topology with respect to the canonical functions $\varphi_i : X_i \to \lim_{\to} X_i$.

Example. Each sphere has an inclusion into higher dimensional spheres. For example, we can identify $\mathbb{S}^{n+1} = \Sigma \mathbb{S}^n$, and define $\mathbb{S}^n \to \Sigma \mathbb{S}^n$ to be the canonical inclusion. This defines a directed system, from which we define

$$\mathbb{S}^\infty = \lim_{\to} \mathbb{S}^n.$$

Suppose $f, g : \mathbb{S}^n \to \mathbb{S}^\infty$. Then these maps factor through $\mathbb{S}^{n+1}$, but $\mathbb{S}^{n+1}$ is n-connected. Therefore $\pi_n(\mathbb{S}^\infty)$ is trivial. But it follows that $\mathbb{S}^\infty$ is aspherical! (I.e. all homotopy groups vanish.) Since $\mathbb{S}^\infty$ is a CW-complex, $\mathbb{S}^\infty$ is equivalently contractible.

Since $\mathbb{S}^\infty$ is contractible, maybe we can use it to understand circle bundles! Indeed, $\mathbb{S}^1$ acts freely on $\mathbb{S}^\infty$, since $\mathbb{S}^1$ acts freely on each $\mathbb{S}^n$, and the action descends to $\mathbb{S}^\infty$ via the quotient. Therefore we obtain a universal bundle

$$\mathbb{S}^\infty \to \mathbb{S}^\infty / \mathbb{S}^1.$$

The space $\mathbb{S}^\infty/\mathbb{S}^1$ can be determined by considering dimension-wise quotients. Concretely, notice that $\mathbb{S}^{2k-1}$ is homotopic to $\mathbb{C}^k - \{0\}$. Each complex line in $\mathbb{C}^k - \{0\}$ is given by a copy of $\mathbb{C} - \{0\}$, which is homotopic to $\mathbb{S}^1$. One can show that the homotopies agree so that

$$\mathbb{S}^{2k-1}/\mathbb{S}^1 = (\mathbb{C}^k - \{0\})/(\mathbb{C} - \{0\}) = \mathbb{CP}^k.$$

In the limit, we have $\mathbb{S}^\infty/\mathbb{S}^1 = \mathbb{CP}^\infty$. This shows that the classifying space of the circle group is $\mathbb{CP}^\infty$, and the universal bundle is $\mathbb{S}^\infty \mapsto \mathbb{CP}^\infty$.

Remark. The classifying space $BU(1) = \mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$. Therefore we observe that $\pi_n(U(1)) = \pi_{n+1}(BU(1))$. We show later that this is a general property.

The fact that $\mathbb{CP}^\infty$ is an Eilenberg-Mac Lane space leads to some even better results. We showed that principal G-bundles are *representable* in the sense that there exists BG such that

$$[-, BG] = \mathrm{Prin}_G(-).$$

Recall from cohomology theory that Eilenberg-Mac Lane spaces represent singular cohomology:

Proposition 1.4.3. Let G be an abelian group. Then singular cohomology with coefficients in G is represented by Eilenberg-Mac Lane spaces:

$$[-, K(G, n)] = H^n(-; G).$$

Therefore for any space X, we have

$$\mathrm{Prin}_{U(1)}(X) = [X, BU(1)] = [X, K(\mathbb{Z}, 2)] = H^2(X; \mathbb{Z}).$$

Moreover, isomorphism classes of principal $U(1)$-bundles determines an isomorphism class of complex line bundles by the associated bundle construction. Therefore we have a canonical isomorphism

$$c_1 : \mathrm{Vect}^1_{\mathbb{C}}(X) \to H^2(X; \mathbb{Z}).$$

The isomorphism c_1 is exactly the *first Chern class*: we have shown that the first Chern class completely determines complex line bundles. In the next chapter, we will define and explore various characteristic classes (incluing the Chern classes).

On the theme of circles, we will next investigate general principal G-bundles over fixed spherical base spaces. By the general theory, we have

$$\pi_n(BG) = [\mathbb{S}^n, BG] = \mathrm{Prin}_G(\mathbb{S}^n).$$

That is, principal G-bundles over $\mathbb{S}^n$ are classified by the nth homotopy group of BG. It turns out that a general result is that

$$\pi_n(G) = \pi_{n+1}(BG)$$

for each n, as we now show.

Proposition 1.4.4. Let G be a topological group. Then

$$\pi_n(G) = \pi_{n+1}(BG).$$

It follows that that looping the classifying space returns the original space up to weak homotopy:

$$\Omega BG \sim G.$$

Finally, if G is discrete, then BG is a $K(G, 1)$.

Proof. To show that $\pi_n(G) = \pi_{n+1}(BG)$, we use the long exact sequence of homotopy associated to a fibration. That is, we have an exact sequence

$$\cdots \to \pi_{n+1}(BG) \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \cdots .$$

But EG is aspherical, so each $\pi_n(EG)$ vanishes. The first claim follows.

Next we use Eckmann-Hilton duality to observe that

$$\pi_n(\Omega BG) = [\mathbb{S}^n, \Omega BG] = [\Sigma \mathbb{S}^n, BG] = [\mathbb{S}^{n+1}, BG] = \pi_{n+1}(BG) = \pi_n(G).$$

Therefore ΩBG is weakly homotopic to G. In particular, if G is a CW complex, then they are homotopic.

Finally note that if G is discrete, then $\pi_0(G) = G$, and $\pi_n(G) = 0$ for all higher n. Therefore BG is a $K(G, 1)$. □

Corollary 1.4.5. $\mathrm{Prin}_G(\mathbb{S}^n) \cong \pi_{n-1}(G)$. *In particular, if* $\mathbb{S}^n = \mathbb{S}^1$*, then isomorphism classes of principal bundles over* $\mathbb{S}^1$ *are determined by the number of components of* G*. This verifies our earlier example where the computed the real and complex* K*-theories the circle.*

Our first example classified principal $U(1)$-bundles. We now extend to $U(n)$ for any n.

Proposition 1.4.6. $B\mathrm{GL}(n, \mathbb{C}) \sim BU(n) = \mathrm{Gr}(n, \mathbb{C}^\infty)$, so in particular

$$\mathrm{Vect}^n_{\mathbb{C}}(X) \cong [X, \mathrm{Gr}(n, \mathbb{C}^\infty)].$$

Remark. The case with $n = 1$ agrees with the first example of this section.

Proof. The conclusion is immediate, provided we prove the first claim. The homotopy equivalence of $B\mathrm{GL}(n, \mathbb{C})$ and $BU(n)$ follows from the homotopy equivalence of $\mathrm{GL}(n, \mathbb{C})$ and $U(n)$. The latter homotopy equivalence is given by the Gram-Schmidt process.

For the second equality, we begin by constructing $EU(n)$. Rather than constructing the space as a direct limit, it is easier to think in terms of an infinite dimensional Hilbert space. For our first example, $\mathbb{S}^\infty$ could have been defined as

$$\mathbb{S}^\infty = \{x \in H : \|x\| = 1\},$$

where H is an infinite dimensional complex Hilbert space. Then $U(1)$ acts by scalar multiplication (i.e. the diagonal action). This gives an equivalent bundle to the one constructed at the start of this section. More generally, we consider

$$EU(n) = \{(x_1, \ldots, x_n) \in H^n : \langle x_i, x_j \rangle = 1\}.$$

This is the space of all orthonormal n-frames in H^n, and has a canonical action of $U(n)$ (again by multiplication). The space of all frames is aspherical, and we have

$$EU(n)/U(n) = \mathrm{Gr}(n, \mathbb{C}^\infty).$$

□

An analogous result holds for real vector bundles, which we do not prove:

Proposition 1.4.7. $B\mathrm{GL}(n,\mathbb{R}) \sim BO(n) = \mathrm{Gr}(n,\mathbb{R}^\infty)$, so in particular

$$\mathrm{Vect}^n_{\mathbb{R}}(X) \cong [X, \mathrm{Gr}(n,\mathbb{R}^\infty)].$$

Example. In the line-bundle case, we have a better grasp of the topology of $\mathrm{Gr}(n,\mathbb{R}^\infty)$. That is,

$$\mathrm{Vect}^1_{\mathbb{R}}(X) \cong [X, \mathrm{Gr}(1,\mathbb{R}^\infty)] = [X, \mathbb{RP}^\infty].$$

But $\mathbb{RP}^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$. Therefore by the earlier theorem on Eilenberg-Mac Lane spaces, we have a canonical isomorphism

$$w_1 : \mathrm{Vect}^1_{\mathbb{R}}(X) \to H^1(X;\mathbb{Z}/2\mathbb{Z}).$$

Therefore real line bundles over a space are determined by the first cohomology! The isomorphism here is called the *first Stiefel-Whitney class*. In the next section we will explore this characteristic class, together with the Chern class and others.

Remark. By the associated bundle construction, we have an isomorphism

$$\mathrm{Prin}_{\mathbb{Z}/2\mathbb{Z}}(X) \cong H^1(X;\mathbb{Z}/2\mathbb{Z}).$$

That is, double covers of a space are determined by the first mod 2 cohomology. The orientable double cover of X is exactly the principal $\mathbb{Z}/2\mathbb{Z}$-bundle corresponding to $0 \in H^1(X;\mathbb{Z}/2\mathbb{Z})$.

While these classifying spaces have been fun, we can quickly see that higher dimensional real and complex vector bundles have only been classified up to a rather strange space: $\mathrm{Gr}(n,\mathbb{R}^\infty)$ in the real case, and $\mathrm{Gr}(n,\mathbb{R}^\infty)$ in the complex case. To obtain a finer understanding of vector bundles, we introduce characteristic classes in the next chapter.

To finish this chapter, we calculate and/or study some examples.

Example. What is $B(\mathbb{Z}/2\mathbb{Z})$? This was looked at earlier in the general case of real vector bundles.

$\mathbb{Z}/2\mathbb{Z}$ has a free action on $\mathbb{S}^1$ by swapping antipodes. Similarly this extends to any $\mathbb{S}^n$ by swapping antipodes, and in particular to $\mathbb{S}^\infty$. But $\mathbb{S}^\infty$ is contractible! Therefore $E(\mathbb{Z}/2\mathbb{Z}) = \mathbb{S}^\infty$, and $B(\mathbb{Z}/2\mathbb{Z}) = \mathbb{S}^\infty/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{RP}^\infty$.

Example. What is $B\mathbb{S}^1$? As derived earlier, this is $\mathbb{CP}^\infty$, and can be obtained as a quotient of the unit sphere in $\mathbb{C}^\infty$. In this case, the action of $\mathbb{S}^1$ on the unit sphere is given by the diagonal action of multiplication.

Example. What is $B\mathrm{SU}(2)$? This is similar to the previous two examples! $\mathrm{SU}(2)$ is homeomorphic to $\mathbb{S}^3$, and can be described as the unit quaternions! Therefore the unit sphere in $\mathbb{H}^\infty$ is again the infinite sphere, but now a diagonal multiplication action is given by $\mathrm{SU}(2)$. The classifying space is given by

$$\mathbb{S}^\infty / \mathrm{SU}(2) = \mathbb{HP}^\infty,$$

the *quaternion projective space*.

Example. What is $B(\mathbb{Z}/n\mathbb{Z})$? We can view $\mathbb{Z}/n\mathbb{Z}$ as a subgroup of $\mathbb{S}^1$. Therefore $\mathbb{Z}/n\mathbb{Z}$ acts freely on the infinite sphere $\mathbb{S}^\infty \subset \mathbb{C}^\infty$. The quotient is the *infinite lens space*; an analogue of the three dimensional lens space: $B(\mathbb{Z}/n\mathbb{Z}) = L^\infty(3)$. In general these lens spaces exist in all odd dimensions.

Example. What is $B\mathbb{Z}$? Since $\mathbb{Z}$ acts freely on $\mathbb{R}$ in the expected way, $B\mathbb{Z} = \mathbb{S}^1$.

These are some common examples we might consider, but a more practical question is the following: sure we can classify principal G-bundles, but what about about arbitrary fibre bundles?

Recall a proposition from earlier: if $\pi : E \to B$ is a fibre bundle with fibre F and structure group G, there exists a principal G bundle $P \to B$ such that $E = P \times_G F$. For $\pi : E \to B$ an arbitrary fibre bundle, the structure group is $\mathrm{Homeo}(F)$ or $\mathrm{Diffeo}(F)$. Therefore by the associated bundle construction, to understand all fibre bundles over a space with given fibre F, it remains to understand $B\mathrm{Homeo}(F)$ and $B\mathrm{Diffeo}(F)$. For certain F such as $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{S}$, and $\mathbb{S}^2$, this space is understood. In general it is infinite dimensional and difficult to understand! Much effort is currently being directed into understanding these spaces.

Chapter 2

Characteristic classes

In the previous chapter we introduced classifying spaces, which give a classification of principal G-bundles and vector bundles in terms of a set theoretic bijection. Next we will aim to obtain a better understanding of classifying spaces, by introducing *characteristic classes*. These are functors from categories of vector bundles or principal G-bundles into a cohomology theory, which measure "twistedness". More formally, we find that characteristic classes must vanish for a bundle to be trivial, and two bundles must have the same characteristic class if they are isomorphic. We first investigate characteristic classes in general, before focussing on the four most important types: the Stiefel-Whitney class, Chern class, Pontryagin class, and Euler class. This chapter largely follows [Coh98, Kot12, Hat03].

2.1 Characteristic classes in general

As mentioned earlier, a *characteristic class* is a functor from categories of vector bundles or principal G-bundles into a cohomology theory. In fact the clearest way to interpret characteristic classes is as *natural transformations*.

Definition 2.1.1. Let G be a topological group. Fix a cohomology theory $\mathbf{HoCW} \to \mathbf{Set}$, $X \mapsto H^*(X)$. This is a contravariant functor. Similarly, $X \mapsto \mathrm{Prin}_G(X)$ is a contravariant functor. A *characteristic class* is a natural transformation

$$c : \mathrm{Prin}_G(-) \mapsto H^*(-).$$

Explicitly, given a map $f : A \to B$ in $\mathbf{HoCW}$, we require the following diagram to commute:

$$\begin{tikzcd}
\mathrm{Prin}_G(B) \arrow[r, "c"] \arrow[d, "f^*"] & H^*(B) \arrow[d, "f^*"] \\
\mathrm{Prin}_G(A) \arrow[r, "c"] & H^*(A).
\end{tikzcd}$$

The map f^* on the left is the pullback bundle map, and the map on the right is the usual induced map on cohomology.

Remark. A priori the coefficients of the cohomology theory, or the cohomology theory itself, is not prescribed - this definition is very general! Usually we use singular cohomology.

The same definition applies to vector bundles:

Definition 2.1.2. A *characteristic class* for vector bundles over a field k is a natural transformation $\mathrm{Vect}_k(-) \to H^*(-)$.

Example. In the previous chapter we established a set theoretic bijection

$$\mathrm{Prin}_{U(1)}(B) \cong [B, K(\mathbb{Z}, 2)] \cong H^2(B; \mathbb{Z}).$$

Explicitly, the maps are given as follows: given $f : B \to K(\mathbb{Z}, 2)$, the corresponding principal $U(1)$-bundle is $f^*(\mathbb{S}^\infty)$. The map $\mathrm{Prin}_{U(1)}(B) \to [B, K(\mathbb{Z}, 2)]$ is

$$P \mapsto f, \text{ where } f \text{ is such that } f^*(\mathbb{S}^\infty) = P.$$

The map $[B, K(\mathbb{Z}, 2)] \to H^2(X; \mathbb{Z})$ is also given canonically: $f : B \to K(\mathbb{Z}, 2)$ is mapped to $f^*(u)$, where $u \in H^2(K(\mathbb{Z}, 2); \mathbb{Z}) = \mathrm{Hom}(G, G)$ is the distinguished identity element.

Define $c_1 : \mathrm{Prin}_{U(1)}(-) \to H^2(-; \mathbb{Z})$ to be the composition of the above two maps. We show that c_1 is a natural transformation. Fix $\varphi : A \to B$, and a principal $U(1)$-bundle $P \to B$. We must show that

$$\varphi^*(c_1(P \to B)) = c_1(\varphi^*(P \to B)).$$

On the left hand side, we have

$$\varphi^*(c_1(P \to B)) = \varphi^*(f^*u),$$

where $f : B \to K(\mathbb{Z}, 2)$ satisfies $f^*(\mathbb{S}^\infty) = P$. But now

$$\varphi^*(f^*u) = (f \circ \varphi)^*u,$$

and $f \circ \varphi$ satisfies $(f \circ \varphi)^*(\mathbb{S}^\infty) = \varphi^*(f^*(\mathbb{S}^\infty)) = \varphi^*(P)$. Therefore

$$(f \circ \varphi)^*u = c_1(\varphi^*(P \to B)).$$

This shows that

$$c_1 : \mathrm{Prin}_{U(1)}(-) \to H^2(-; \mathbb{Z})$$

is a characteristic class. (We call this the *first Chern class*, as mentioned in the previous chapter.) Moreover, we have natural isomorphisms

$$\mathrm{Vect}^1_{\mathbb{C}}(-) \cong \mathrm{Prin}_{\mathrm{GL}(1,\mathbb{C})}(-) \cong \mathrm{Prin}_{U(1)},$$

so that

$$c_1 : \mathrm{Vect}^1_{\mathbb{C}}(-) \to H^2(-;\mathbb{Z})$$

is a natural transformation. We will extend this to $c : \mathrm{Vect}_{\mathbb{C}}(-) \to H^2(-;\mathbb{Z})$.

The same reasoning applies to the case of $U(1)$ replaced with $O(1) = \mathbb{Z}/2\mathbb{Z}$. This gives rise to the first Stiefel-Whitney class

$$w_1 : \mathrm{Vect}^1_{\mathbb{R}}(-) \to H^1(-;\mathbb{Z}/2\mathbb{Z}).$$

The two most important properties of characteristic classes are that they are obstructions to two vector bundles or principal G-bundles being isomorphic, and in particular an obstruction to a bundle being trivial.

Proposition 2.1.3. Let c be a characteristic class. Then

- two (vector/principal) bundles $E_1, E_2 \to B$ are isomorphic only if $c(E_1) = c(E_2)$, and
- $E_1 \to B$ is trivial only if $c(E_1) = 0$ (for values of $c(E_1)$ living in non-zero degrees).

The first claim is immediate from the definition, since characteristic classes are defined on isomorphism classes! The second claim comes from naturality: the trivial bundle is obtained as a pullback of a bundle over a point, and by the dimension axiom of a cohomology theory together with naturality, $c(E_1)$ must then vanish in non-zero degrees.

It is now natural to ask how many characteristic classes there are. Perhaps they don't even exist! This has a surprisingly easy answer.

Theorem 2.1.4. *Let R be a commutative ring. Define $Char_G(R)$ to be the set of characteristic classes*

$$\mathrm{Prin}_G(-) \to H^*(-;R).$$

Given any space X, $H^*(X;R)$ is a ring, with addition given by the usual sum in cohomology, and multiplication by the cup product. Moreover, this induces the structure of a commutative ring on $Char_G(R)$ pointwise.

There is an isomorphism of rings

$$\Phi : Char_G(R) \to H^*(BG;R).$$

Proof. This is a direct application of Yoneda's lemma. Recall that there is a natural isomorphism

$$[-, BG] \cong \mathrm{Prin}_G(-).$$

Therefore

$$\mathrm{Char}_G(R) = \mathrm{Nat}(\mathrm{Prin}_G(-), H^*(-;R)) \cong \mathrm{Nat}(\mathrm{Hom}(-, BG), H^*(-;R)) \cong H^*(BG;R)$$

where the last isomorphism is exactly the Yoneda lemma. (We have written $\mathrm{Hom}(-;BG)$ to denote $[-, BG]$ to emphasise that $[X, Y]$ is exactly the set of morphism $X \to Y$ in the homotopy category.) Concretely, one can show that $\Phi(c) = c(EG)$ realises the isomorphism, where $EG \to BG$ is the universal bundle. $\square$

This shows that characteristic classes are tractable in some sense. Next we investigate some very important examples.

2.2 The Stiefel-Whitney class

We established in the previous chapter that real line bundles over a space B are classified by a map

$$w_1 : \mathrm{Vect}^1_{\mathbb{R}}(B) \to H^1(B; \mathbb{Z}/2\mathbb{Z}).$$

But each real line bundle is obtained by pulling back a universal bundle:

$$\begin{tikzcd} f^*(\mathbb{R}^\infty) \arrow[r] \arrow[d] & \mathbb{R}^\infty \arrow[d, "p"] \\ B \arrow[r, "f"] & BO(1) = \mathbb{RP}^\infty \end{tikzcd}$$

By naturality of characteristic classes, $w_1 : \mathrm{Vect}^1_{\mathbb{R}}(B) \to H^1(B; \mathbb{Z}/2\mathbb{Z})$ is obtained by pulling back the universal class

$$w_1 : \mathrm{Vect}^1_{\mathbb{R}}(BO(1)) \to H^1(BO(1); \mathbb{Z}/2\mathbb{Z}).$$

In fact, by the last theorem of the previous section, w_1 is given by some element of $H^1(BO(1); \mathbb{Z}/2\mathbb{Z})$. Wrapping this into a definition, we have the following:

Definition 2.2.1. $\mathbb{RP}^\infty$ has cohomology ring $\mathbb{Z}/2\mathbb{Z}[a]$, with generator $a \in H^1(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z})$. This generator is called the *first Stiefel-Whitney class*. Given a real line bundle $E \to B$ with classifying map $f : B \to \mathbb{RP}^\infty$, $f^*(a) \in H^1(B; \mathbb{Z}/2\mathbb{Z})$ is the *first Stiefel-Whitney class* of E.

The *Stiefel-Whitney classes* generalise this to higher degrees in cohomology, as we generalise our gauge group from $O(1)$ to $O(n)$. Since

$$H^*(BO(1); \mathbb{Z}/2\mathbb{Z}) = H^*(\mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a],$$

we have that

$$H^*(BO(1)^n; \mathbb{Z}/2\mathbb{Z}) = H^*(\prod_n \mathbb{RP}^\infty; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a_1, \ldots, a_n].$$

Definition 2.2.2. Let $k = \mathbb{R}, \mathbb{C}$ be understood. For each $n \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{\infty\}$, we write γ^j_n to denote the tautological line bundle

$$\gamma^j_n : k^{j+1} - \{0\} \to \mathrm{Gr}(n, k^{j+1}).$$

In particular, γ^1_1 denotes the tautological line bundle over the real or complex projective line.

With this definition, we can construct rank n real vector bundles $\oplus_n \gamma_1^\infty$, which have a classifying map

$$f_n : (\mathbb{RP}^\infty)^n \to \mathrm{Gr}(n, \mathbb{R}^\infty).$$

This induces a map

$$f_n^* : H^*(\mathrm{Gr}(n, \mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z}) \to H^*((\mathbb{RP}^\infty)^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a_1, \ldots, a_n].$$

It turns out that f_n^* is injective, and its image is the free algebra generated by elementary symmetric functions $\sigma_1, \ldots, \sigma_n$.

Theorem 2.2.3. *The cohomology of $BO(n)$ is a polynomial ring*

$$H^*(BO(n); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\sigma_1, \ldots, \sigma_n], \quad \sigma_i \in H^1(BO(n); \mathbb{Z}/2\mathbb{Z}).$$

*Each $\sigma_i \in H^i(BO(n); \mathbb{Z}/2\mathbb{Z})$ is termed the i*th Stiefel-Whitney class*.*

We do not give a proof of this fact, but it can be proven using the *Leray-Hirsch theorem* which we also state without proof. (A proof is available in Hatcher's *Algebraic topology.*)

Theorem 2.2.4 (Leray-Hirsch)**.** *Let $\pi : E \to B$ be a fibre bundle with fibre F. Let R be a principal ideal domain. Let $i : F \to E$ be an inclusions of a fibre. Suppose $H^*(F; R)$ is a finitely generated free R-module, and there exists $c_1, \ldots, c_N \in H^*(E; R)$ whose pullbacks i^*c_i to each fibre form a basis for $H^*(F; R)$, then $H^*(E; R)$ is a free $H^*(B; R)$ module. Moreover, there is an isomorphism*

$$H^*(B; R) \otimes_R H^*(F; R) \xrightarrow{\cong} H^*(E; R)$$

given by $\sum b_j i^*(c_j) \mapsto \sum \pi^*(b_j) c_j$.

Definition 2.2.5. Given a vector real bundle $E \to B$ of rank n, there is a classifying map $f : B \to BO(n)$. The *ith Stiefel-Whitney class* of E is defined by $w_i(E) = f^*(\sigma_i)$, where $\sigma_i \in H^1(BO(n); \mathbb{Z}/2\mathbb{Z})$ are generators (Stiefel-Whitney classes) of $H^*(BO(n); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$. The *total Stiefel-Whitney class* of E is the polynomial

$$w(E) := 1 + w_1(E) + \cdots + w_n(E).$$

Proposition 2.2.6. The Stiefel-Whitney classes satisfy the following properties:

1. Naturality: For any map $A \to B$, and vector bundle $E \to B$, $w(f^*(E)) = f^*(w(E))$.

2. Normalisation: if $(\mathbb{R}^2 - \{0\}) \to \mathbb{RP}^1$ is the tautological line bundle, then $w_1(E) = a$, for $a \in H^1(\mathbb{RP}^1; \mathbb{Z}/2\mathbb{Z})$ a generator.

3. Rank: For $E \to B$ an arbitrary vector bundle of rank n, $w_0(E) = 1 \in H^0(B)$, and for $i > n$, $w_i(E) = 0$.

4. Product formula: for vector bundles $E, F \to B$, $w(E \oplus F) = w(E) \smile w(F)$. More explicitly, for each i,
$$w_i(E \oplus F) = \sum_{j \leq i} w_j(E) \smile w_{i-j}(F).$$

Proof. The first three properties are all immediate. The naturally and rank conditions are part of the definition. As for normalisation, $H^1(\mathbb{RP}^1; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and we know that isomorphism classes of vector bundles over $\mathbb{RP}^1$ are in bijective correspondence with $H^1(\mathbb{RP}^1; \mathbb{Z}/2\mathbb{Z})$. The trivial vector bundle corresponds to 0, while the tautological vector bundle corresponds to the generator a.

The non-trivial fact that needs proving is the product formula. Recall that $BO(n) = \mathrm{Gr}(n, \mathbb{R}^\infty)$. Let $p_n : BO(n) \times BO(m) \to BO(n)$ be the projection map, and similarly for p_m. Recall the tautological bundles $\gamma_n^\infty : EO(n) \to BO(n)$ and similarly for m. These define a rank $n + m$ bundle
$$p_n^* \gamma_n^\infty \oplus p_m^* \gamma_m^\infty : EO(n) \oplus EO(m) \to BO(n) \times BO(m)$$
with classifying map
$$f : BO(n) \times BO(m) \to BO(m+n).$$
(This is really saying that the structure group $O(n+m)$ reduces to $O(n) \times O(m)$ for this sum bundle.) The classifying map induces a commutative diagram

$$\begin{tikzcd}
BO(1)^n \times BO(1)^m \arrow[r, "\widetilde{f}"] \arrow[d, "h_n \times h_m"] & BO(1)^{n+m} \arrow[d, "h_{n+m}"] \\
BO(n) \times BO(m) \arrow[r, "f"] & BO(n+m)
\end{tikzcd}$$

Here h_k is the classifying map of the product bundle $\oplus_{i=1}^k \gamma_1^\infty : EO(1)^n \to BO(1)^n$. The map $\widetilde{f}$ is the natural homeomorphism.

We noted earlier that $H^*(BO(1)^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a_1, \ldots, a_n]$, and that h_n induces an inclusion
$$h_n^*(H^*(BO(n); \mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z}[\sigma_1, \ldots, \sigma_n] \subset \mathbb{Z}/2\mathbb{Z}[a_1, \ldots, a_n],$$
where σ_i are elementary polynomials in the a_i. By the Kunneth formula, dualising the diagram gives

$$\begin{tikzcd}
H^*(BO(1)^n) \times H^*(BO(1)^m) & H^*(BO(1)^{n+m}) \arrow[l, "\widetilde{f}^*"'] \\
H^*(BO(n)) \times H^*(BO(m)) \arrow[u, "h_n^* \times h_m^*"] & H^*(BO(n+m)), \arrow[l, "f^*"'] \arrow[u, "h_{n+m}^*"']
\end{tikzcd}$$

where all cohomology is in $\mathbb{Z}/2\mathbb{Z}$. The top map is an isomorphism, and the map on the right is an inclusion. Therefore f^* is also an inclusion. Writing

$$H^*(BO(n+m)) = \mathbb{Z}/2\mathbb{Z}[\sigma_1, \ldots, \sigma_{n+m}] \subset H^*(BO(1)^n) \times H^*(BO(1)^m),$$

each σ_i must be expressed as a product of the elementary symmetric polynomials τ_i and η_i generating $H^*(BO(n))$ and $H^*(BO(m))$ respectively. This forces

$$f^*(\sigma_i) = \sum_{i=j+k} \tau_j \smile \eta_k.$$

We now apply this to arbitrary sum bundles. Let $E, D \to B$ be vector bundles of rank n and m. These have classifying maps f_E and f_D. The sum bundle $E \oplus D$ has classifying map $f_{E\oplus D}$. Recall the classifying map $f : BO(n) \times BO(m) \to BO(n+m)$.

Let $f_1 \times f_2 : B \to BO(n) \times BO(m)$ be the canonical map. Then $f \circ (f_1 \times f_2) : B \to BO(n+m)$ corresponds to the vector bundle $E \oplus D$ (under the natural bijection of maps into classifying spaces and vector bundles). Therefore $f \circ (f_1 \oplus f_2)$ is homotopic to $f_{E\oplus D}$. By naturality, this gives

$$w_i(E \oplus D) = f^*_{E\oplus D}(\sigma_i) = (f_E \times f_D)^*\Big(\sum_{i=j+k} \tau_j \smile \eta_k\Big) = \sum_{i=j+k} w_j(E) \smile w_k(D).$$

□

Proposition 2.2.7. The Stiefel-Whitney classes are determined uniquely by naturality, normalisation, rank, and the product formula.

Proof. Let $E \to B$ be a real vector bundle of rank n. Then there is a map $f : B \to BO(n)$ such that $E = f^*(\gamma_n^\infty)$. By naturality, $w(E) = f^*(w(\gamma_n^\infty))$, so it suffices to determine $w(\gamma_n^\infty)$. But recall that we have an inclusion

$$h^* : H^*(BO(n); \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^*(BO(1)^n; \mathbb{Z}/2\mathbb{Z}),$$

and that $h^*(\gamma_n^\infty) = \oplus_{i=1}^n p_i^* \gamma_1^\infty$. Therefore, again by naturality, it suffices to determine $w(\oplus_{i=1}^n p_i^* \gamma_1^\infty)$. But by the product formula, it suffices to determine $w(\gamma_1^\infty)$. By the rank axiom,

$$w_0(\gamma_1^\infty) = 1, \quad w_i(\gamma_1^\infty) = 0, i \geq 2.$$

Therefore only $w_1(\gamma_1^\infty) \in H^1(\mathbb{RP}^1; \mathbb{Z}/2\mathbb{Z})$ is not yet constrained. However, by the normalisation axiom, this must be non-zero. Since $H^1(\mathbb{RP}^1; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, there is a unique non-zero element a. This shows that $w_1(\gamma_1^\infty)$ is uniquely determined. □

Finally we consider some general properties of the Stiefel-Whitney class. One such general property expresses the first Steifel-Whitney class of any real vector bundle in terms of its determinant bundle.

Proposition 2.2.8. Let $E \to B$ be a real vector bundle of rank n. Then its *determinant bundle* is $\det E = \Lambda^n E \to B$. We have

$$w_1(E) = w_1(\det E).$$

Proof. Let $f : B \to BO(n)$ be the classifying map of E. Then $w(E) = f^*(w(\gamma_n^\infty))$. By the *splitting principle* we have an inclusion

$$h^* : H^*(BO(n); \mathbb{Z}/2\mathbb{Z}) \to H^*(BO(1)^n; \mathbb{Z}/2\mathbb{Z}),$$

so that $h^*(w(\gamma_n^\infty)) = w(\oplus_{i=1}^n p_i^* \gamma_1^\infty)$ where p_i are projection maps $BO(1)^n \to BO(1)$. By the Whitney sum formula, this gives

$$w_1(E) = f^*(w_1(\oplus_{i=1}^n p_i^* \gamma_1^\infty)) = f^*(\sum_{i=1}^n w_1(\gamma_1^\infty)).$$

But $\sum_n w_1(\gamma_1^\infty) = w_1(\otimes^n \gamma_1^\infty) = w_1(\Lambda^n \gamma_n^\infty)$, giving the desired result. $\square$

We observed near the end of the previous chapter that w_1 detects orientability. Explicitly, we mentioned that the *orientable double cover* of a manifold X is the principal bundle corresponding to $H^1(X; \mathbb{Z}/2\mathbb{Z})$. More generally, we have the following result:

Proposition 2.2.9. Let $E \to B$ be a vector bundle. Then E is orientable if and only if $w_1(E) = 0$.

Proof. An orientation corresponds to a reduction of the structure group from $O(n)$ to $SO(n)$.

Let $E \to B$ be a real vector bundle of rank n. Consider the short exact sequence of groups

$$SO(n) \to O(n) \xrightarrow{\det} O(1).$$

This induces an exact sequence on classifying spaces, which in turn induces an exact sequence

$$[X, BSO(n)] \to [X, BO(n)] \to [X, BO(1)].$$

The classifying map $f : X \to BO(n)$ lifts to a classifying map $X \to BSO(n)$ if and only its image vanishes in $[X, BO(1)] = H^1(BO(1); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. We have a commutative diagram

$$\begin{tikzcd}
{[BO(n), BO(n)]} \arrow[r, "\det_*"] \arrow[d, "f^*"] & {[BO(n), BO(1)] = H^1(BO(n); \mathbb{Z}/2\mathbb{Z})} \arrow[d, "f^*"] \\
{[X, BO(n)]} \arrow[r, "\det_*"] & {[X, BO(1)] = H^1(X; \mathbb{Z}/2\mathbb{Z}).}
\end{tikzcd}$$

We want to show that $w_1(E) = 0$ if and only if $\det_* f = 0$.

Consider the identity map $\mathrm{id}_{BO(n)} \in [BO(n), BO(n)]$. Then $\det_* \mathrm{id}_{BO(n)}$ is the classifying map of the determinant bundle of the universal bundle over $BO(n)$. By the previous proposition, together with the correspondence

$$[B, BO(1)] \longrightarrow H^1(B; \mathbb{Z}/2\mathbb{Z}), \quad [\text{classifying map } f \text{ of } E \to B] \mapsto w_1(E),$$

we have

$$\det{}_*\mathrm{id}_{BO(n)} = [\text{classifying map of } \det \gamma_n^\infty \to B] = w_1(\det \gamma_n^\infty) = w_1(\gamma_n^\infty).$$

By the definition of the Stiefel-Whitney class of an arbitrary vector bundle of rank n, we then have

$$w_1(E) = f^* w_1(\gamma_n^\infty) = f^* \det{}_*\mathrm{id}_{BO(n)}.$$

By commutativity of the diagram, we can alternatively write

$$w_1(E)\det{}_* f^* \mathrm{id}_{BO(n)} = \det{}_* f.$$

Therefore $w_1(E)$ vanishes if and only if $\det_* f$ vanishes. By exactness of the sequence introduced at the start of the proof, this is the statement that $f : X \to BO(n)$ lifts to a class $X \to BSO(n)$ if and only if $w_1(E) = 0$. □

Remark. In particular, a manifold is orientable if and only if $w_1(TM) = 0$.

The last property we investigate is additivity of the first Stiefel-Whitney class for line bundles. This is in fact a corollary of the previous result.

Proposition 2.2.10. Let $E_1, E_2 \to B$ be line bundles. Then

$$w_1(E_1 \otimes E_2) = w_1(E_1) + w_1(E_2).$$

Proof. For E_1, E_2 line bundles, we can take $O(1) = \{1, -1\}$, with fibre $\mathbb{R}$. Then clutching functions h_1, h_2 for E_1 and E_2 are given by multiplication by 1 or -1 on each overlap of local trivialisations.

Now $w(E_1 \oplus E_2) = 0$ if and only if $E_1 \oplus E_2$ is orientable. But this holds if and only if $h_1 \otimes h_2 \in SO(1) = \{1\}$. Equivalently, at any point, we require both h_1 and h_2 to have the same sign. Therefore both h_1 and h_2 have image in $SO(1)$, or neither do. That is, both $w(E_1)$ and $w(E_2)$ are 0 or 1. Since $1 + 1$ and $0 + 0$ are the unique ways of writing 0 as a sum of two numbers mod 2, we are done. □

2.3 The Chern class

The Chern class is essentially the complex analogue to the Stiefel-Whitney class. All results we state now have the same proofs as in the Stiefel-Whitney case, so no proofs are given here.

Theorem 2.3.1. *There is a canonical inclusion*

$$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[\sigma_1,\dots,\sigma_n] \subset \mathbb{Z}[a_1,\dots,a_n] = H^*(BU(1)^n;\mathbb{Z}),$$

where the a_i are generators of $H^1(BU(1);\mathbb{Z})$, and σ_i is the ith elementary symmetric polynomial in the a_i. The inclusion is induced by the classifying map $BU(1)^n \to BU(n)$ of the rank n complex vector bundle $\oplus_{i=1}^n \gamma_1^\infty$.

Definition 2.3.2. Let $E \to B$ be a complex vector bundle of rank n. The *ith Chern class* of E is defined by

$$c_i(E) = f^*(\sigma_i) \in H^{2i}(B;\mathbb{Z}),$$

where $f : B \to BU(n)$ is the classifying map of E. The *total Chern class* is defined to be

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(B;\mathbb{Z}).$$

Theorem 2.3.3. *The Chern class is uniquely determined by the following axioms:*

1. *Naturality: For any map $A \to B$, and vector bundle $E \to B$, $c(f^*(E)) = f^*(c(E))$.*
2. *Normalisation: if γ_1^1 is the tautological complex line bundle over $\mathbb{CP}^1$, then $c_1(E) = a$, for $a \in H^1(\mathbb{CP}^1;\mathbb{Z})$ a generator.*
3. *Rank: For $E \to B$ an arbitrary complex vector bundle of rank n, $c_0(E) = 1 \in H^0(B)$, and for $i > n$, $c_i(E) = 0$.*
4. *Product formula: for complex vector bundles $E, F \to B$, $c(E \oplus F) = c(E) \smile c(F)$. More explicitly, for each i,*

$$c_i(E \oplus F) = \sum_{j \le i} c_j(E) \smile c_{i-j}(F).$$

We now explore some additional properties of the Chern class. Analogously to the Stiefel-Whitney classes, we have the following three properties:

Proposition 2.3.4. Let $E \to B$ be a complex vector bundle. Then

- $c_1(\det E) = c_1(E)$.
- The structure group $U(n)$ of E reduces to an $SU(n)$ structure if and only if $c_1(E) = 0$.

- If E_1, E_2 are line bundles, then $c_1(E_1 \otimes E_2) = c_1(E_1) + c_1(E_2)$.

Proofs are analogous to ones we saw in the Stiefel-Whitney case. However, we introduce and prove two more properties.

Proposition 2.3.5. Let E^* denote the dual of a complex vector bundle $E \to B$. Then $c_i(E^*) = (-1)^i c_i(E)$.

Proof. We again use the splitting principle. First we prove the special case of a line bundle, and then extend to the general case. Let $L \to B$ be a complex line bundle. Then there is a natural isomorphism $\mathrm{Hom}(L, L) \cong L \otimes L^*$ of vector bundles. But $\mathrm{Hom}(L, L) \to B$ is a trivial line bundle, because it admits the global section $\sigma(x) = (x, \mathrm{id})$, and this induces a global trivialisation. Therefore $L \otimes L^*$ is trivial. But now

$$0 = c_1(L \otimes L^*) = c_1(L) + c_1(L^*),$$

so $c_1(L^*) = -c_1(L)$.

Next for the general case, suppose $E \to B$ is a complex vector bundle of rank n. By the splitting principal, there is a space A and $f : A \to B$ such that the pullback bundle $f^*(E)$ is a Whitney sum bundle of line bundles, and $f^* : H^*(B; \mathbb{Z}) \to H^*(A; \mathbb{Z})$ is injective. Write $f^*(E) = L_1 \oplus \cdots \oplus L_n$. Now

$$f^*(c(E)) = c(f^*(E)) = c(L_1 \oplus \cdots \oplus L_n) = \prod_{i=1}^{n} (1 + c_1(L_i)).$$

Similarly, we have

$$f^*(c(E^*)) = \prod_{i=1}^{n} (1 - c_1(L_i)).$$

Therefore, using the fact that f^* is injective, expanding these formulae give $c_i(E^*) = (-1)^i c_i(E)$ as required. $\square$

Proposition 2.3.6. Let E be a complex vector bundle of rank n. Then it is also a real vector bundle of rank $2n$. With this interpretation, we have

$$w_{2i}(E) = c_i(E) \mod 2, \quad w_{2i+1}(E) = 0.$$

Proof. We prove this by showing that the the Stiefel-Whitney classes satisfy the definition of the Chern class (reduced mod 2). That is, we define

$$c_i' = w_{2i} \in H^{2i}(-, \mathbb{Z}/2\mathbb{Z})$$

for each i, and show that c' satisfies the axioms of naturality, normalisation, rank, and the product formula. Then c_i' is necessarily the reduction of c_i, mod 2.

Naturality, rank, and the product formula are all immediate. Therefore it remains to prove the normalisation axiom. Consider the tautological bundle

$$\gamma_1^1 : (\mathbb{C}^2 - \{0\}) \to \mathbb{CP}^2.$$

We later show that the top Chern class of a complex vector bundle is the *Euler class*, and this also reduces to the top Stiefel-Whitney class mod 2. Therefore $c_1(\gamma_1^1) \mod 2 = e(\gamma_1^1) \mod 2 = w_2(\gamma_1^1)$. In particular $w_2(\gamma_1^1)$ is non-zero, so the result follows. Of course we haven't even defined the Euler class at this stage! The Euler class will be introduced at the start of the next section. □

2.4 Applications of the Stiefel-Whitney and Chern classes

The Stiefel-Whitney class is an obstruction to embedding (and in fact immersing) manifolds in larger manifolds. As an example, we show that $\mathbb{RP}^{2^k}$ cannot be immersed in $\mathbb{R}^m$ for $m < 2^{k+1} - 1$. Recall that Whitney famously proved that all n manifolds can be immersed in $\mathbb{R}^{2n-1}$! Therefore this shows that Whitney's immersion theorem gives a tight bound.

Suppose N^n is a smooth immersed submanifold of M^m. Then

$$\nu N \oplus TN = TM,$$

where νN is the normal bundle of N. By the product formula, we have

$$w(\nu N) w(TN) = w(TM).$$

But writing $w(\nu N) = 1 + w_0(\nu N) + \cdots$, any non-zero $w_i(\nu N)$ forces the rank of νN to be at least i (by the rank axiom). This forces $m - n \geq i$. Therefore N can be immersed in M only if M has dimension at least i more than N! We now give an explicit example.

Example. $w(\gamma_1^n) = 1 + a$, where $a \in H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is the generator. It follows that

$$w(T\mathbb{RP}^n) = (1+a)^{n+1} \in H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$$

as we now show. (Note that the class a^{n+1} is zero by cohomological restrictions.) We claim without proof that

$$T\mathbb{RP}^n \oplus \epsilon_1 = \oplus_{n+1} \gamma_1^n,$$

where ϵ_1 is a trivial line bundle. By the product formula, the claim follows. In particular, this means that

$$w_i(T\mathbb{RP}^n) = \binom{n+1}{i} a^i \in H^i(\mathbb{RP}^n).$$

Now suppose $\mathbb{RP}^n$ is immersed in $\mathbb{R}^m$ for some m. By the product formula, this gives

$$(1+a)^{n+1} w(\nu \mathbb{RP}^n) = w(T\mathbb{RP}^n) w(\nu \mathbb{RP}^n) = w(T\mathbb{R}^m) = 1.$$

Therefore we have an explicit formula for $w(\nu\mathbb{RP}^n)$, namely $1/(1+a)^{n+1}$. Computing $(1+a)^{n+1}$ gives

$$(1+a)^{n+1} = 1 + a + a^{2^k}$$

provided that $n = 2^k$. We study this specific example. But now

$$(1 + a + a^{2^k})(1 + a + a^2 + \cdots + a^{2^k - 1}) = 1 \in H^*(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}),$$

using the fact that $H^i(\mathbb{RP}^{2^k})$ vanishes for $i > 2^k$. This means that

$$w(\nu\mathbb{RP}^{2^k}) = 1 + a + a^2 + \cdots + a^{2^k - 1}.$$

In particular, $w_{2^k-1}(\nu\mathbb{RP}^{2^k}) \neq 0$, so by the rank axiom, $\nu\mathbb{RP}^{2^k}$ must have rank at least $2^k - 1$. In other words, $\dim \mathbb{R}^m - \dim \mathbb{RP}^{2^k} \geq 2^k - 1$. Therefore $\mathbb{RP}^{2^k}$ cannot be immersed in $\mathbb{R}^m$ for $m < 2^{k+1} - 1$ as required.

We noted at the start that if a characteristic class on a vector bundle vanishes, then the bundle is trivial. While this is not strictly an application, we note here that the converse does not hold.

Example. Consider the standard embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Then the normal bundle to $\mathbb{S}^n$ is a trivial line bundle ϵ_1. Therefore

$$w(T\mathbb{S}^n) = w(T\mathbb{S}^n)w(\epsilon_1) = w(T\mathbb{S}^n)w(\nu\mathbb{S}^n) = w(T\mathbb{R}^{n+1}) = w(\epsilon_{n+1}) = 1.$$

Hence the tangent bundles of all spheres have trivial Stiefel-Whitney class.

2.5 The Euler class

In the previous section we promised that we'd introduce the Euler class, which was used to prove that the top Chern and top Stiefel-Whitney classes were equal mod 2. We do this now. The main ingredient required for the definition of the Euler class is the *Thom isomorphism theorem*.

Definition 2.5.1. Let $p : E \to B$ be a real vector bundle of rank n. Each fibre is isomorphic to $\mathbb{R}^n$, so by taking a one-point compactification of each fibre, we obtain an n-sphere bundle $S(E) \to B$. We can further quotient $S(E)$ by B so that all of the newly added points are identified. This is the *Thom space* $T(E)$.

Remark. If B is compact, then $T(E)$ is the one-point compactification of E.

Theorem 2.5.2. *Let $p : E \to B$ be a real orientable vector bundle of rank n. Then for all k there is an isomorphism*

$$\Phi : H^k(B; \mathbb{Z}) \to \widetilde{H}^{k+n}(T(E); \mathbb{Z}).$$

(The right side is reduced cohomology.)

This is a global generalisation of the suspension isomorphism $\widetilde{H}^k(X,A) \cong \widetilde{H}^{k+n}(\mathbb{S}^n \wedge X, A)$. The theorem can be stated without reference to Thom spaces by the following calculations:

Observe that $T(E) = S(E)/B$, so that $\widetilde{H}^n(T(E)) = H^n(S(E), B)$. Next consider the triple $B \subset (S(E) - E_0) \subset S(E)$. Here $B \subset S(E) - E_0$ be embedding B as the "points at infinity", while $E_0 \cong B$ is the zero section of $S(E)$. By the long exact sequence in homology, we have

$$\begin{aligned} \cdots &\to H^{n-1}(S(E) - E_0, B) \\ &\to H^n(S(E), S(E) - E_0) \to H^n(S(E), B) \to H^n(S(E) - E_0, B) \to \cdots . \end{aligned}$$

But $S(E) - E_0$ deformation retracts to B, so $H^n(S(E), S(E) - E_0) \cong H^n(S(E), B)$! By the excision theorem, each $H^n(S(E), S(E) - E_0)$ is isomorphic to $H^n(E, E - E_0)$. Therefore the Thom isomorphism theorem can be stated by making use of an isomorphism

$$\widetilde{H}^n(T(E); \mathbb{Z}) \cong H^n(E, E - E_0; \mathbb{Z}).$$

We do this now:

Theorem 2.5.3. *Let $p : E \to B$ be an oriented vector bundle. Then there exists a unique class $u \in H^n(E, E - E_0; \mathbb{Z})$, such that for any fibre F, the restriction of u to a class in $H^n(F, F - 0; \mathbb{Z})$ is the orientation class of F. Moreover, for each k,*

$$H^k(E; \mathbb{Z}) \to H^{k+n}(E, E - E_0; \mathbb{Z}), \quad x \mapsto x \smile u$$

is an isomorphism. The class u is called the Thom class. *If E is not oriented, the theorem remains true with $\mathbb{Z}$ replaced with $\mathbb{Z}/2\mathbb{Z}$.*

Since the projection $p : E \to B$ induces an isomorphism $p^* : H^*(B) \to H^*(E)$ in cohomology, the Thom isomorphism $\Phi : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E - E_0; \mathbb{Z})$ is given by $b \mapsto p^*(b) \smile u$.

Using this map, we can define the *Euler class.*

Definition 2.5.4. Let $p : E \to B$ be an oriented real vector bundle of rank n. Let $u \in H^n(E, E - E_0; \mathbb{Z})$ be the Thom class. The *Euler class* of E, denoted by $e(E)$, is the image of u under the map

$$H^n(E, E - E_0; \mathbb{Z}) \to H^n(E; \mathbb{Z}) \to H^n(B; \mathbb{Z})$$

induced by the inclusion $B \hookrightarrow E$.

Proposition 2.5.5. The Euler class is a characteristic class.

Proof. We must show that $e : \mathrm{Vect}_{\mathbb{R}}(-) \to H^*(-)$ is a natural transformation. In other words, if $p : E \to B$ is a real vector bundle of rank n, and $f : A \to B$ is a map, then $e(f^*(E)) = f^*(e(E)) \in H^n(f^*(E); \mathbb{Z})$.

For now, suppose the Thom class is natural: $f^*(u(E)) = u(f^*(E))$. Consider the following diagram:

$$\begin{tikzcd} \widetilde{H}^n(T(E)) \arrow[r, "f^*"] \arrow[d, "i^*"] & \widetilde{H}^n(T(f^*E)) \arrow[d, "i^*"] \\ H^n(B) \arrow[r, "f^*"] & H^n(A) \end{tikzcd}$$

This diagram commutes, because each of the arrows is induced by maps i and f which commute. (Note that i and f really denote two maps each.) The image of $u(E)$ by first mapping down and then to the right is $f^*(e(E))$. The image of $u(E)$ by first mapping to the right and then down is $i^*(f^*(u(E)))$. But we assume $f^*(u(E)) = u(f^*(E)) \in \widetilde{H}^n(T(f^*E))$, so that $i^*(f^*(u(E))) = i^*(u(f^*E)) = u(f^*E)$. Therefore

$$f^*(u(E)) = u(f^*(E))$$

as required.

It remains to verify that $f^*(u(E)) = u(f^*(E))$, i.e, that the Thom class is natural. Let $f : A \to B$ be as above. There exist (unique) Thom classes $u(E)$ and $u(f^*E)$. By the definition of the pull-back bundle, $f^* : H^n(E, E - E_0) \to H^n(f^*E, f^*E - f^*E_0)$ sends the orientation class of each fibre to the orientation class. Therefore uniqueness ensures that $f^*u(E) = u(f^*E)$. This completes the proof. $\square$

As with the Chern and Stiefel-Whitney classes, the Euler class satisfies several analogous axioms.

Proposition 2.5.6. Let $p : E \to B$ be a real vector bundle of rank n. The Euler class satisfies the following properties:

- Naturality: if $f : A \to B$, then $e(f^*(E)) = f^*(e(E))$.
- Normalisation: If E admits a non-vanishing section, then $e(E) = 0$.
- Whitney sum formula: if $E' \to B$ is another oriented real vector bundle, then $e(E \oplus E') = e(E) \smile e(E')$.
- Orientation: if $\overline{E}$ is E equipped with the opposite orientation, then $e(\overline{E}) = -e(E)$.

Proof. We have already established naturality, and the orientation condition is immediate. It remains to verify normalisation and the sum formula.

Suppose E admits a non-vanishing section. We have a composition

$$B \xrightarrow{\sigma} (E - E_0) \hookrightarrow E \xrightarrow{p} B$$

which is actually the identity map. This induces maps

$$H^n(B) \xrightarrow{p^*} H^n(E) \xrightarrow{\varphi} H^n(E - E_0) \xrightarrow{\sigma^*} H^n(B).$$

Recall that $e(E)$ is defined to be the image of $u(E)$ under the composition

$$H^n(E, E - E_0) \xrightarrow{\psi} H^n(E) \xrightarrow{i^*} H^n(B),$$

and i^* is the inverse of p^* (since i^* is induced by the inclusion of B into E as the zero section). But now

$$(\varphi \circ p^*)(e(E)) = (\varphi \circ \psi)(u(E)) = 0,$$

since $\varphi \circ \psi = 0$ from the long exact sequence of relative cohomology. Therefore

$$e(E) = (\sigma^* \circ \varphi \circ p^*)(e(E)) = \sigma^*(0) = 0.$$

It now remains to prove the Whitney sum formula. Let $E_1 \to B_1$ and $E_2 \to B_2$ be oriented real vector bundles of rank n and m. Then I claim the Thom class $u(E_1 \times E_2)$ is given by $u(E_1) \otimes u(E_2) \in H^n(T(E_1)) \otimes H^m(T(E_2)) \cong H^{n+m}(T(E_1 \times E_2))$.

The isomorphism $H^n(T(E_1)) \otimes H^m(T(E_2)) \cong H^{n+m}(T(E_1 \times E_2))$ is given by the Thom isomorphism theorem. To see that $u(E_1 \times E_2) = u(E_1) \otimes u(E_2)$, it suffices to show that $u(E_1) \otimes u(E_2)$ induces the orientation class on each fibre, and then the result follows from the uniqueness of the Thom class. On each fibre, $u(E_1) \otimes u(E_2)$ restricts to $u_x \otimes u_y$ for $(x, y) \in B_1 \times B_2$. The fibres of $T(E_1)$ are homeomorphic to $\mathbb{S}^n$, and the fibres of $T(E_2)$ to $\mathbb{S}^m$. Tracing the isomorphisms, one can show that

$$u_x \otimes u_y \in H^n(\mathbb{S}^n) \otimes H^m(\mathbb{S}^m) \cong H^{n+m}(\mathbb{S}^n \wedge \mathbb{S}^m) = H^{n+m}(\mathbb{S}^{n+m})$$

is orientation class. It follows from the definition of the Euler class that

$$e(E_1 \times E_2) = e(E_1) \otimes e(E_2).$$

(Technically there may be sign difficulties: we cautiously write $e(E_1 \times E_2) = (\pm 1)^{nm} e(E_1) \otimes e(E_2)$. Then the only issue is if n, m are odd. But in these cases, we show in the a subsequent proposition that the right hand side has order two!)

The Whitney sum bundle is obtained as a pullback: this is the special case with $B := B_1 = B_2$, where we pull back both sides of the equation to $H^{n+m}(B; \mathbb{Z})$ by means of the diagonal embedding $B \to B_1 \times B_2$. Then we have

$$e(E_1 \oplus E_2) = e(E_1) \smile e(E_2).$$

$\square$

In the above proof, we promised that we would prove the following fact:

Proposition 2.5.7. Let $E \to B$ be an oriented vector bundle of odd rank. Then $e(E)$ has rank 2.

Proof. If E has odd rank, $\iota : E \to E$ defined by $(x, v) \mapsto (x, -v)$ is an orientation reversing automorphism of E. Therefore $e(E) = -e(\overline{E}) = -e(E)$ by the naturality and orientation axioms. $\square$

We observed that all characteristic classes of complex vector bundles can be expressed as polynomials of Chern classes. Since a complex vector bundle is itself an oriented real vector bundle, the Euler class of a complex vector bundle should be determined by the Chern classes. We see that this is indeed the case, and the analogous relationship holds with Stiefel-Whitney classes as well. (The proof is not included.)

Proposition 2.5.8. Let $p : E \to B$ be an oriented real or complex vector bundle. In the former case, the reduction of $e(E)$ mod 2 gives the top Stiefel-Whitney class. In the latter case, $e(E)$ is exactly the top Chern class.

Finally we mention two important results, the first of which we state without proof, which gives the Euler class a more geometric interpretation.

Theorem 2.5.9. *Let $p : E \to M$ be a real vector bundle of rank n, with M a smooth manifold of dimension d. Let $\sigma : M \to E$ be a smooth section that transversely intersects the zero section E_0. Then $Z = E_0 \cap \sigma(M)$ is a submanifold of M with codimension n, representing a homology class $[Z] \in H_{d-n}(M; \mathbb{Z})$. The Euler class $e(E)$ is the Poincaré dual of $[Z]$.*

Using this characterisation, we can understand compute a very well known invariant using the Euler class of the tangent bundle. Namely, the Euler class of the tangent bundle of an orientable manifold evaluated on the fundamental class is the Euler characteristic of the manifold. To prove this result, we make use of the Poincaré Hopf index theorem. First we must introduce some definitions.

Definition 2.5.10. Let X, Y be closed connected oriented manifolds of dimension m. Orientations correspond exactly to choices of generator $[X]$ and $[Y]$ for $H_m(X), H_m(Y)$. A map $f : X \to Y$ induces a homomorphism $f_* : H_m(X) \to H_m(Y)$. The *degree* of f is defined by

$$f_*([X]) = \deg(f)[Y].$$

Now suppose V is a vector field on a closed oriented manifold M. Then at any $p \in M$, there is a disk $D \subset M$ such that $p \in M$ is the only zero of $V|_M$. (Note that p need not be a zero!) There is a canonical map $\varphi : D \cong \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ defined by $\varphi(x) = V(x)/\|V(x)\|$. The degree of (V, p) is defined by

$$\deg(V, p) = \deg \varphi.$$

With this definition out of the way, we can introduce the celebrated Poincaré-Hopf index theorem.

Proposition 2.5.11 (Poincaré-Hopf)**.** Let M be a closed manifold, and V a vector field (i.e. a section of TM) with isolated zeroes. Then

$$\sum_{V(p)=0} \deg(V,p) = \chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

We now give a proof outline of an important application of the Euler class:

Theorem 2.5.12. *Let M be a closed oriented n-manifold. The Euler characteristic of M is given by the pairing*

$$\chi(M) = \langle e(M), [M] \rangle.$$

Proof. Recall from earlier that $e(TM)$ is Poincaré dual to $[Z]$, where $Z = V(M) \cap M$ is the transverse intersection of the zero section M with a vector field V. This means that $e(M) \frown [M] = [Z]$. But now if $1 \in \mathbb{Z} = H^0(M;\mathbb{Z})$, we have

$$\langle e(M), [M] \rangle = \langle 1 \smile e(M), [M] \rangle = \langle 1, e(M) \cap [M] \rangle = \langle 1, [Z] \rangle = [V(M)] \cdot [M].$$

The expression on the right is the signed count of intersections of $V(M)$ and M, which we can alternatively write as $\sum_{V(p)=0} I(V,p)$, where $I(V,p)$ is the sign of the intersection of V and M at p. One can show that $I(V,p) = \deg(V,p)$, so the desired result follows from the Poincaré-Hopf theorem. $\square$

2.6 The Pontryagin class

The last characteristic class we introduce is the *Pontryagin class*. In the same way that the Euler class gives an integral refinement of the Stiefel-Whitney class in a specific degree (top degree), the Pontryagin class gives integral refinements for other specific degrees. Concretely, we complexify the vector bundle, and declare the Pontryagin class to be the corresponding Chern class. We now provide details.

Definition 2.6.1. Let $E \to B$ be a real vector bundle. The *complexification* of E, denoted by $E_{\mathbb{C}}$, can be defined to be the tensor bundle $E \otimes \mathbb{C} \to B$ where the $\mathbb{C}$ factor comes from the trivial vector bundle $B \times \mathbb{C} \to B$. Equivalently, it can be defined to be the sum bundle $E \oplus E \to B$ equipped with a complex structure: $J : E \oplus E \to E \oplus E, J(x,y) = (-y,x)$.

We now define the Pontryagin class:

Definition 2.6.2. Let $p : E \to B$ be a real vector bundle. The ith Pontryagin class is defined by

$$p_i(E) = (-1)^n c_{2i}(E_{\mathbb{C}}) \in H^{4i}(B; \mathbb{Z}).$$

To motivate this definition, we show that the odd Chern classes c_{2i+1} of the complexification can be understood in terms of Stiefel-Whitney classes of E. Therefore the only new information is introduced from the even Chern classes, $c_{2i}(E_{\mathbb{C}})$.

Proposition 2.6.3. Let $p : E \to B$ be a real vector bundle. Then for any i, $2c_{2i+1}(E_{\mathbb{C}}) = 0$. In particular, it is determined by the Stiefel-Whitney classes of E.

Proof. Let $E_{\mathbb{C}}$ be the complexification $E \otimes \mathbb{C}$. The conjugate bundle is given by $E \otimes \overline{C}$. The map $x \otimes z \mapsto x \otimes \overline{z}$ is an isomorphism of complex vector bundles, so $E_{\mathbb{C}}$ and $\overline{E_{\mathbb{C}}}$ have the same Chern classes. But by a general property of Chern classes, we also know that $c_i(E_{\mathbb{C}}) = (-1)^i(\overline{E_{\mathbb{C}}})$. Therefore

$$c_i(E_{\mathbb{C}}) = (-1)^i c_i(E_{\mathbb{C}}).$$

This proves that for i odd, $c_i(E_{\mathbb{C}})$ has order two. But then it is uniquely determined by its reduction mod 2 in $H^{2i}(B; \mathbb{Z}/2\mathbb{Z})$. Such a characteristic class is necessarily a polynomial is Stiefel-Whitney classes of $E_{\mathbb{C}}$. As a real vector bundle, this is a sum-bundle $E \oplus E$, so by the Whitney sum formula, $c_i(E_{\mathbb{C}})$ is a polynomial in Stiefel-Whitney classes of E. $\square$

Next we describe how the Pontryagin class relates to other characteristic classes.

Proposition 2.6.4. For a real vector bundle $E \to B$, $p_i(E)$ maps to $w_{2i}(E) \smile w_{2i}(E)$ under the reduction $H^{4i}(B; \mathbb{Z}) \to H^{4i}(B; \mathbb{Z}/2\mathbb{Z})$. Moreover, for an orientable real $2n$-dimensional vector bundle $E \to B$, the Euler class satisfies

$$p_n(E) = e(E) \smile e(E).$$

Proof. The first of these properties is immediate from the definition. That is, $c_{2i}(E_{\mathbb{C}})$ reduces mod 2 to $w_{4i}(E \oplus E)$, and by the Whitney sum formula, this is given by

$$w_{4i}(E \oplus E) = w_{2i}(E) \smile w_{2i}(E) + 2 \sum_{j<k, j+k=4i} w_j(E) \smile w_k(E) = w_{2i}(E) \smile w_{2i}(E).$$

The result with the Euler class takes a bit more work - the proof can be found in Hatcher's notes on characteristic classes. $\square$

2.7 Exercises

Exercise 2.7.1. (Hatcher's K-theory) Show that every class in $H^{2k}(\mathbb{CP}^\infty)$ can be realised as the Euler class of a vector bundle over $\mathbb{CP}^\infty$ that is a sum of complex line bundles.

Solution. Recall that $\mathbb{CP}^\infty$ is a $K(\mathbb{Z},2)$. We first look at the case of $\alpha \in H^{2k}(\mathbb{CP}^\infty;\mathbb{Z})$ for $k>1$. Then $H^{2k}(\mathbb{CP}^\infty;\mathbb{Z})$ is trivial! But the trivial class in degree $2k$ is the Euler class of the trivial bundle $\mathbb{CP}^\infty \times \mathbb{C}^k$.

Next consider the case of $\alpha \in H^2(\mathbb{CP}^\infty) = \mathbb{Z}$. Then $\alpha = \pm n$ for some non-negative integer n. Define given a vector bundle γ, define $\gamma^{-n} = (\gamma^*)^{\otimes n}$, and $\gamma^n = \gamma^{\otimes n}$. Since the first Chern class of a complex line bundle is additive with respect to the tensor product, we have

$$\alpha = c_1(\gamma^\alpha)$$

provided that $c_1(\gamma) = 1$. For this, we simply take γ to be the tautological bundle over $\mathbb{CP}^\infty$. $\square$

Exercise 2.7.2. An important relation satisfied by Stiefel-Whitney classes is *Wu's formula*:

$$\mathrm{Sq}^i(w_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} \smile w_{j+t}.$$

Prove Wu's formula in the special case of

$$\mathrm{Sq}^1(w_j) = \sum_{t=0}^{1} \binom{j+t-2}{t} w_{1-t} \smile w_{j+t} = w_1 \smile w_j + (j-1)w_{j+1}.$$

Solution. For notational brevity, we write $u \smile v = uv$. By the splitting principal and naturality of Steenrod squares and Stiefel-Whitney classes, we work in

$$H^*(BO(1)^n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[u_1,\dots,u_n],$$

so that w_j maps to the jth symmetric polynomial in the u_i. That is,

$$w_j = \sum_{1\le k_1<\cdots<k_j\le n} u_{k_1}\cdots u_{k_j}.$$

Then

$$\mathrm{Sq}^1(w_j) = \sum_{1\le k_1<\cdots<k_j\le n} \mathrm{Sq}^1(u_{k_1}\cdots u_{k_j}).$$

We now isolate a term $\mathrm{Sq}^1(u_{k_1}\cdots u_{k_j})$. Inductively applying the Cartan formula, together with the fact that $\mathrm{Sq}^1(u_k) = u_k^2$, gives

$$\mathrm{Sq}^1(u_{k_1}\cdots u_{k_j}) = u_{k_1}\,\mathrm{Sq}^1(u_{k_2}\cdots u_{k_j}) + u_{k_1}^2 u_{k_2}\cdots u_{k_j} = \cdots = (u_{k_1}+\cdots+u_{k_j})u_{k_1}\cdots u_{k_j}.$$

In particular,

$$\mathrm{Sq}^1(w_j) = \sum_{1\le k_1<\cdots<k_j\le n} (u_{k_1}+\cdots+u_{k_j})u_{k_1}\cdots u_{k_j}.$$

We now compare the right hand side to the following expression:

$$(u_{k_1} + \cdots + u_{k_n}) \sum_{1 \le k_1 < \cdots < k_j \le n} u_{k_1} \cdots u_{k_j} = w_1 w_j.$$

By grouping terms, this can be written as

$$\begin{aligned} w_1 w_j &= (u_{k_1} + \cdots + u_{k_n}) \sum_{1 \le k_1 < \cdots < k_j \le n} u_{k_1} \cdots u_{k_j} \\ &= \sum_{1 \le k_1 < \cdots < k_j \le n} (u_{k_1} + \cdots + u_{k_j}) u_{k_1} \cdots u_{k_j} \\ &\quad + \sum_{1 \le k_1 < \cdots < k_j \le n} \Big(\sum_{k \notin \{k_1, \ldots, k_j\}} u_k \Big) u_{k_1} \cdots u_{k_j} \\ &= \mathrm{Sq}^1(w_j) + (n-j) \binom{n}{j} \Big/ \binom{n}{j+1} \sum_{1 \le k_1 < \cdots < k_{j+1} \le n} u_{k_1} \cdots u_{k_{j+1}} \\ &= \mathrm{Sq}^1(w_j) - (j-1) w_{j+1}. \end{aligned}$$

The last line requires some further clarification: notice that

$$(n-j)\binom{n}{j} = \frac{(n-j)n!}{j!(n-j)!} = \frac{(j+1)n!}{(j+1)!(n-(j+1))!} = (j+1)\binom{n}{j+1}.$$

But working mod 2, we have $j + 1 = j - 1 = -(j - 1)$. This establishes the chain of equalities above. Finally by rearranging the first and last lines, we have

$$\mathrm{Sq}^1(w_j) = w_1 w_j + (j-1) w_{j+1}$$

as required. □

Exercise 2.7.3. (Hatcher's K-theory) Show that $c_{2i+1}(E_{\mathbb{C}}) = \beta(w_{2i}(E) w_{2i+1}(E))$ where β denotes the Bockstein homomorphism associated to

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Solution. Fix a real vector bundle $E \to B$. Consider the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\text{mod } 2"] & \mathbb{Z} \arrow[r, "\text{mod } 2"] \arrow[d, "\text{mod } 4"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, "\text{id}"] & 0 \\ 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\text{mod } 2"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

We apply the cohomology functor $H^*(B; -)$ to the above diagram, to obtain the following commutative diagram, whose rows are exact:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{4i+1}(B;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\beta"] \arrow[d, "\mathrm{id}"] & H^{4i+2}(B;\mathbb{Z}) \arrow[r, "\times 2"] \arrow[d, "\bmod 2"] & H^{4i+2}(B;\mathbb{Z}) \arrow[r] \arrow[d, "\bmod 4"] & \cdots \\
\cdots \arrow[r] & H^{4i+1}(B;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\beta_2"] & H^{4i+2}(B;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\times 2"] & H^{4i+2}(B;\mathbb{Z}/4\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

Here β is the Bockstein homomorphism as given in the exercise statement, while β_2 is the Bockstein homomorphism corresponding to the exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

We are finally ready to consider $c_{2i+1}(E_{\mathbb{C}}) \in H^{4i+2}(B;\mathbb{Z})$. First recall that $c_{2i+1}(E_{\mathbb{C}}) \bmod 2 = w_{4i+2}(E_{\mathbb{C}}) \in H^{4i+2}(B;\mathbb{Z}/2\mathbb{Z})$. Next since $E_{\mathbb{C}} = E \oplus E$ as real vector bundles, by the Whitney sum formula,

$$c_{2i+1}(E_{\mathbb{C}}) \mod 2 = w_{2i+1}(E)^2.$$

When we introduced the Pontryagin class, we showed that $c_{2i+1}(E_{\mathbb{C}})$ has order 2. (Recall that we only considered even-degree Chern classes to define Pontryagin classes.) Since the mod 2 map doesn't destroy 2-torsion, $c_{2i+1}(E_{\mathbb{C}})$ is the unique lift of $w_{2i+1}(E)^2$ with the same order as $w_{2i+1}(E)^2$.

To show that $c_{2i+1}(E_{\mathbb{C}}) = \beta(w_{2i}(E)w_{2i+1}(E))$, it suffices to show that it is a lift of $w_{2i+1}(E)^2$, and that it has the same order as $w_{2i+1}(E)^2$. For the former, since the left square in the previous diagram commutes, it suffices to show that $\beta_2(w_{2i}(E)w_{2i+1}(E)) = w_{2i+1}(E)^2$. But β_2 is actually just the first Steenrod square, Sq^1. By the special case of Wu's formula proved in the previous exercise, together with the Cartan formula,

$$\begin{aligned}
\beta_2(w_{2i}w_{2i+1}) &= w_1 w_{2i} w_{2i+1} + (2i-1)w_{2i+1}^2 + w_1 w_{2i} w_{2i+1} + (2i) w_{2i} w_{2i+2} \\
&= w_{2i+1}^2.
\end{aligned}$$

Therefore $\beta(w_{2i}(E)w_{2i+1}(E))$ reduces mod 2 to w_{2i+1}^2 as required. Finally, since β has domain $H^{4i+1}(B;\mathbb{Z}/2\mathbb{Z})$, the order of $\beta(w_{2i}(E)w_{2i+1}(E))$ divides 2. If w_{2i+1}^2 is non-trivial, $\beta(w_{2i}(E)w_{2i+1}(E))$ is necessarily non-trivial, and must have order 2. If w_{2i+1}^2 is trivial, then again by commutativity $\beta(w_{2i}(E)w_{2i+1}(E))$ must be trivial (since if it had order 2, then its image under the mod 2 map would be non-trivial). Therefore $\beta(w_{2i}(E)w_{2i+1}(E))$ is indeed the unique lift of $w_{2i+1}(E)^2$ of the same order, which is $c_{2i+1}(E_{\mathbb{C}})$. $\square$

Exercise 2.7.4. (Hatcher's K-theory) For an oriented $(2k+1)$-dimensional vector bundle E show that $e(E) = \beta(w_{2k}(E))$ with β as in the preceding exercise.

Solution. The proof is essentially the same as for the previous exercise, so we only provide an outline. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{2k}(B;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\beta"] \arrow[d, "\mathrm{id}"] & H^{2k+1}(B;\mathbb{Z}) \arrow[r, "\times 2"] \arrow[d, "\bmod 2"] & H^{2k+1}(B;\mathbb{Z}) \arrow[r] \arrow[d, "\bmod 4"] & \cdots \\
\cdots \arrow[r] & H^{2k}(B;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\beta_2"] & H^{2k+1}(B;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\times 2"] & H^{2k+1}(B;\mathbb{Z}/4\mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

We know from general theory that $e(E) \in H^{2k+1}(B;\mathbb{Z})$ reduces modulo 2 to $w_{2k+1}(E)$. Moreover, since $2k+1$ is odd, $e(E)$ has order dividing 2. Therefore as in the previous proof, $e(E)$ is the unique lift of $w_{2k+1}(E)$ with the same order as $w_{2k+1}(E)$. As in the previous proof, it now suffices to show that

$$\beta(w_{2k}) \mod 2 = \beta_2(w_{2k}) = w_{2k+1}.$$

Since $\beta_2 = \mathrm{Sq}^1$, by Wu's formula we have

$$\beta_2(w_{2k}) = w_1 w_{2k} + (2k-1)w_{2k+1} = w_{2k+1}.$$

This is because $2k-1 \equiv 1 \mod 2$, and w_1 vanishes by orientability. $\square$

Exercise 2.7.5. Show that $T\mathbb{RP}^n \cong \mathrm{Hom}(\gamma_1^n, \gamma_1^{n\perp})$. Conclude that $T\mathbb{RP}^n \oplus \epsilon_1 = \oplus_{n+1}\gamma_1^n$.

Solution. We first establish some notation. Let

$$E = \{([u], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in [u]\}.$$

Then the tautological bundle over $\mathbb{RP}^n$ is $\gamma_1^n : E \to \mathbb{RP}^n$ induced from the trivial bundle. Moreover, we define

$$E^\perp = \{([u], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \perp u\}.$$

Again there is an induced bundle $\gamma_1^{n\perp} : E^\perp \to \mathbb{RP}^n$. The fibre E_x of γ_1^n over a point x is the line L_x in $\mathbb{R}^{n+1}$ determined by x. Similarly the fibre $E_x^\perp$ of $\gamma_1^{n\perp}$ over any point x is $L_x^\perp$.

Recall that there is a double cover $p : \mathbb{S}^n \to \mathbb{RP}^n$ defined by identifying antipodal points. We use this to study $T\mathbb{RP}^n$ in terms of $T\mathbb{S}^n$. On one hand, we know that

$$T\mathbb{S}^n = \{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : \|x\| = 1, x \cdot v = 0\}.$$

The derivative $Dp : T\mathbb{S}^n \to T\mathbb{RP}^n$ sends (x, v) and $(-x, -v)$ to the same vector in $T\mathbb{RP}^n$. Since the double cover is a local diffeomorphism, $Dp(x)$ is an isomorphism at each $x \in \mathbb{S}^n$. Therefore

$$T\mathbb{RP}^n = \{\{(x, v), (-x, -v)\} : x, v \in \mathbb{R}^{n+1}, \|x\| = 1, x \cdot v = 0\}.$$

Given x, a pair $\{(x, v), (-x, -v)\}$ is of course uniquely determined by v. But now define $\ell : L_x \to L_x^\perp$ by $\ell(tx) = tv$, where $L_x = \mathrm{span}\{x\} \subset \mathbb{R}^{n+1}$. This is a linear map in $\mathrm{Hom}(L_x, L_x^\perp)$, and conversely any such linear map determines a unique pair $\{(x, v), (-x, -v)\}$. Thus $T_x\mathbb{RP}^n \cong \mathrm{Hom}(L_x, L_x^\perp) \cong \mathrm{Hom}(E_x, E_x^\perp)$, and as x is allowed to vary, we have

$$T\mathbb{RP}^n \cong \mathrm{Hom}(\gamma_1, \gamma_1^\perp).$$

But now ϵ_1 denotes a trivial line bundle. We claim that

$$T\mathbb{RP}^n \oplus \epsilon_1 = \oplus_{n+1}\gamma_1^n.$$

Recall that any real vector bundle is isomorphic to its dual, by choosing a global metric g and sending (x, v) to $(x, g(v, -))$. Next note that ϵ_1 can be taken to be the trivial Hom-bundle $\mathrm{Hom}(\gamma_1^n, \gamma_1^n) \to \mathbb{RP}^n$. Therefore

$$\begin{aligned} T\mathbb{RP}^n \oplus \epsilon_1 &= \mathrm{Hom}(\gamma_1^n, \gamma_1^{n\perp}) \oplus \mathrm{Hom}(\gamma_1^n, \gamma_1^n) \\ &= \mathrm{Hom}(\gamma_1^n, \gamma_1^{n\perp} \oplus \gamma_1^n) \\ &= \mathrm{Hom}(\gamma_1^n, \epsilon_{n+1}) \\ &= \oplus_{n+1} (\gamma_1^n)^* = \oplus_{n+1} \gamma_1^n. \end{aligned}$$

□

Remark. The above reasoning applies to complex projective space as well:

$$T\mathbb{CP}^n \oplus \epsilon_1 = \oplus_{n+1} \gamma_1^n$$

where γ is the tautological bundle over $\mathbb{CP}^n$.

Exercise 2.7.6. Compute the Stiefel-Whitney and Chern classes of $\mathbb{RP}^n$ and $\mathbb{CP}^n$ respectively. Which real and complex projective spaces are parallelisable?

Solution. We provide details for the Stiefel-Whitney class calculations, but only state the Chern class calculations as they are analogous. By the Whitney sum formula, we have

$$w(T\mathbb{RP}^n) = w(T\mathbb{RP}^n) w(\epsilon_1) = \prod_{n+1} w(\gamma_1^n) = (1 + a)^{n+1},$$

where $w_1(\gamma_1^n) = a$ is the non-trivial element of $\mathbb{Z}/2\mathbb{Z} = H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$. Therefore by the binomial formula,

$$w_i(T\mathbb{RP}^n) = \binom{n+1}{i} a^i.$$

In fact, the same result holds for Chern classes:

$$c_i(T\mathbb{CP}^n) = \binom{n+1}{i} b^i, \quad \langle b \rangle = \mathbb{Z} = H^2(\mathbb{CP}^2; \mathbb{Z}).$$

For a $\mathbb{RP}^n$ to be parallelisable, we require all of its Stiefel-Whitney class to be trivial. From our calculation, this means we need $\binom{n+1}{i} \equiv 0$ for $1 \leq i \leq n$. By combinatorics, this is exactly when $n = 2^k - 1$ for some k. The same argument applies to $\mathbb{CP}^n$. (Note that these are necessary but not sufficient conditions. It turns out that the only parallelisable real projective spaces are in dimensions $1, 3, 7$.) □

Exercise 2.7.7. Which products of spheres are parallelisable?

Solution. We prove that $\mathbb{S}^n \times \mathbb{S}^m$ is parallelisable if and only if at least one of n, m is odd.

First suppose that one of n, m is odd. Without loss of generality, assume n is odd. Then $\mathbb{S}^n$ has Euler characteristic zero. Since $\chi(\mathbb{S}^n) = \langle e(T\mathbb{S}^n), [\mathbb{S}^n]\rangle$, and $[\mathbb{S}^n]$ is the generator of $H_n(\mathbb{S}^n; \mathbb{Z})$, this means $e(T\mathbb{S}^n)$ vanishes. Thus $T\mathbb{S}^n$ admits a non-vanishing section. This gives a decomposition

$$T\mathbb{S}^n = E \oplus \epsilon_1$$

where $E \to \mathbb{S}^n$ is a real vector bundle of rank $n-1$, and ϵ_1 is the trivial line bundle.

We can write $T(\mathbb{S}^n \times \mathbb{S}^m)$ as $p_1^* T\mathbb{S}^n \oplus p_2^* T\mathbb{S}^m$ where $p_1 : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{S}^n$ is the projection map and analogously for p_2. But then we have

$$\begin{aligned} T(\mathbb{S}^n \times \mathbb{S}^m) &= p_1^* T\mathbb{S}^n \oplus p_2^* T\mathbb{S}^m \\ &= p_1^*(E \oplus \epsilon_1) \oplus p_2^* T\mathbb{S}^m \\ &= p_1^* E \oplus (\epsilon_1 \oplus p_2^* T\mathbb{S}^m) \\ &= p_1^* E \oplus \epsilon_{m+1} \\ &= (p_1^* T\mathbb{S}^n \oplus \epsilon_1) \oplus \epsilon_{m-1} \\ &= \epsilon_{n+1} \oplus \epsilon_{m-1} = \epsilon_{n+m}. \end{aligned}$$

This proves that if at least one of n, m are odd, then $\mathbb{S}^n \times \mathbb{S}^m$ is parallelisable. Conversely, suppose both n and m are even. Then

$$\chi(\mathbb{S}^n \times \mathbb{S}^m) = \chi(\mathbb{S}^n)\chi(\mathbb{S}^m) = 2 \cdot 2 \neq 0.$$

Since the Euler characteristic is non-trivial, the Euler class of $T(\mathbb{S}^n \times \mathbb{S}^m)$ is non-trivial, so it is a non-trivial bundle. This proves the converse. $\square$

Exercise 2.7.8. Which spheres are parallelisable?

Solution: This is difficult! It appears I need the Bott periodicity theorem to answer this question. This will be explored in the following chapter! $\triangle$

Exercise 2.7.9. (Ciprian) Find a G-bundle which is not a principal G-bundle.

Solution. We proved earlier that a principal G-bundle is trivial if and only if it admits a global section. On the other hand, any vector bundle admits a global section (namely the zero section). Therefore a non-trivial vector bundle is an example of a G-bundle which is not principal.

For example, $\mathbb{R}$ is a Lie group, and the mobius strip is a non-trivial $\mathbb{R}$-bundle over $\mathbb{S}^1$. Similarly $\mathbb{R}^2$ is a Lie group, and $T\mathbb{S}^2$ is a non-trivial $\mathbb{R}^2$-bundle over $\mathbb{S}^2$.

We can also recreate this with discrete groups: For example, consider $\mathbb{Z}/n\mathbb{Z}$ for n at least 3. Then one can define a non-trivial $\mathbb{Z}/n\mathbb{Z}$ bundle over the circle by permuting $n-1$ points in $\mathbb{Z}/n\mathbb{Z}$ as we wrap around $\mathbb{S}^1$, but leaving one point p unchanged. Then a global section is given by $\sigma(x) = (x, p)$, but the bundle is non-trivial. Therefore it cannot be a principal $\mathbb{Z}/n\mathbb{Z}$-bundle. $\square$

Exercise 2.7.10. (Ciprian) Given homogeneous polynomials p_i of degree d_i in $n+1$ variables, for $i = 1, \dots, n-2$, let

$$S = S(d_1, ..., d_{n-2}) = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : p_i(z_0, \dots, z_n) = 0, \forall i\}.$$

One can show that for generic p_i, the subset S is a smooth, simply connected, four-dimensional submanifold of $\mathbb{CP}^n$, and that its diffeomorphism type depends only on the degrees d_i, not on the particular polynomials p_i. The manifold S is called the *complete intersection surface* of multidegree $(d_1, \dots, d_{n-2})$.

(a) Compute the Chern class of S.

(b) Compute the Euler characteristic and signature of S.

(c) Show that S is spin if and only if $\sum d_i - (n+1)$ is even.

Solution. (a) We wish to understand the tangent bundle of S in terms of its normal bundle and the tangent bundle of $\mathbb{CP}^n$. Suppose $M_1, M_2 \subset X$ are submanifolds intersecting transversely. Then $N(M_1 \cap M_2) = NM_1 \oplus NM_2$. Since S is obtained as the intersection of $n-2$ submanifolds

$$M_i = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n : p_i(z_0, \dots, z_n) = 0\},$$

it suffices to understand NM_i for some i to determine NS.

Recall that isomorphism classes of line bundles are in one to one correspondence with their first Chern classes (and NM_i is a line bundle over M_i). But then $e(NM_i) = c_1(NM_i)$, and $e(NM_i)$ is Poincaré dual to $[M_i \cap \sigma(M_i)]$ where $\sigma(M_i)$ is a generic section of NM_i. This is really just the self intersection $[M_i \cdot M_i]$. By algebraic geometry, for M_i a hypersurface determined by a degree d_i polynomial in $\mathbb{CP}^n$, this can be shown to be $d_i[M_i \cdot \mathbb{CP}^{n-1}]^*$, where multiplication by d_i uses the $\mathbb{Z}$-module structure of $H^2(M_i; \mathbb{Z})$. In particular, it follows that

$$c(NM_i) = 1 + d_i[M_i \cdot \mathbb{CP}^{n-1}]^*.$$

But now since NS is the direct sum of the NM_i restricted to S, we have

$$c(NS) = \prod_{i=1}^{n-2}(1 + d_i[M_i \cdot \mathbb{CP}^{n-1}]^*|_S) = \prod_{i=1}^{n-2}(1 + d_i\eta)$$

where $\eta = [S \cdot \mathbb{CP}^{n-1}]^*$. On the other hand, it is already understood that $c(T\mathbb{CP}^n) = (1+b)^{n+1}$ where $b \in H^2(\mathbb{CP}^n; \mathbb{Z})$ is the generator, $[\mathbb{CP}^{n-1}]^*$. Therefore

$$c(T\mathbb{CP}^n|_S) = (1 + [\mathbb{CP}^{n-1}]^*|_S)^{n+1} = (1+\eta)^{n+1}.$$

It follows that

$$c(TS)\prod_{i=1}^{n-2}(1 + d_i\eta) = (1+\eta)^{n+1}.$$

Since S is a 4-manifold, we can write $c(TS) = 1 + c_1(TS) + c_2(TS)$ by the rank axiom. On the other hand, we have

$$\prod_{i=1}^{n-2}(1 + d_i\eta) = \sum_{j=0}^{n-2}\sigma_j\eta^j$$

where σ_j is the jth elementary polynomial in $d_1, \ldots, d_{n-2}$. The last term expands by the binomial theorem, giving

$$(1+\eta)^{n+1} = \sum_{j=0}^{n}\binom{n+1}{j}\eta^j.$$

Note that the η^{n+1} term dies by rank considerations. Collecting these three expressions we have

$$(1 + c_1(TS) + c_2(TS))\sum_{j=0}^{n-2}\sigma_j\eta^j = \sum_{j=0}^{n}\binom{n+1}{j}\eta^j.$$

This forces

$$c_1(TS) = c_1(T\mathbb{CP}^n|_S) - c_1(NS) = (n+1)\eta - \sigma_1\eta = (n+1-d_1-\cdots-d_{n-2})\eta$$

and

$$\begin{aligned} c_2(TS) &= c_2(T\mathbb{CP}^n|_S) - c_2(NS) - c_1(NS)c_1(TS) \\ &= \frac{n(n+1)}{2}\eta^2 - \sigma_2\eta^2 - \sigma_1\eta(n+1-d_1-\cdots-d_{n-2})\eta \\ &= \left(\frac{n(n+1)}{2} - (n+1)(d_1+\cdots+d_{n-2}) + \left(\sigma_2 + \sum_i d_i^2\right)\right)\eta^2. \end{aligned}$$

(b) We now use the above identity to determine the Euler characteristic and signature of S. The Euler characteristic is given by

$$\chi(S) = \langle c_2(TS), [S]\rangle.$$

Using the expression for $c_2(TS)$ obtained above gives

$$\chi(S) = \left(\frac{n(n+1)}{2} - (n+1)(d_1+\cdots+d_{n-2}) + \left(\sigma_2 + \sum_i d_i^2\right)\right)\langle\eta^2, [S]\rangle.$$

It remains to determine $\langle\eta^2, [S]\rangle$. This can be shown to be $d_1 + \cdots d_{n-2}$ by using the earlier result that the self intersection of each M_i is d_i times the intersection of M_i with $\mathbb{CP}^{n-1}$. For notational brevity, we write D to denote the sum of the d_i. Then

$$\chi(S) = D\frac{n(n+1)}{2} - D^2(n+1) + D(D^2 - \sigma_2).$$

For example, if each $d_i = 1$, then we have

$$\chi(S) = (n-2)\frac{n(n+1)}{2} - (n-2)^2(n+1) + (n-2)\frac{(n-2)(n-1)}{2} = 3$$

independent of n!

Next we determine the signature of S. By the Hirzebruch signature theorem, $\sigma(S) = \frac{1}{3}\langle p_1(TS), [S]\rangle$, where p_1 is the first Pontryagin class. By definition, the Pontryagin class is given by

$$p_1(TS) = -c_2(TS \otimes \mathbb{C}) = -c_2(TS \oplus T^*S) = \sum_{i=0}^{2} c_i(TS) \smile c_{2-i}(T^*S).$$

Since $c_i(TS) = (-1)^i c_i(T^*S)$, this gives $p_1(TS) = c_1^2(TS) - 2c_2(TS)$. In our case, a calculation gives

$$p_1(TS) = \frac{n(n-1)}{2} - D(n+1) + \sigma_2.$$

In particular, the signature of S is given by

$$\frac{1}{3}\langle p_1(TS), [S]\rangle = \frac{D}{3}\Big(n + 1 - \sum_i d_i^2\Big).$$

(c) An orientable manifold M is spin if and only if $w_2(TM)$ vanishes. In this case S is orientable, so S is spin if $w_2(TS)$ vanishes. Since the second Stiefel-Whitney class is the mod 2 reduction of $c_1(TS)$, S is equivalently spin if and only if $c_1(TS)$ is even. We showed above that $c_1(TS) = (n+1-D)\eta$, so S is spin if and only if $D - (n+1)$ is even, which is what we wanted to show.

Finally, a remark that was not requested in the exercise statement: the homeomorphism class of a closed simply connected smooth 4-manifold X is determined by $e(TX), p_1(TX)$, and $w_2(TX)$. We calculated each of these invariants above, and so we have shown that the homeomorphism class of the surface S is independent of the choice of polynomials (depending only of n and d_i). $\square$

Chapter 3

K-theory

3.1 Basic definitions

In the previous two chapters, we studied vector bundles using classifying spaces and characteristic classes. In each case, bundles are studied for a given dimension. In *K-theory*, we study all k-vector bundles (independent of dimension) over a given space X at the same time (by associating to X a certain ring called the K-theory of X). K-theory is the prototypical example of a *generalised cohomology theory*, namely a theory in which the cohomology of a point is not necessarily concentrated in degree 0. The primary sources for this chapter are [Coh98, Hat03].

We recall some definitions from earlier in these notes:

Given an abelian monoid, its *completion* is an abelian group as described in the following proposition. It is the smallest abelian group containing the given monoid.

Proposition 3.1.1. Let M be an abelian monoid. There exists an abelian group $G(M)$, unique up to isomorphism, and a monoid homomorphism $\iota : M \to G(M)$ such that any morphism $M \to N$ of monoids extends uniquely to a morphism $G(M) \to N$ of monoids. In other words, for any f as in the diagram below, there exists a unique $\widetilde{f}$ making the diagram commute.

$$\begin{tikzcd} M \arrow[r, "\iota"] \arrow[d, "f"] & G(M) \arrow[dl, dotted, "\widetilde{f}"] \\ N & \end{tikzcd}$$

The abelian group $G(M)$ is called the *Grothendieck completion* of M, and can be thought of as the smallest group containing M.

Definition 3.1.2. Fix a space X, and $k = \mathbb{R}, \mathbb{C}$. Recall that the Whitney sum defines an operation

$$\mathrm{Vect}^n_k(X) \times \mathrm{Vect}^m_k(X) \to \mathrm{Vect}^{n+m}(X),$$

which turns $\mathrm{Vect}_k(X) = \bigoplus_n \mathrm{Vect}_k^n(X)$ into an abelian monoid. The *K-theory* $K_k(X)$ of X is defined to be the Grothendieck completion of $\mathrm{Vect}_k(X)$. The standard notation is to write

$$K(X) \coloneqq K_{\mathbb{C}}(X), \quad KO(X) \coloneqq K_{\mathbb{R}}(X).$$

We think of $K(X)$ as being the space of isomorphism classes of complex vector bundles over X. There is a reduced version, which is defined to be the space of stable isomorphism classes of complex vector bundles:

Definition 3.1.3. Fix a space X, and $k = \mathbb{R}, \mathbb{C}$. Two vector bundles $E_1, E_2 \to X$ are said to be *stably isomorphic* if there exist trivial bundles ϵ_1 and ϵ_2 such that $E_1 \oplus \epsilon_1 \cong E_2 \oplus \epsilon_2$. The *reduced K-theory* $\widetilde{K}_k(X)$ of X is defined to be the group of stable isomorphism classes of k-vector bundles over X, under Whitney sum. The standard notation is to write

$$\widetilde{K}(X) \coloneqq \widetilde{K}_{\mathbb{C}}(X), \quad \widetilde{KO}(X) \coloneqq \widetilde{K}_{\mathbb{R}}(X).$$

Proposition 3.1.4. The reduced K-theory is really a group (without requiring the Grothendieck completion).

Proof. The reduced K-theory is easily seen to be an abelian monoid under the Whitney sum. It remains to prove that inverses exist.

This is true provided that the base space X is compact and Hausdorff. The approach is to first show that any vector bundle $E \to X$ is a subbundle of a trivial bundle $X \times k^n \to X$ for some n. We then use this to find a complement of E inside $X \times k^n$.

Let $\{U_i\}$ be a finite open cover of X, so that each U_i is also the chart of a local trivialisation $f_i : \pi^{-1}(U_i) \to U_i \times k^m$ of X. (This can be done by compactness.) Next since X is paracompact and Hausdorff, we can find a partition of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$. We use this to extend f_i to a global bundle morphism $E \to X \times k^m$, by

$$e \mapsto \varphi_i(\pi(e)) f_i(e).$$

This map is not injective, since for any fibre $\pi(F) \notin U_i$, all points in F map to the same point in $X \times k^m$. To remedy this, we glue together all of the $\varphi_i f_i$ to give a map into the direct sum bundle:

$$E \to X \times \left(\bigoplus_i k^m \right) = X \times k^n.$$

Since at least one φ_i is non-zero above any $x \in X$, this map is injective. Therefore E is a subbundle of $X \times k^n$.

Next we use partitions of unity again to recall that any vector bundle admits a Riemmanian metric. In particular, $X \times k^n$ admits a metric. The orthogonal complement of E in $X \times k^n$ is another vector bundle, and it satisfies $E \oplus E^\perp = X \times k^n = \epsilon_n$.

Therefore $E \oplus E^\perp$ is stably isomorphic to the zero bundle ϵ_0. We have shown the existence of inverses as required. $\square$

Proposition 3.1.5. Given a space X, the abelian group $K(X)$ is naturally a ring. The multiplicative structure is induced from tensor products of vector bundles. The class of the trivial line bundle is the multiplicative unit.

With this established, we have an alternative definition of $\widetilde{K}(X)$ which also induces a multiplicative structure. Consider the map

$$r : K(X) \to K(x_0)$$

which sends each class to its restriction to x_0. This is easily seen to be a ring homomorphism. Moreover, $\ker r$ can be identified with $\widetilde{K}(X)$.

Proposition 3.1.6. K and $\widetilde{K}$ are contravariant functors (from topological spaces, in particular compact manifolds, to rings). The map is induced by pullbacks.

Proof. For $\widetilde{K}$ the result is immediate, since it is literally the space of stable isomorphism classes of vector bundles. For K, we must use the universal property. The corresponding diagram looks like this:

$$\begin{tikzcd}
\mathrm{Vect}_k(Y) \arrow[r, "i_Y"] \arrow[d, "f^*"] \arrow[dr, dotted] & K_k(Y) \arrow[d, dotted, "\widetilde{i_X f^*}"] \\
\mathrm{Vect}_k(X) \arrow[r, "i_X"] & K_k(X)
\end{tikzcd}$$

The vertical map on the right is a group homomorphism (which exists and is unique). One can show that it is a ring homomorphism. □

Finally for K-theories to really be useful, we need some ways to relate K-theories beyond just the functorially induced map. One such example is the *exterior product.*

Definition 3.1.7. Let X_1, X_2 be spaces. Then there are canonical projections $p_i : X_1 \times X_2 \to X_i$. By means of pullback, any vector bundle over X_i defines a vector bundle over $X_1 \times X_2$. In particular, a pair of vector bundles (E_1, E_2) over X_1 and X_2 respectively defines a vector bundle over $X_1 \times X_2$ by

$$(E_1, E_2) \mapsto p_1^*(E_1) \otimes p_2^*(E_2).$$

This induces a map

$$\mu : K(X_1) \otimes K(X_2) \to K(X_1 \times X_2)$$

called the *exterior product.*

Proposition 3.1.8. The exterior product is a ring homomorphism.

Proof. The tensor product $K(X_1) \otimes K(X_2)$ is a ring, with multiplication defined by $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$. We show that μ is multiplicative. Let $a \otimes b, c \otimes d \in K(X_1) \otimes K(X_2)$. Then

$$\begin{aligned}\mu((a \otimes b)(c \otimes d)) &= \mu(ac \otimes bd) \\ &= p_1^*(ac) \otimes p_2^*(bd) \\ &= p_1^*(a) \otimes p_1^*(c) \otimes p_2^*(b) \otimes p_2^*(d) \\ &= (p_1^*(a) \otimes p_2^*(b)) \otimes (p_1^*(c) \otimes p_2^*(d)) = \mu(a \otimes b) \otimes \mu(c \otimes d).\end{aligned}$$

□

As short hand, we write

$$a * b := \mu(a \otimes b).$$

We now state our last theorem for this section.

Theorem 3.1.9. *The exterior product*

$$\mu : K(X) \otimes K(\mathbb{S}^2) \to K(X \times \mathbb{S}^2)$$

is an isomorphism. More explicitly, taking H to be the canonical line bundle over $\mathbb{S}^2 = \mathbb{CP}^1$, there is a natural ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(\mathbb{S}^2)$. Then

$$\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times \mathbb{S}^2)$$

is an isomorphism.

This is in fact the core of the Bott periodicity theorem, and takes a lot of work to prove. We give a very brief outline here.

- Suppose $E \to \mathbb{S}^2$ is a vector bundle. We can decompose $\mathbb{S}^2$ as $D^2 \sqcup_{\mathbb{S}^1} D^2$. Then E restricts to trivial bundles over each copy of D^2. Therefore the only data required to specify $E \to \mathbb{S}^2$ is a clutching function $\mathbb{S}^1 \to \mathrm{GL}_n(\mathbb{C})$.

- We now generalise this construction: let $E \to X$ be a vector bundle. Let f be an automorphism of the product bundle $E \times \mathbb{S}^1 \to X \times \mathbb{S}^1$. Then the data (E, f) determines a vector bundle $[E, f]$ obtained by gluing trivial bundles $E \times D^2 \to X \times D^2$ along $X \times \mathbb{S}^1$ by f. We call f a *clutching function.*

- $f : X \times \mathbb{S}^1 \to X \times \mathbb{S}^1$ is said to be a *Laurent polynomial clutching function* if it is of the form

  $$f(x, z) = \sum_{|i| \leq n} a_i(x) z^i,$$

  where $a_i : E \to E$ restricts to linear maps on each fibre. One can show that every vector bundle $[E, f]$ is isomorphic to $[E, \ell]$ where ℓ is a Laurent polynomial clutching function. Moreover, any two homotopic Laurent polynomial clutching functions are homotopic via a Laurent polynomial clutching function. The main idea in the proof of this fact is to find appropriate coefficient functions a_i. These are obtained by a familiar formula from Fourier analysis:

$$a_n(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x, e^{i\theta})e^{-in\theta}\, d\theta.$$

- We have reduced clutching functions to Laurent polynomial clutching functions. These further reduce to *polynomial* clutching functions by writing $\ell = z^{-m}q$ for sufficiently large m. Then $[E, \ell] \cong [E, q] \otimes H^{-m}$, so to understand $[E, \ell]$ it suffices to understand $[E, q]$.

- Next we reduce polynomial clutching functions to linear clutching functions. If q is a polynomial clutching function of degree at most n, one can show that

$$[E, q] \oplus [nE, 1] \cong [(n+1)E, L^n q],$$

  where $L^n q$ is a linear clutching function. This is really a linear algebra fact that can be established on fibres.

- It remains to understand clutching functions $a(x)z + b(x)$. It turns out that given $[E, a(x)z + b(x)]$, there is a splitting $E \cong E_+ \oplus E_-$ with

$$[E, a(x)z + b(x)] \cong [E_+, 1] \oplus [E_-, z].$$

- We are finally ready to proceed with the proof that

$$\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times \mathbb{S}^2)$$

  is an isomorphism. We must show surjectivity and injectivity. For surjectivity, using the reductions above, any $[E, f]$ over $X \times \mathbb{S}^2$ can be written as

$$[E, f] = ((n+1)E)_+ * H^{-m} + ((n+1)E)_- * H^{1-m} - nE * H^{-m}.$$

  To show that μ is injective, we must define a map $\nu : K(X \times \mathbb{S}^2) \to K(X) \otimes \mathbb{Z}[H]/(H-1)^2$ so that $\nu\mu = \mathrm{id}$. It turns out that the map

$$\mu : [E, z^{-m}q] = ((n+1)E)_- \otimes (H-1) + E \otimes H^{-m}$$

  works.

3.2 K-theory as cohomology, and the Bott periodicity theorem

K-theory is often referred to as the prototypical generalised cohomology theory. We describe this notion now, in two steps.

Proposition 3.2.1. Let $A \hookrightarrow X$ be an inclusion. Then the composition $A \to X \to X/A$ induces an exact sequence

$$\widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A).$$

Moreover, this further induces a long exact sequence

$$\cdots \widetilde{K}(\Sigma X) \to \widetilde{K}(\Sigma A) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A).$$

Proof. For the first part, we must show that

$$\operatorname{im} q^* = \ker i^*.$$

We break this down into two smaller arguments: namely, showing that $i^* \circ q^* = 0$ (i.e. $\operatorname{im} q^* \subset \ker i^*$), and showing that $\ker i^* \subset \operatorname{im} q^*$. For the former, notice that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "q'"] & X \arrow[d, "q"] \\ A/A \arrow[r, "i'"] & X/A \end{tikzcd}$$

Therefore

$$i^* \circ q^* = (q \circ i)^* = (i' \circ q')^* = q'^* \circ i'^*.$$

But A/A has the homotopy type of a point, so all vector bundles over A/A are trivial. In particular, all vector bundles are stably trivial, so $\widetilde{K}(A/A)$ is trivial! Therefore i'^* has trivial image, and the composition $i^* \circ q^*$ must be trivial.

For the other direction, suppose $E \to X$ lives in the kernel of i^*. This means the restriction of E to a bundle over A is stably trivial. Without loss of generality, we can take E to be trivial over A. Let $h : p^{-1}(A) \to A \times k^n$ be the trivialisation. We have a projection map

$$E \to X \to X/A$$

which is not a vector bundle, since the fibre over the point A in X/A is $A \times k^n$ rather than k^n. However, using h, we can define a projection map

$$E/h \to X/A$$

where E/h is the quotient of E by the relation $h^{-1}(x,v) \sim h^{-1}(y,v)$ for all $x, y \in A$. Moreover, E/h is a vector bundle provided that E/h is trivial in a neighbourhood of A/A. This is indeed true, since if $E \to X$ is trivial over A, then it is trivial over a neighbourhood of A which deformation retracts onto A.

Consider the following diagram:

$$\begin{tikzcd} E \arrow[r, "q'"] \arrow[d, "p"] & E/h \arrow[d, "p'"] \\ X \arrow[r, "q"] & X/A \end{tikzcd}$$

Our goal is to show that $E \to X$ (which lies in $\ker i^*$) is in fact in the image of q^*. To this end, we show that $E = q^*(E/h)$. This follows from the fact that q' defines an isomorphism on fibres in the above commutative diagram.

This completes the proof that $A \hookrightarrow X \to X/A$ induces an exact sequence

$$\widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A).$$

Next we extend this result to give a long exact sequence. This is a rather quick argument, which relies on the following fact: *if $A \subset X$ is contractible, then the quotient map $X \to X/A$ induces an isomorphism $\widetilde{K}(X/A) \to \widetilde{K}(X)$.* In particular, cones of spaces are contractible! Consider the following sequence of inclusions, where the vertical maps are quotients.

$$\begin{tikzcd} A \arrow[r] & X \arrow[r] & X \cup CA \arrow[r] \arrow[d] & (X \cup CA) \cup CX \arrow[r] \arrow[d] & ((X \cup CA) \cup CX) \cup C(X \cup CA) \arrow[r] \arrow[d] & \cdots \\ & & X/A & \Sigma A & \Sigma X & \end{tikzcd}$$

To extend the diagram to the right, we add the cone over the space in the preceding inclusion. The vertical quotient is then the quotient by this cone. One can show that for any spaces Y, Z, there are homeomorphisms as shown in the following sequence:

$$Y \to Y \cup CZ \to \frac{(Y \cup CZ) \cup CY}{CY} \cong \Sigma Z \cong \frac{Y \cup CZ}{Y}.$$

Therefore by the result proved above, we have an exact sequence

$$\widetilde{K}((Y \cup CZ)/Y) \to \widetilde{K}(Y \cup CZ) \to \widetilde{K}(Y).$$

But now by the italicised fact, the vertical map induces an isomorphism of reduced K-theories. Therefore we have an exact sequence

$$\widetilde{K}((Y \cup CZ) \cup CY) \to \widetilde{K}(Y \cup CZ) \to \widetilde{K}(Y).$$

By taking $Y_2 = Y \cup CZ$, and $Z_2 = Y$, we immediately obtain an exact sequence

$$\widetilde{K}(((Y \cup CZ) \cup CY) \cup C(Y \cup CZ)) \to \widetilde{K}((Y \cup CZ) \cup CY) \to \widetilde{K}(Y \cup CZ).$$

Inductively we obtain a long exact sequence. By using the italicised fact, each of the unions of cones can be replaced with an appropriate suspension, giving the desired long exact sequence. $\square$

Example. Consider the sequence

$$A \to A \vee B \to B.$$

This induces an exact sequence of reduced K-theories

$$\widetilde{K}(B) \to \widetilde{K}(A \vee B) \to \widetilde{K}(A).$$

This is a split short exact sequence, as sections and retractions of the two maps above are induced by the dual sequence $B \to A \vee B \to A$. Therefore we have an isomorphism

$$\widetilde{K}(A \vee B) \cong \widetilde{K}(A) \oplus \widetilde{K}(B).$$

Proposition 3.2.2. There is a reduced version of the external product,

$$\beta : \widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y).$$

Proof. The reduced external product will arise as a restriction of both the domain and codomain as in the following diagram:

$$\begin{tikzcd}
K(X) \otimes K(Y) \arrow[r, "\cong"] \arrow[d] & (\widetilde{K}(X) \otimes \widetilde{K}(X)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \arrow[d] \\
K(X \times Y) \arrow[r, "\cong"] & \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}
\end{tikzcd}$$

The map on the right respects the factorisation, with the restrictions to $\widetilde{K}(X), \widetilde{K}(Y)$, and $\mathbb{Z}$ all given by identity maps. The reduced external product is obtained by removing these factors. We do not prove the result, but give an example calculation.

The smash product is the quotient

$$X \wedge Y = \frac{X \times Y}{X \vee Y}.$$

By the long exact sequence of reduced K-theory, we have an exact sequence

$$\widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X \vee Y) \cong \widetilde{K}(X) \oplus \widetilde{K}(Y).$$

The second map admits a section:

$$\widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \times Y), \quad (a, b) \mapsto p_1^* a + p_2^* b.$$

In particular, the last map is surjective, so the exact sequence extends on the right by zero. Extending the above sequence to the left, we also have

$$\widetilde{K}(\Sigma(X \times Y)) \to \widetilde{K}(\Sigma(X \vee Y)) \cong \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y) \to \widetilde{K}(X \wedge Y).$$

Similarly to above, the first map admits a section. This means the map

$$\widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y) \to \widetilde{K}(X \wedge Y)$$

is the zero map. In summary we have a short exact sequence

$$0 \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y) \to 0$$

where the second non-trivial map splits. Therefore the above is a split short exact sequence, giving $\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$. Next we recall that

$$\widetilde{K}(X \times Y) = \ker i^*, \quad i^* : K(X \times Y) \to K(x) \cong \mathbb{Z}.$$

This is itself part of a split short exact sequence, so that

$$K(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}.$$

$\square$

At the end of the previous section, we mentioned the *fundamental product theorem*, namely that $\mu : K(X) \otimes K(\mathbb{S}^2) \to K(X \times \mathbb{S}^2)$ is an isomorphism. We next rephrase this, and adapt it to the reduced exterior product.

Theorem 3.2.3. *There is a natural ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(\mathbb{S}^2)$. Here H represents the canonical line bundle over $\mathbb{CP}^1 \cong \mathbb{S}^2$; the aforementioned morphism is the observation that $(H-1)^2 = 0$. Combining this with the exterior product, we have a natural map*

$$\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times \mathbb{S}^2).$$

This is an isomorphism for X compact and Hausdorff.

In reduced K-theory, this gives an isomorphism

$$\beta : \widetilde{K}(X) \to (\widetilde{\Sigma}^2 X), \quad \beta : a \mapsto (H-1) * a.$$

Theorem 3.2.4 (Bott periodicity theorem)**.**

$$\widetilde{K}(\mathbb{S}^n) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z} & n \text{ even.} \end{cases}$$

Proof. This follows from the fact that $\Sigma^2\mathbb{S}^n = \mathbb{S}^{n+2}$. Therefore it suffices to compute the reduced K-theories of $\mathbb{S}^0$ and $\mathbb{S}^1$. (For the purposes of K-theory), we take vector bundles to be locally trivial but to not necessarily have constant rank. In practice this means that any vector bundles over a connected space have a well defined rank, but vector bundles over a disconnected space need not have a rank. In particular, stable isomorphism classes of vector bundles over a two point space (i.e. $\mathbb{S}^0$) do not have ranks, but they have well defined *differences of ranks* between the two fibres. Therefore $\widetilde{K}(\mathbb{S}^0) \cong \mathbb{Z}$.

On the other hand, every vector bundle over $\mathbb{S}^1$ is trivial, so the group of stable isomorphism class of vector bundles over $\mathbb{S}^1$ is the trivial group, giving $\widetilde{K}(\mathbb{S}^1) \cong 0$. Now the theorem follows from the isomorphism

$$\widetilde{K}(\mathbb{S}^n) \cong \widetilde{K}(\Sigma^2\mathbb{S}^n) \cong \widetilde{K}(\mathbb{S}^{n+1}).$$

$\square$

An application of this is that it tells us about the homotopy groups of $BU = \lim_{\to} BU(n)$! In particular, we can compute some homotopy groups of $U(n)$. We first determine the homotopy groups of BU. To this end, note that

$$[X, BU] \cong \widetilde{K}(X).$$

Therefore by plugging in $X = \mathbb{S}^n$, we can compute the homotopy groups of BU. Specifically,

$$\pi_n(BU) = \begin{cases} 0 & n \text{ odd} \\ \mathbb{Z} & n \text{ even.} \end{cases}$$

But now we know that $\pi_n(BU) = \pi_{n-1}(U)$, so this tells us the homotopy groups of the direct limit $U = \lim_{\to} U(n)$. Finally we recall that there is a fibration $U(n) \to U(n+1) \to U(n+1)/U(n) \cong \mathbb{S}^{2n+1}$. Since $\mathbb{S}^{2n+1}$ is $2n$-connected, this shows that there is a stable range in which $\pi_k(U(n)) \cong \pi_k(U(n+1))$, specifically when $k \leq 2n - 1$. Therefore we have the following table:

	π_0	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	$\cdots$
$U(1)$	0	$\mathbb{Z}$	0	0	0	0	0	0	0	0	$\cdots$
$U(2)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?	?	?	?	?	?	?
$U(3)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	?	?	?	?	?
U	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$

There is a real analogue to this theorem, in which we compute the homotopy groups of the orthogonal group. (This is the *real Bott periodicity theorem.*) Specifically,

$$\pi_k(O) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k \equiv 0 \mod 8 \\ \mathbb{Z}/2\mathbb{Z} & k \equiv 1 \mod 8 \\ 0 & k \equiv 2 \mod 8 \\ \mathbb{Z} & k \equiv 3 \mod 8 \\ 0 & k \equiv 4 \mod 8 \\ 0 & k \equiv 5 \mod 8 \\ 0 & k \equiv 6 \mod 8 \\ \mathbb{Z} & k \equiv 7 \mod 8, \end{cases}$$

where $O = \lim_{\to} O(n)$.

Finally we go back to K-theory has a cohomology theory. We proved earlier that given a sequence $A \to X \to X/A$ of spaces, we have a long exact sequence

$$\cdots \to \widetilde{K}(\Sigma X) \to \widetilde{K}(\Sigma A) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A).$$

To turn this into a cohomology theory, we define

$$\widetilde{K}^{-n}(X) \coloneqq \widetilde{K}(\Sigma^n X).$$

By the Bott periodicity theorem, there are isomorphisms $\widetilde{K}^n(X) \to \widetilde{K}^{n-2}(X)$ for any X and n. We can extend the K-theory to positive n using this isomorphism. The long exact sequence of cohomology forms a loop as follows:

$$\begin{tikzcd} \widetilde{K}^1(X/A) \arrow[r] & \widetilde{K}^1(X) \arrow[r] & \widetilde{K}^1(A) \cong K^{-1}(A) \arrow[d] \\ \widetilde{K}^0(A) \arrow[u] & \widetilde{K}^0(X) \arrow[l] & \widetilde{K}^0(X/A). \arrow[l] \end{tikzcd}$$

There is also an unreduced version of the groups $\widetilde{K}^i(X)$, giving rise to a generalised cohomology theory. We define

$$K^n(X) \coloneqq \widetilde{K}^n(X_+),$$

where $X_+ = X \cup \{*\}$ for some disjoint point $*$. Then we have $K^0(X) = K(X)$ and $\widetilde{K}^1(X) = K^1(X)$. Moreover, we again have a long exact sequence (which induces a loop):

$$\begin{tikzcd} K^1(X,A) \arrow[r] & K^1(X) \arrow[r] & K^1(A) \arrow[d] \\ K^0(A) \arrow[u] & K^0(X) \arrow[l] & K^0(X,A). \arrow[l] \end{tikzcd}$$

Specifically $K^0(A)$ and $K^0(X)$ have an additional $\mathbb{Z}$ component, while the rest of the groups are equal to their reduced counterparts.

3.3 Bott periodicity: applications and examples

We've computed the K-theory of spheres via the Bott periodicity theorem, so next we use this to compute the K-theory of projective space.

Theorem 3.3.1. $K^q(\mathbb{CP}^n) = \begin{cases} \mathbb{Z}^{n+1} & q \text{ even} \\ 0 & q \text{ odd.} \end{cases}$

This follows immediately from a more general result, obtained by calculating the K-theory inductively by building the space a cell at a time.

Proposition 3.3.2. Let X be a finite cell complex with n cells, in which all cells have even dimension. Then $K^0(X) = \mathbb{Z}^n$, and $K^1(X) = 0$.

Proof. We proceed by induction. Let

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n = X$$

be a chain of cell complexes which build up to give X. (The subscript denotes the number of cells in each X_i.) For the base case, note that X_1 has the homotopy type of a point. Therefore $\widetilde{K}(X_1) = 0$, from which it follows that $K^0(X_1) = \mathbb{Z}$. On the other hand, $K^1(X_1) = \widetilde{K}^1(X_1) = \widetilde{K}(\Sigma X_1) = 0$, since ΣX_1 again has the homotopy type of a point. This proves the base case.

For the inductive step, fix i and suppose $K^0(X_i) = \mathbb{Z}^i, K^1(X_i) = 0$. Then X_{i+1} is obtained by adding an even dimensional cell to X_i, so $X_{i+1}/X_i \cong \mathbb{S}^{2m_i}$ for some m_i. The long exact sequence of K-theory now gives the following exact loop:

$$\begin{tikzcd} K^1(\mathbb{S}^{2m_i}) \arrow[r] & K^1(X_{i+1}) \arrow[r] & K^1(X_i) \arrow[d] \\ K^0(X_i) \arrow[u] & K^0(X_{i+1}) \arrow[l] & K^0(\mathbb{S}^{2m_i}). \arrow[l] \end{tikzcd}$$

By the Bott periodicity theorem and inductive hypothesis, this can be written as

$$\begin{tikzcd} 0 \arrow[r] & K^1(X_{i+1}) \arrow[r] & 0 \arrow[d] \\ \mathbb{Z}^i \arrow[u] & K^0(X_{i+1}) \arrow[l] & \mathbb{Z}. \arrow[l] \end{tikzcd}$$

Therefore $K^1(X_{i+1}) = 0$ and $K^0(X_{i+1}) = \mathbb{Z}^{i+1}$. This completes the inductive step. □

Next we study one of the most famous applications of the Bott periodicity theorem: the classification of real division algebras (as well as a classification of parallelisable spheres). This follows from a result on the *Hopf invariant* using *Adams operations.* We give a proof outline now:

1. First we axiomatically define the Adams operations $\Psi^k : K(X) \to K(X)$ and outline their construction. These provide additional structure to K-theory. These satisfy some important properties, which are implied by the *splitting principal* for K-theory.

2. Next we introduce the *Hopf invariant* in K-theoretic terms. This is an integer h assigned to a map $\mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$. We use the Adams operations to prove that the Hopf invariant of any such map can be equal to ± 1 only if $n = 2, 4, 8$.

3. Next we introduce the notion of an H-space. We show that $\mathbb{R}^n$ admits a division algebra structure only if $\mathbb{S}^{n-1}$ is an H-space. Similarly, we show that $\mathbb{S}^{n-1}$ is parallelisable only if it is an H-space.

4. To tie everything together, we prove that non-trivial even-dimensional spheres are not H-spaces. Next given an odd dimensional sphere $\mathbb{S}^{2n-1}$, we prove that if it is an H-space, then $\mathbb{S}^{2n}$ is the codomain of a map with Hopf invariant 1. This proves that the only spheres that could be H-spaces are $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$, and $\mathbb{S}^7$. (We will see that each of these really are H-spaces.) It follows that the only real division algebras are a subset of $\{\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8\}$. Similarly the only possible parallelisable spheres are $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$.

5. Finally we note that each of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ are indeed division algebras, and each of $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$ are parallelisable.

1. We now define and construct the Adams operations.

Definition 3.3.3. The Adams operations are ring homomorphisms $\Psi^k : K(X) \to K(X)$ (for non-negative integers) characterised by the following properties:

1. Naturality: $\Psi^k f^* = f^* \Psi^k$ for all maps $f : X \to Y$.

2. Normalisation: $\Psi^k(L) = L^k$ if L is a line bundle.

In addition, the Adams operations satisfy the following properties:

3. Product formula: $\Psi^k \circ \Psi^\ell = \Psi^{k\ell}$.

4. $\Psi^p(\alpha) = \alpha^p \mod p$ for p prime.

These are a type of power operation, similar to the Steenrod squares. The idea is that $\Psi^k(\alpha)$ will be to $\Lambda^k(\alpha)$ what the power sum $\sum a_i^k$ is to $\sigma_k(a_i)$, for some roots a_i of a polynomial. Here the Λ refers to the exterior power. As the constructions suggests, Adams operations exist at the generality of λ-*rings* which are rings admitting an exterior-power type operation.

To define the Adams operations we define an auxiliary function (which is the λ-structure on $K(X)$.) Let $K(X)[[t]]$ denote the ring of formal power series with coefficients in $K(X)$. Define

$$\lambda_t : \mathrm{Vect}(X) \to K(X)[[t]], \quad \lambda_t([E]) = \sum_{k=0}^{\infty} [\Lambda^k E] t^k.$$

Here $[\Lambda^k E]$ denotes the isomorphism class of the kth exterior power of E. Note that the constant term is always the unit: $[\Lambda^0 E] = 1$. Therefore

$$\lambda_t(E) = 1 + tP(t) \in 1 + tK(X)[[t]].$$

This has inverse $1 - tP(t) + t^2P(t)^2 - t^3P(t)^3 + \cdots \in K(X)[[t]]$. Thus, more generally, we define

$$\lambda_t : K(X) \to K(X)[[t]], \quad \lambda_t([E - F]) = \lambda([E])\lambda([F])^{-1}.$$

Next we define the total Adams operation using the λ-operation:

$$\Psi_t : K(X) \to K(X)[[t]], \quad \Psi_t(E) = \mathrm{rank}(E) - t\frac{\mathrm{d}}{\mathrm{d}t} \log \lambda_{-t}(E).$$

(The second term uses the formal logarithmic derivative.) We write this as

$$\Psi_t(E) = \sum_{k=0}^{\infty} \Psi^k(E) t^k,$$

and the $\Psi^k(E)$ are the kth Adams operations.

To prove that the Adams operations are uniquely determined by the first two properties, as well satisfying the subsequent two properties, one can use the *splitting lemma* for K-theory.

Proposition 3.3.4 (Splitting lemma)**.** Let $E \to X$ be a vector bundle, with X compact Hausdorff. There exists a compact Hausdorff space $F(E)$ and a map $p : F(E) \to X$ such that $p^* : K^*(X) \to K^*(F(E))$ is injective, and $p * (E)$ splits as a sum of line bundles.

We do not prove this result, but it follows from a K-theoretic Leray-Hirsch theorem. We will also omit the proof of uniqueness of the Adams operations, or that they satisfy the additional claimed properties, but these proofs are straight forward. They are analogous to proofs using the splitting principal with characteristic classes.

2. Next we introduce the Hopf invariant. The goal is to use the Adams operations to prove that the Hopf invariant is ± 1 only for maps with codomain $\mathbb{S}^{2,4,8}$.

Definition 3.3.5 (Hopf invariant)**.** Let $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ be a continuous map. We can glue a disk D^{4n} to $\mathbb{S}^{2n}$ along f, to obtain a space C_f. The quotient $C_f/\mathbb{S}^n$ is then $\mathbb{S}^{4n}$. Therefore we have an exact sequence

$$0 = \widetilde{K}^1(\mathbb{S}^{2n}) \to \widetilde{K}^0(\mathbb{S}^{4n}) \xrightarrow{p^*} \widetilde{K}^0(C_f) \xrightarrow{i^*} \widetilde{K}^0(\mathbb{S}^{2n}) \to \widetilde{K}^1(\mathbb{S}^{4n}) = 0.$$

This is a short exact sequence! Let

$$\alpha = p^*(*_{2n}(H-1)) \in \widetilde{K}(C_f),$$
$$\beta \in i^{*-1}(*_n(H-1)) \in \widetilde{K}(C_f).$$

Note that, from the Bott periodicity theorem, the reduced exterior products of $2n$ and n copies of $H-1$ respectively are canonical generators of $\widetilde{K}^0(\mathbb{S}^{4n})$ and $\widetilde{K}^0(\mathbb{S}^{2n})$, where H is the canonical line bundle over $\mathbb{S}^2$. The *Hopf invariant* of f is the integer h such that

$$\beta^2 = h\alpha.$$

To see that this is well defined, first we note that such an h exists by exactness: we know that β^2 maps to 0 in $\widetilde{K}(\mathbb{S}^{2n})$. We must also show that the definition is invariant under the choice of β. Suppose β' is another element of $\widetilde{K}(C_f)$ that maps to $*_n(H-1)$. Then $\beta - \beta' \in \ker i^*$, so there exists m such that $\beta' = \beta + m\alpha$. But now $(\beta+m\alpha)^2 = \beta^2 + 2m\alpha\beta$ since $\alpha^2 = 0$. But α maps to 0 in $\widetilde{K}(\mathbb{S}^{2n})$, so $\alpha\beta$ does too. Therefore by exactness there exists some integer k such that $k\alpha = \alpha\beta$. But now

$$k\alpha\beta = \alpha\beta^2 = \alpha(h\alpha) = h\alpha^2 = 0.$$

Therefore

$$k(2m\alpha\beta) = 2m \cdot 0 = 0.$$

If k is non-zero, this implies that $2m\alpha\beta = 0$. If k is zero, then again $2m\alpha\beta = 2m(k\alpha) = 0$. In either case, this proves that

$$\beta'^2 = (\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta = \beta^2.$$

Therefore the Hopf invariant is well defined.

The most important ingredient in the classification of real division algebras and parallelisable spheres is the following proposition:

Theorem 3.3.6 (Adam)**.** *If a map $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ has Hopf invariant ± 1, then $n = 1, 2$, or 4.*

We now give the proof of Adam's theorem in several parts. The first lemma is purely number theoretic, the second concerns the behaviour of Adams operators on even dimensional spheres, and the third part is the main body of the proof which relies on the previous two lemmas.

Lemma 3.3.7. Suppose 2^n divides $3^n - 1$. Then $n \in \{1, 2, 4\}$.

Proof. We give a proof by induction. Specifically, we prove the following: *Let $n = 2^\ell(2m+1)$. Then the highest power of 2 dividing $3^n - 1$ is 2 for $\ell = 0$, and $2^{\ell+2}$ for positive ℓ.*

Base case 1: if $\ell = 0$, then $n = 2m + 1$. Therefore

$$3^n - 1 = 3(9^m) - 1 \equiv 3(1^m) - 1 \equiv 2 \mod 4.$$

Therefore 2^1 divides $3^n - 1$, but no higher power of 2 does.

Base case 2: if $\ell = 1$, then $n = 2(2m + 1)$. Therefore

$$3^n - 1 = 3^{2(2m+1)} - 1 = (3^{2m+1} - 1)(3^{2m} + 1).$$

By base case 1, the highest power of 2 dividing the first factor $3^{2m} - 1$ is 1. On the other hand,

$$3^{2m+1} + 1 = 3(9^m) + 1 \equiv 3(1^m) + 1 \equiv 4 \mod 8.$$

Therefore the highest power of 2 dividing $3^{2m+1} + 1$ is 2^2. Altogether, this gives that $2^3 = 2^{1+2}$ is the highest power of 2 diving $3^n - 1$.

Inductive step: let $\ell > 0$, and suppose that the highest power of 2 dividing $3^n - 1$ is $2^{\ell+2}$. Now we pass from ℓ to 2ℓ, which is equivalently passing from n to $2n$. Then $3^{2n} - 1 = (3^n - 1)(3^n + 1)$. Since n is even, $3^n + 1 \equiv 2 \mod 4$. Therefore the highest power of 2 dividing the first factor is $2^{\ell+2}$, while the highest power dividing the second factor is 2^1. Combining these proves the base case.

This completes the induction - but why does it prove the lemma? Suppose that 2^n divides $3^n - 1$. By the induction, the highest power of 2 dividing $3^n - 1$ is $2^{\ell+2}$. Therefore $n \leq \ell + 2$. But now we know that $2^\ell \leq 2^\ell m = n \leq \ell + 2$, so

$$2^\ell \leq \ell + 2.$$

This is only possible for $\ell \leq 2$. Therefore $n \leq 4$. These cases can be manually checked: if $n = 3$, then 8 does not divide 26. Therefore $n \in \{1, 2, 4\}$ as required. $\square$

Next we study the structure of Adams operations on the K-theory of spheres. We find that they end up exactly being powers!

Lemma 3.3.8. The Adams operations restrict to maps $\Psi^k : \widetilde{K}(X) \to \widetilde{K}(X)$. Moreover, for X an even-dimensional sphere $\mathbb{S}^{2n}$, we have

$$\Psi^k(\gamma) = k^n \gamma.$$

Proof. The Adams operations give natural maps $K(X) \to K(X)$ and $K(x_0) \to K(x_0)$ given $x_0 \in X$. We can define $\widetilde{K}(X)$ to be the kernel of $K(X) \to K(x_0)$. By naturality of Ψ^k, there is an induced map $\Psi^k : \widetilde{K}(X) \to \widetilde{K}(X)$.

Now we study some properties of this induced Adams operation on reduced K-theory. Recall that the external product $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y), (\gamma, \delta) \mapsto \gamma * \delta$ was defined by $p_1^*(\gamma)p_2^*(\delta)$, where p_i are projection maps from $X \times Y$ to X and Y. Using naturality of Ψ^k and this definition, it can be shown that

$$\Psi^k(\gamma * \delta) = \Psi^k(\gamma) * \Psi^k(\delta).$$

Finally we show that $\Psi^k(\gamma) = k^n\gamma$, for $\gamma \in \widetilde{K}(\mathbb{S}^{2n})$. We proceed by induction. For the base case, i.e. when $n = 1$, recall that $\alpha = H - 1$ is a generator of $\widetilde{K}(\mathbb{S}^2)$ where H is the class of the canonical line bundle. Therefore it suffices to show that $\Psi^k(\alpha) = k\alpha$. This is a straight forward calculation using the axiomatic properties of Adams operations:

$$\Psi^k(\alpha) = \Psi^k(H-1) = H^k - 1 = (1+\alpha)^k - 1 = 1 + k\alpha - 1 = k\alpha.$$

Note that we used that $\alpha^n = 0$ for $n \geq 2$. For the inductive step, we use the isomorphism $\widetilde{K}(\mathbb{S}^2) \otimes \widetilde{K}(\mathbb{S}^{2n-2}) \to \widetilde{K}(\mathbb{S}^{2n})$ from the Bott periodicity theorem. Assume the result holds for $\widetilde{K}(\mathbb{S}^{2n-2})$. Then by the previous result that $\Psi^k(\gamma * \delta) = \Psi^k(\gamma) * \Psi^k(\delta)$, we have

$$\Psi^k(\alpha * \beta) = \Psi^k(\alpha) * \Psi^k(\beta) = k\alpha * k^{n-1}\beta = k^n\alpha * \beta.$$

By the Bott periodicity theorem, each $\gamma \in \widetilde{K}(\mathbb{S}^{2n})$ is of the form $\alpha * \beta$, so this completes the proof. $\square$

We are finally ready to prove the main result. We re-state the result here.

Theorem 3.3.9 (Adams)**.** *If a map $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ has Hopf invariant ± 1, then $n = 1, 2$, or 4.*

Proof. Recall that we defined a space C_f to be $\mathbb{S}^{2n}$ with a disk D^{2n} glued along f. Recall further that the Hopf invariant is the integer h such that

$$\beta^2 = h\alpha,$$

where β maps to the generator of $\widetilde{K}(\mathbb{S}^{2n})$, and α is the image of the generator of $\widetilde{K}(\mathbb{S}^{4n})$, given the exact sequence

$$\widetilde{K}(\mathbb{S}^{4n}) \to \widetilde{K}(C_f) \to \widetilde{K}(\mathbb{S}^{2n}).$$

By naturality of the Adams operations, the previous proposition immediately gives

$$\Psi^k(\alpha) = k^{2n}\alpha, \quad \Psi^k(\beta) = k^n\beta + \mu_k\alpha,$$

where μ_k is an integer that has yet to be determined.

Recall that $\Psi^k\Psi^\ell = \Psi^{k\ell} = \Psi^\ell\Psi^k$. In particular, $\Psi^2\Psi^3 = \Psi^3\Psi^2$. Therefore applying these operations to β, we have

$$(\Psi^2\Psi^3)(\beta) = \Psi^2(3^n\beta + \mu_3\alpha) = 3^n2^n\beta + 3^n\mu_2\alpha + 4^n\mu_3\alpha,$$
$$(\Psi^3\Psi^2)(\beta) = \Psi^3(2^n\beta + \mu_2\alpha) = 2^n3^n\beta + 2^n\mu_3\alpha + 6^n\mu_2\alpha,$$

Since these must be equal, dropping the β term and factoring out the α, we must have

$$3^n(3^n - 1)\mu_2 = (6^n - 3^n)\mu_2 = (4^n - 2^n)\mu_3 = 2^n(2^n - 1)\mu_3.$$

We have yet to assume that f has Hopf index ± 1. We use this premise now: if f has Hopf index ± 1, then $\beta^2 = \pm\alpha$. But we know that $\Psi^2(\beta) = \beta^2 \mod 2$ by a general property of Adams operations. By our premise this gives $\Psi^2(\beta) = \alpha \mod 2$. On the other hand, we have $\Psi^2(\beta) = 2^n\beta + \mu_2\alpha$. It follows that

$$2^n\beta + \mu_2\alpha \equiv \alpha \mod 2.$$

Equivalently, μ_2 is odd. Since $3^n(3^n - 1)\mu_2 = 2^n(2^n - 1)\mu_3$, it must be the case that 2^n divides $3^n(3^n - 1)\mu_2$. We've shown that μ_2 is odd, and 3^n is always odd, so we must have $2^n \mid (3^n - 1)$. This happens only if $n \in \{1, 2, 4\}$ by the earlier number theoretic lemma. $\square$

3. Now we introduce H-spaces, and show that:

$$\mathbb{R}^n \text{ admits a division algebra structure} \Rightarrow \mathbb{S}^{n-1} \text{ admits an } H\text{-space structure.}$$
$$\mathbb{S}^{n-1} \text{ is parallelisable} \Rightarrow \mathbb{S}^{n-1} \text{ admits an } H\text{-space structure.}$$

Later we will show that $\mathbb{S}^{n-1}$ is an H-space only if $\mathbb{S}^n$ is the codomain of a map with Hopf index ± 1! The classification of real division algebras and parallelisable spheres will follow.

Definition 3.3.10. An *H-space* is a topological unital magma. That is, a topological space X together with a continuous binary operation $X \times X \to X$ with a two-sided identity element (but not associative, and does not necessarily have inverses).

Proposition 3.3.11. If $\mathbb{R}^n$ has a division algebra structure, then $\mathbb{S}^{n-1}$ admits an H-space structure.

Proof. Consider $\mathbb{S}^{n-1}$ as the unit sphere in $\mathbb{R}^n$. Define

$$\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad u \cdot v = uv/\|uv\|,$$

where uv is understood to be the product using the division algebra structure. This is well defined and continuous. Moreover, the unit of the division algebra structure gives the unit in of the H-space structure. $\square$

Proposition 3.3.12. If $\mathbb{S}^{n-1}$ is parallelisable, then it admis an H-space structure.

Proof. If $\mathbb{S}^{n-1}$ is parallelisable, there exist vector fields $v_1, \ldots, v_{n-1}$ such that $v_1(x), \ldots, v_{n-1}(x)$ defines a basis of $\mathbb{R}^{n-1}_x$ at every $x \in \mathbb{S}^{n-1}$. This copy of $\mathbb{R}^{n-1}_x$ embeds in $\mathbb{R}^n$ as a subspace in a canonical way. By the Gram-Schmidt process, we can assume $\{x, v_1(x), \ldots, v_{n-1}(x)\}$ defines an orthonormal basis of $\mathbb{R}^n$ for every x. Moreover, by a global change of coordinates, we can take $\{e_1, v_1(e_1), \ldots, v_{n-1}(e_1)\}$ to be the standard basis for $\mathbb{R}^n$.

For each $x \in \mathbb{S}^{n-1}$, let $\alpha_x \in \mathrm{SO}(n)$ be the map sending the standard basis to $x, v_1(x), \ldots, v_{n-1}(x)$. Then we define a map

$$\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}, \quad (x, y) \mapsto \alpha_x(y).$$

This is again continuous. In fact, it is an H-space structure since $\alpha_x(e_1) = x$ by definition (of α_x), and $\alpha_{e_1}(x) = x$ since α_{e_1} is also the identity by definition. $\square$

We have established that (the difficult direction of) the classification of real division algebras and parallelisable spheres will follow from a classification of spheres that are H-spaces.

4. We are entering the final stages of the proof! We have established the double arrows below, and it remains to prove the dotted arrow.

$$\begin{tikzcd}
\mathbb{R}^n \text{ is a division algebra} \arrow[r, Rightarrow] & \mathbb{S}^{n-1} \text{ is an } H\text{-space} \arrow[d, dotted] & \mathbb{S}^{n-1} \text{ is parallelisable} \arrow[l, Rightarrow] \\
& n \in \{1, 2\mathbb{Z}\}, \text{ and for } n \text{ even there is a map } \mathbb{S}^{2n-1} \to \mathbb{S}^n \text{ with Hopf index } \pm 1. \arrow[d, Rightarrow] & \\
& n \in \{2, 4, 8\} &
\end{tikzcd}$$

Assume $\mathbb{S}^{n-1}$ is an H-space. We must prove two things: first that n is even (or 1), and next the existence of a map $\mathbb{S}^{2n} \to \mathbb{S}^n$ with Hopf index ± 1 given that n is even.

We work through this result step by step.

Proposition 3.3.13. If $\mathbb{S}^{n-1}$ is an H-space, either $n = 1$ or n is even.

Proof. If $n = 1$, $\mathbb{S}^0 \cong \mathbb{Z}/2\mathbb{Z}$ which is an H-space when given the discrete topology. For $n > 1$, assume for a contradiction that there is an H-space structure

$$\mu : \mathbb{S}^{2k} \times \mathbb{S}^{2k} \to \mathbb{S}^{2k}$$

with $k \geq 1$. By the Bott periodicity theorem, there exists γ, α, β such that

$$K(\mathbb{S}^{2k}) \cong \mathbb{Z}[\gamma]/\langle \gamma^2 \rangle, \quad K(\mathbb{S}^{2k} \times \mathbb{S}^{2k}) \cong \mathbb{Z}[\alpha, \beta]/\langle \alpha^2, \beta^2 \rangle.$$

The map μ^* sends γ to some combination $m_1\alpha + m_2\beta + m_3\alpha\beta$ where the m_i have yet to be determined. In fact, we now show that $m_1 = m_2 = 1$.

There exists $e \in \mathbb{S}^{2k}$ which is a 2-sided identity of μ. Let $i : \mathbb{S}^{2k} \to \mathbb{S}^{2k} \times \mathbb{S}^{2k}$ be the inclusion into $\mathbb{S}^{2k} \times \{e\}$. This gives a (non-exact) sequence

$$K(\mathbb{S}^{2k}) \xrightarrow{\mu^*} K(\mathbb{S}^{2k} \times \mathbb{S}^{2k}) \xrightarrow{i^*} K(\mathbb{S}^{2k}).$$

In fact, the composition $\mu \circ i$ is the identity, so the above sequence also composes to give the identity. The map i^* sends α to γ and β to 0. But we know that $\gamma = i^*(\mu^*\gamma) = i^*(m_1\alpha + m_2\beta + m_3\alpha\beta) = m_1\gamma$, so it must be the case that $m_1 = 1$. Similarly, considering the inclusion $\mathbb{S}^{2k} \to \{e\} \times \mathbb{S}^{2k}$ shows that $m_2 = 1$.

But now $\gamma^2 = 0$, so it must be the case that $\mu^*(\gamma^2) = 0$. However, we instead have that

$$\mu^*(\gamma^2) = (\alpha + \beta + m_3\alpha\beta)^2 = 2\alpha\beta \neq 0.$$

This is a contradiction, so the only possible H-spaces are $\mathbb{S}^{n-1}$ for $n \in \{1, 2\mathbb{Z}\}$. $\square$

Now to finish relating the notion of H-spaces to Hopf indices, we must show that whenever $\mathbb{S}^{n-1}$ is an H-space (for $n = 2k$), there is a map $\mathbb{S}^{4k-1} \to \mathbb{S}^{2k}$ with Hopf index ± 1. We now explain how we construct this map, before computing its Hopf index.

Definition 3.3.14. Let $f : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. We define the *associated map* $\hat{f} : \mathbb{S}^{2n-1} \to \mathbb{S}^{n-1}$ as follows:

- Decompose $\mathbb{S}^{2n-1}$ as
$$\begin{aligned}\partial(D^n \times D^n) &= (\partial D^n \times D^n) \sqcup_{\partial D^n \times \partial D^n} (D^n \times \partial D^n) \\ &= (\mathbb{S}^{n-1} \times D^n) \sqcup_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} (D^n \times \mathbb{S}^{n-1}).\end{aligned}$$
- Decompose $\mathbb{S}^n$ as $D^n_+ \sqcup_{\mathbb{S}^{n-1}} D^n_-$.
- The map $\hat{f}$ will be an extension of f: for $(x, y) \in \mathbb{S}^{n-1} \times D^n$, define $\hat{f}_+$ by
$$\hat{f}_+(x, y) = |y| g(x, y/|y|) \in D^n_+.$$
For $(x, y) \in D^n \times \mathbb{S}^{n-1}$, define $\hat{f}$ by
$$\hat{f}_-(x, y) = |x| g(x/|x|, y) \in D^n_-.$$
Then $\hat{f}_+$ and $\hat{f}_-$ are continuous, and agree on overlaps. Therefore they glue to give a map $\hat{f} : \mathbb{S}^{2n-1} \to \mathbb{S}^n$.

We finally show that the map associated to an H-space structure has Hopf index ± 1.

Proposition 3.3.15. Let $\mu : \mathbb{S}^{2n-1} \times \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ be an H-space structure. Then $\hat{\mu} : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ has Hopf index ± 1.

Proof. To prove this result, we use a giant commutative diagram. We start by introducing the spaces and maps one by one.

- Recall that, to define the Hopf index of $\hat{\mu} : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$, we introduce an auxiliary space $C_{\hat{\mu}}$ defined by gluing a copy of D^{4n} to $\mathbb{S}^{2n}$ along $\hat{\mu}$. There is a map
$$\widetilde{K}(C_{\hat{\mu}}) \times \widetilde{K}(C_{\hat{\mu}}) \to \widetilde{K}(C_{\hat{\mu}})$$
which is simply the product map.
- Next we consider relative K-theory. The subspace D^{2n}_- of $\mathbb{S}^{2n}$ is of course a subspace of $C_{\hat{\mu}}$. Modding out $C_{\hat{\mu}}$ by D^{2n}_- is a homotopy equivalence, and induces an isomorphism on reduced K-theories. The same result holds for D^{2n}_+. Therefore there is a canonical isomorphism
$$\widetilde{K}(C_{\hat{\mu}}, D^{2n}_-) \otimes \widetilde{K}(C_{\hat{\mu}}, D^{2n}_+) \xrightarrow{\cong} \widetilde{K}(C_{\hat{\mu}}) \times \widetilde{K}(C_{\hat{\mu}}).$$

- The previous two maps will define two edges of a commutative square. We define the other two edges. Recall the reduced external product

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y).$$

Taking $X = C_{\hat{\mu}}/D_-^{2n}$ and $Y = C_{\hat{\mu}}/D_+^{2n}$, the external product induces a map

$$\widetilde{K}(C_{\hat{\mu}}, D_-^{2n}) \otimes \widetilde{K}(C_{\hat{\mu}}, D_+^{2n}) \to \widetilde{K}(C_{\hat{\mu}}, \mathbb{S}^{2n}).$$

Finally a map $\widetilde{K}(C_{\hat{\mu}}, \mathbb{S}^{2n}) \to \widetilde{K}(C_{\hat{\mu}})$ is induced by the quotient $C_{\hat{\mu}} \to C_{\hat{\mu}}/\mathbb{S}^{2n}$.

- Next we consider a somewhat non-trivial map. We have a map $D^{2n} \times D^{2n} = D^{4n} \to C_{\hat{\mu}}$ induced by $\hat{\mu}$. When we quotient $C_{\hat{\mu}} \to C_{\hat{\mu}}/D_-^{2n}$, only the boundary $\partial(D^{2n} \times D^{2n})$ is being killed. Therefore there is an induced map $(D^{2n} \times D^{2n})/(\partial D^{2n} \times D^{2n}) \to C_{\hat{\mu}}/D_-^{2n}$. This defines a map

$$\Phi^* : \widetilde{K}(C_{\hat{\mu}}, D_-^{2n}) \to \widetilde{K}(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}).$$

There are analogous maps with domain $\widetilde{K}(C_{\hat{\mu}}, D_+^{2n})$ and $\widetilde{K}(C_{\hat{\mu}}, \mathbb{S}^{2n})$, giving maps

$$\begin{aligned} \Phi^* \otimes \Phi^* : \widetilde{K}(C_{\hat{\mu}}, D_-^{2n}) \otimes \widetilde{K}(C_{\hat{\mu}}, D_+^{2n}) &\to \widetilde{K}(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \\ &\qquad \otimes \widetilde{K}(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n}), \\ \Phi^* : \widetilde{K}(C_{\hat{\mu}}, \mathbb{S}^{2n}) &\xrightarrow{\cong} \widetilde{K}(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})). \end{aligned}$$

Notice that the second map is really induced by the identity map on $\mathbb{S}^{4n}$, so it is an isomorphism.

- Again to turn this into a second commutative square, we require a map

$$\begin{aligned} \widetilde{K}(D^{2n} \times D^{2n}, \partial D^{2n} \times D^{2n}) \otimes \widetilde{K}(D^{2n} \times D^{2n}, D^{2n} \times \partial D^{2n}) \\ \to \widetilde{K}(D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})). \end{aligned}$$

Once again, such a map is provided by the reduced exterior product.

In summary, we have established the existence of the following commutative diagram (where D denotes D^{2n}, and $\mathbb{S}$ denotes $\mathbb{S}^{2n}$):

$$\begin{tikzcd}
\widetilde{K}(C_{\hat{\mu}}) \otimes \widetilde{K}(C_{\hat{\mu}}) \arrow[r] & \widetilde{K}(C_{\hat{\mu}}) \\
\widetilde{K}(C_{\hat{\mu}}, D_-) \otimes \widetilde{K}(C_{\hat{\mu}}, D_+) \arrow[r] \arrow[u, "\cong"] \arrow[d, "\Phi^* \otimes \Phi^*"] & \widetilde{K}(C_{\hat{\mu}}, \mathbb{S}) \arrow[u] \arrow[d, "{\Phi^*, \cong}"] \\
\widetilde{K}(D \times D, \partial D \times D) \otimes \widetilde{K}(D \times D, D \times \partial D) \arrow[r] & \widetilde{K}(D \times D, \partial(D \times D)).
\end{tikzcd}$$

We begin by getting a better grasp on the bottom horizontal map. I claim that it is an isomorphism - we will see this from the Bott periodicity theorem. By the excision theorem, there is an isomorphism

$$\widetilde{K}(D \times D, \partial D \times D) \to \widetilde{K}(D \times \{e\}, \partial D \times \{e\}).$$

Similarly we obtain an isomorphism on the second factor of the tensor product in the bottom left of the diagram. But now $\widetilde{K}(D \times \{e\}, \partial D \times \{e\})$ is just $\widetilde{K}(\mathbb{S}^{2n})$, while $\widetilde{K}(D \times D, \partial(D \times D)) = \widetilde{K}(\mathbb{S}^{4n})$. By the Bott periodicity theorem, the bottom map factors through isomorphisms as

$$\begin{tikzcd}
\widetilde{K}(C_{\hat{\mu}}) \otimes \widetilde{K}(C_{\hat{\mu}}) \arrow[r] & \widetilde{K}(C_{\hat{\mu}}) \\
\widetilde{K}(C_{\hat{\mu}}, D_-) \otimes \widetilde{K}(C_{\hat{\mu}}, D_+) \arrow[r] \arrow[u, "\cong"] \arrow[d, "\Phi^* \otimes \Phi^*"] & \widetilde{K}(C_{\hat{\mu}}, \mathbb{S}) \arrow[u] \arrow[d, "{\Phi^*, \cong}"] \\
\widetilde{K}(D \times D, \partial D \times D) \otimes \widetilde{K}(D \times D, D \times \partial D) \arrow[r] \arrow[d, "\cong"] & \widetilde{K}(D \times D, \partial(D \times D)) \\
\widetilde{K}(D \times \{e\}, \partial D \times \{e\}) \otimes \widetilde{K}(\{e\} \times D, \{e\} \times \partial D). \arrow[ur, "\cong"'] &
\end{tikzcd}$$

We are now ready to proceed with the proof. Consider the diagram

$$\widetilde{K}(\mathbb{S}^{4n}) \xrightarrow{p^*} \widetilde{K}(C_{\hat{\mu}}) \xrightarrow{i^*} \widetilde{K}(\mathbb{S}^{2n}).$$

We wish to show that

$$\beta^2 = \pm\alpha,$$

where β is a lift of a generator of $\widetilde{K}(\mathbb{S}^{2n})$, and α is the image of a generator of $\widetilde{K}(\mathbb{S}^{4n})$. To this end, fix a generator a of $\widetilde{K}(\mathbb{S}^{2n}) \otimes \widetilde{K}(\mathbb{S}^{2n}) = \widetilde{K}(D \times \{e\}, \partial D \times \{e\}) \otimes \widetilde{K}(\{e\} \times D, \{e\} \times \partial D)$. By definition, this lifts to $\beta \otimes \beta$ in the top left of the diagram. On the other hand, a maps to a generator of $\widetilde{K}(\mathbb{S}^{4n}) \cong \widetilde{K}(C_{\hat{\mu}}, \mathbb{S})$. By definition, this maps to $\pm\alpha$ in the top right of the diagram. Therefore

$$\beta^2 = \pm\alpha$$

as required. $\square$

This completes the proof that $\mathbb{S}^{n-1}$ is an H-space only if $\mathbb{S}^{n-1}$ is the zero sphere, or $\mathbb{S}^n$ is the codomain of a map with Hopf index ± 1. Several pages ago we showed that the only spheres which are codomains of maps with Hopf index ± 1 are $\mathbb{S}^2, \mathbb{S}^4$, and $\mathbb{S}^8$. Therefore, we have proven that:

$$\text{If } \mathbb{S}^{n-1} \text{ is an } H\text{-space, then } n \in \{1, 2, 4, 8\}.$$

We also proved earlier that if $\mathbb{R}^n$ is a division algebra, then $\mathbb{S}^{n-1}$ is an H-space. Similarly if $\mathbb{S}^{n-1}$ is parallelisable, then $\mathbb{S}^{n-1}$ is an H-space. Therefore:

$$\text{If } \mathbb{R}^n \text{ is a division algebra, then } n \in \{1, 2, 4, 8\}.$$

$$\text{If } \mathbb{S}^n \text{ is parallelisable, then } n \in \{0, 1, 3, 7\}.$$

5. Finally we must show that each of $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ are really division algebras, and that $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$ are really parallelisable.

The former is classical. $\mathbb{R}$ is a field. $\mathbb{R}^2 \cong \mathbb{C}$ which is also a field. The former are both vector spaces over $\mathbb{R}$ as well, so they are division algebras. $\mathbb{R}^4 \cong \mathbb{H}$, the quaternions. Finally, $\mathbb{R}^8 \cong \mathbb{O}$, the octonions.

For the parallelisability of spheres, note that $\mathbb{S}^0$ is trivially parallelisable. $\mathbb{S}^1 \cong U(1)$ and $\mathbb{S}^3 \cong SU(2)$, which are both Lie groups. Therefore they are parallelisable. Finally, for $\mathbb{S}^7$, we cannot use the Lie group trick (we will soon see that $\mathbb{S}^7$ is not a Lie group!) However, we use the octonions to show that $\mathbb{S}^7$ is parallelisable.

Identify $\mathbb{S}^7$ with unit octonions,

$$\mathbb{S}^7 = \{a + bl : a, b \in \mathbb{H}, |a|^2 + |b|^2 = 1\}.$$

Seven global vector fields are given by

$$v_1(x) = ix, v_2(x) = jx, v_3(x) = kx, v_4(x) = lx, v_5(x) = ilx, v_6(x) = jlx, v_7(x) = klx.$$

One can further show that these are orthogonal, so they form a global frame! That is, $T\mathbb{S}^7$ admits a global trivialisation.

This completes our classification of division algebras and parallelisable spheres:

$$\mathbb{R}^n \text{ admits a division algebra structure if and only if } n \in \{1, 2, 4, 8\}.$$

$$\mathbb{S}^n \text{ is parallelisable if and only if } n \in \{0, 1, 3, 7\}.$$

To finish off this section, we remark that while $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$, and $\mathbb{S}^7$ are the unique parallelisable spheres, only $\mathbb{S}^0, \mathbb{S}^1$, and $\mathbb{S}^3$ are Lie groups:

Proposition 3.3.16. Amongst spheres, the only Lie groups are $\mathbb{S}^0, \mathbb{S}^1$, and $\mathbb{S}^3$.

Proof. A reasonably elementary proof can be given independent of the parallelisability result. (By the above result, it suffices to show that $\mathbb{S}^7$ is not a Lie group to prove above proposition. However, we will give a self contained proof.)

The proof outline is as follows:

1. Suppose $\mathbb{S}^n$ is a Lie group for some n. Show that it cannot be abelian if $n > 1$. Note that $\mathbb{S}^0$ and $\mathbb{S}^1$ are abelian Lie groups.

2. Suppose M is a closed non-abelian Lie group. Show that $H^3(M)$ is non-trivial. Deduce that for $n > 1$, only $\mathbb{S}^3$ could be a Lie group. Note that $\mathbb{S}^3$ is indeed a Lie group.

1.First suppose $\mathbb{S}^n$ is abelian. Then its Lie algebra is isomorphic to $\mathbb{R}^n$ with a trivial Lie bracket. The Lie group $\mathbb{R}^n$ under addition also has Lie algebra $\mathbb{R}^n$ with the trivial bracket, so it must be the universal cover of $\mathbb{S}^n$ (by the correspondence between Lie algebras and Lie groups). But for $n > 1$, $\mathbb{S}^n$ is its own universal cover! Therefore $n \leq 1$.

On the other hand, $\mathbb{S}^0 = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{S}^1 = U(1)$ are clearly abelian Lie groups. This is part of a more general result that every connected abelian Lie group is a product of circles and lines (i.e. is of the form $\mathbb{T}^n \times \mathbb{R}^m$.

2. Next we suppose G is a closed non-abelian Lie group. Its Lie algebra admits an inner product $\langle -, - \rangle$ which is invariant under the adjoint action. We define a map

$$t(x, y, z) = \langle [x, y], z \rangle.$$

This is multilinear, and clearly changes sign when x and y are swapped. In fact, one can show that it is totally antisymmetric! That is, t is a 3-form (which is often called the *Cartan 3-form*. Note that t is not the 0-form, since we can choose x and y with $[x, y]$ non-zero (by the requirement that G be non-abelian). Therefore

$$t(x, y, [x, y]) = \langle [x, y], [x, y] \rangle = \|[x, y]\|^2 \leq 0,$$

so t is not the 0-form.

We have defined 3-form on T_eM, and using left translation on G, this extends to a 3-form on G. By definition, this is left invariant. One can also show that it is right invariant, so it is biinvariant. But every biinvariant form is closed, so t defines an element in $H^3_{\text{de Rham}}(G)$. In particular, the third de Rham cohomology is non-trivial! This is isomorphic to singular cohomology, so we must have $H^3(G, \mathbb{Z}) \neq 0$.

In the specific case of $G = \mathbb{S}^n$, this is only possible for $n = 3$. Conversely, $\mathbb{S}^3$ really is a Lie group because $\mathbb{S}^3 = SU(2)$. $\square$

3.4 K-theory versus singular cohomology

In this section, we introduce the *Chern character* to establish a rigorous relationship between K-theory and singular cohomology.

Consider the total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \cdots$ of a vector bundle $E \to B$. This is an element of $H^*(B)$. Therefore is a map $\text{Vect}_{\mathbb{C}} \to H^*(B)$ defined by the total Chern class. By the Whitney sum formula, we also have $c(E \oplus F) = c(E) \cup c(F)$. But this is somewhat problematic - since $H^*(B)$ and $K(B)$ are both rings, ideally we would have a ring homomorphism from $K(B)$ to $H^*(B)$. However, c takes sums in $K(B)$ to products in

$H^*(B)$. To remedy this, we aim to create a map ch satisfying $ch(E \oplus F) = ch(E) + ch(F)$. Essentially we want to exponentiate the total chern class. Recalling that

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

we require some notion of a power. This takes us back to Adams operations! In the previous section we defined Adams operations using logarithmic derivatives, but we mentioned that Adams operations are to Exterior powers what $\sum a_i^k$ is to $\sigma_k(a_i)$. By using logarithmic derivatives, we obtain the most elementary definition of Adams operations. However, by using the splitting lemma for K-theory and *Newton polynomials* we can obtain a slightly more intuitive definition (which we will also compare to the definition of the Chern character ch).

Recall that $\sigma_k(t_1, \ldots, t_n)$ denotes the kth elementary symmetric polynomial in n variables. One can show that there exists polynomials s_k called the *Newton polynomials* so that

$$s_k(\sigma_1, \ldots, \sigma_k) = t_1^k + \cdots + t_n^k.$$

Therefore $s_k(\sigma_1, \ldots, \sigma_k)$ is essentially a *k-th power* operation. We now give an alternative definition of Adams operations. By the splitting lemma for K-theory and naturality of Adams operations, it suffices to define Adams operations on Whitney sums of line bundles. We declare that $\Psi^k(L_1 \oplus \cdots \oplus L_n) = s_k(\lambda_1, \ldots, \lambda_k)$ where $\lambda_i = \sigma_i(L_1, \ldots, L_n)$.

To define the Chern character, we instead consider $s_k(c_1, \ldots, c_k)$. Recall that one construction of $c_i(E)$ is in terms of pulling back c_i of a Whitney sum of line bundles! And for a Whitney sum, we define $c_i(L_1 \oplus \cdots \oplus L_n)$ to be the ith symmetric polynomial in the generators of $H^*(BU(1)^n; \mathbb{Z})$. Using the definition of e^x as given above, we combine these power operations as

$$ch(E) = \sum_{k \geq 0} s_k(c_1(E), \ldots, c_k(E))/k!$$

where s_0 is defined to return the rank of E.

Proposition 3.4.1. $ch : K(X) \to H^*(X; \mathbb{Q})$ is a ring homomorphism (where we take $H^*(X; \mathbb{Q})$ to mean the direct product of individual cohomology groups).

Proof. By the definition of $K(X)$ as the Grothendieck closure of $\mathrm{Vect}_{\mathbb{C}}(X)$, it suffices to show that $ch : \mathrm{Vect}_{\mathbb{C}}(X) \to H^*(X; \mathbb{Q})$ is a ring homomorphism. This is not hard to do from the explicit definition. □

Since K-theory is 2-periodic, we define $K^*(X) = K^0(X) \oplus K^1(X)$. (Similarly for reduced K-theory). The main result with Chern characters is the following theorem, which states that torsion free parts of K-theory and singular cohomology agree!

Theorem 3.4.2. *Let X be a finite cell complex. Then*

$$ch : K^*(X) \otimes \mathbb{Q} \to H^*(X; \mathbb{Q})$$

is an isomorphism of rings.

Proof. We give a proof outline. First one can show that $ch : \widetilde{K}(\mathbb{S}^{2n}) \to \widetilde{H}^{2n}(\mathbb{S}^{2n}; \mathbb{Q})$ is injective, with image the subgroup $\widetilde{H}^{2n}(\mathbb{S}^{2n}; \mathbb{Z}) \subset \widetilde{H}^{2n}(\mathbb{S}^{2n}; \mathbb{Q})$. This follows inductively by using the Bott periodicity theorem (since the result is trivial for the zero sphere). From here, we have an isomorphism $ch : K^0(X) \otimes \mathbb{Q} \to H^{\text{even}}(X; \mathbb{Q})$ whenever X is an even dimensional sphere. One can define a map $K^1(X) \otimes \mathbb{Q} \to H^{\text{odd}}(X; \mathbb{Q})$ since odd rational singular cohomology is isomorphic to even rational singular cohomology of the suspension! This is trivially an isomorphism for even dimensional spheres.

Next to achieve the result for a general space X, we use induction. A finite cell complex K' is obtained by gluing a cell to a subcomplex K, and then K'/K is a sphere. This induces a short exact sequence. The iductive step follows from the short five-lemma. □

While this shows that torsion free parts of K-theory agree with that of singular cohomology, we have yet to exhibit an example of a space whose (integral) K-theory disagrees with its singular cohomology. Note that we computed K-theory for spheres and complex projective space. These were

- $K^0(\mathbb{S}^{2n}) = \mathbb{Z} \oplus \mathbb{Z}, K^1(\mathbb{S}^{2n}) = 0$. Therefore $K^*(\mathbb{S}^{2n}) = \mathbb{Z} \oplus \mathbb{Z} = H^*(\mathbb{S}^{2n})$.
- $K^0(\mathbb{S}^{2n+1}) = \mathbb{Z}, K^1(\mathbb{S}^{2n+1}) = \mathbb{Z}$. Therefore $K^*(\mathbb{S}^{2n}) = \mathbb{Z} \oplus \mathbb{Z} = H^*(\mathbb{S}^{2n})$.
- $K^0(\mathbb{CP}^n) = \mathbb{Z}^{n+1}, K^1(\mathbb{CP}^n) = 0$. Therefore $K^*(\mathbb{CP}^n) = \mathbb{Z}^{n+1}$. On the other hand, $\mathbb{CP}^n$ has singular cohomology $\mathbb{Z}, 0, \mathbb{Z}, 0, \ldots, 0, \mathbb{Z}$ up to dimension $2n$, so that $H^*(\mathbb{CP}^n) = \mathbb{Z}^{n+1}$ as well!

There we wish to find a space so that either its K-theory is torsion free, or its singular cohomology is torsion free. An example is given by Lie groups - the proof is outside the scope of what I'm about to type, but one can show that $K(G)$ is always torsion free for G a simply connected Lie group. However, one can find Lie groups whose integral singular cohomology have torsion.

A more manageable example is the real projective plane, $\mathbb{RP}^2$. None the less, the K theory of $\mathbb{RP}^2$ is actually difficult to compute! In a later chapter concerning equivariant cohomology we will introduce required machinery to compute the K-theory of $\mathbb{RP}^2$.

Chapter 4

(Co)bordism theory

In this chapter we develop yet another significant (co)homology theory - namely (co)bordism theory. Via the Pontryagin-Thom construction, we relate cobordism to stable homotopy. In particular, the framed cobordism group associated to a point is isomorphic to the stable homotopy groups of spheres. We will also compute several low dimensional examples. As with the previous chapters, several sources will be used, but the primary source is [Mil01].

4.1 Unoriented bordism as a homology theory

Closed n-manifolds M_1 and M_2 are said to be *cobordant* (or *bordant*) if there is a compact $(n+1)$-manifold N with boundary $\partial N = M_1 \sqcup M_2$. The terminology is derived from *bord*, which is French for *boundary.* Notice that $M_1 \sqcup M_2$ is the boundary of N, while N is the coboundary of $M_1 \sqcup M_2$. This is why we usually say cobordism rather than bordism. Hereafter we will use the term bordism as we are developing a homology theory. We use cobordism in the contravariant setting.

In this section we will develop unoriented bordism as a homology theory. This corresponds to "bordism without structure", and in the following section we will consider additional structure as well as the Pontryagin-Thom construction, and cobordism as a cohomology theory. More concretely, we will define the notion of a G-bordism, and we will observe that unoriented bordism is O-bordism, oriented bordism is SO-bordism, and framed bordism 1-bordism. As mentioned above, we will not delve into this generality for now and simply consider unoriented bordism.

Definition 4.1.1. Let M be a closed n-manifold. Then M is *null-bordant* if there a compact $(n+1)$-manifold N with boundary $\partial N = M$. Two n-manifolds M and M' are *bordant* if $M \sqcup M'$ is null-bordant.

One can show that bordism is an equivalence relation. Disjoint union descends to a well defined operation on bordism classes. Under this operation, the class of null-bordant

n-manifolds is a two sided identity. Moreover, every manifold is bordant to itself, so every bordism class has order two (and in particular an inverse) under disjoint union. It follows that the collection of bordism classes of n-manifolds forms a group.

Definition 4.1.2. For each n, $\mathcal{N}_n = \{\text{closed } n\text{-manifolds}\}/\text{bordism}$ is an abelian group under disjoint union called the *unoriented bordism group*. It is also denoted by Ω_n^O (for reasons which will become clear later).

Since every element in the unoriented bordism group has order 2, it is actually a $\mathbb{Z}/2\mathbb{Z}$-vector space.

Example. $\mathcal{N}_0 = \mathbb{Z}/2\mathbb{Z}$.

Every closed 0-manifold is a finite collection of points. Therefore closed 0-manifolds are classified by cardinality. On the other hand, every compact 1-manifold is a disjoint union of arcs and circles. The boundary of each arc has cardinality two. Therefore a zero manifold M is null-bordant if and only if it has parity 0. The above claim follows.

Example. $\mathcal{N}_1 = 0$.

Every closed 1-manifold is a disjoint union of circles, so these are again classified by cardinality. Let S_n denote the 1-manifold of n disjoint circles. Then $\mathbb{S}^2 - (\sqcup_n D^2)$ is a compact 2-manifold with boundary S_n. Therefore all closed 1-manifolds are null-bordant, proving our claim.

Example. $\mathcal{N}_2 = \mathbb{Z}/2\mathbb{Z}$.

Every closed 2-manifold is a disjoint union of manifolds of the form $\#^n T$ or $\#^n T \# \mathbb{RP}^2$, where T denotes the 2-torus. It is immediate that all manifolds of the form $\#^n T$ are null-bordant (as they bound the boundary-connected-sums of n solid tori). Therefore all disjoint unions of them belong to a single bordism class, the zero class. We claim that the bordism class of a 2-manifold depends on the parity of the number of disjoint-union components with a factor of $\mathbb{RP}^2$. (That is, whenever $\mathbb{RP}^2$ occurs an even number of times, the manifold is null-bordant. Otherwise it is not null-bordant, and belongs to a unique other bordism class.)

Let M be a closed 2-manifold. Write $M = \sqcup_i \#^{n_i} T \#^{m_i} \mathbb{RP}^2$, where m_i is 0 or 1 for each i. Suppose $2a$ of the m_i are non-zero, for some integer a. We now construct a compact 3-manifold N with boundary M. For each $j \leq a$, let P_j be the projective cylinder $\mathbb{RP}^2 \times [0,1]$. This is a 3-manifold with boundary $\mathbb{RP}^2 \sqcup \mathbb{RP}^2$. Assign each i with m_i non-zero to a boundary component of the P_j. Let Q_{n_i} denote the boundary connected sum of n_i solid tori. For each i with m_i non-zero, take the boundary connected sum of Q_{n_i} with the boundary component of P_j corresponding to i. The disjoint union of all of these manifolds is a compact manifold with boundary M as required.

We have shown that whenever there is an even number of $\mathbb{RP}^2$ factors, a surface is null-bordant. It follows that any two closed surfaces with an odd number of $\mathbb{RP}^2$ factors also belongs to the same bordism class, since their disjoint union then has an even number of $\mathbb{RP}^2$ factors. Therefore to show that $\mathcal{N}_2 = \mathbb{Z}/2\mathbb{Z}$ it remains to show that $\mathbb{RP}^2$ is not null-bordant. We use a well known result that all closed odd dimensional manifolds have Euler characteristic zero. Suppose for a contradiction that M is a compact 3-manifold with boundary $\mathbb{RP}^2$. Then gluing a double of M to itself along its boundary, we obtain a closed odd dimensional manifold M'. But now $\chi(M') = \chi(M) + \chi(M) - \chi(\mathbb{RP}^2)$, so $\chi(\mathbb{RP}^2) = 2\chi(M)$. This is absurd, since $\chi(\mathbb{RP}^2) = 1$.

In the last example, we cited the fact that closed odd dimensional manifolds have trivial Euler characteristic. This can be proven using Morse theory as follows. Let M be a closed manifold with odd dimension n. Let $f : M \to \mathbb{R}$ be a Morse function. By the Morse inequalities,

$$\chi(M) = \sum_{i=0}^{n} (-1)^i N_i,$$

where N_i is the number of index i critical points of f. Now consider the Morse function $-f$. For each i, $-f$ has N_{n-i} critical points of index i. Since n is odd,

$$\chi(M) = \sum_{i=0}^{n} (-1)^i N_{n-i} = -\sum_{i=0}^{n} (-1)^i N_i = -\chi(M).$$

Hence $\chi(M) = 0$ as required.

Examples in higher dimensions get more difficult! We will compute these using the Pontryagin-Thom construction in the following section.

If (W, M_1, M_2) is a cobordism of m-manifolds, and (V, N_1, N_2) is a cobordism of n-manifolds, then $W \times V$ is a cobordism from $M_1 \times N_1$ to $M_2 \times N_2$. Therefore there is map

$$\mathcal{N}_n \otimes \mathcal{N}_m \to \mathcal{N}_{n+m}, \quad [M] \otimes [N] \mapsto [M \times N].$$

This turns $\mathcal{N}_* = \bigoplus_n \mathcal{N}$ into a graded ring, called the *bordism ring*.

So far we have defined individual bordism groups and the bordism ring. However, this looks nothing like a homology theory! We want a way of assigning bordism groups to any given space.

Definition 4.1.3. Let X be a space, and $f : M \to X$ a continuous map from a closed smooth n-manifold M. The pair (M, f) is called a *singular manifold*. We say that singular manifolds (M_1, f_1) and (M_2, f_2) are *bordant* if there is a bordism W from M_1 to M_2, and there is a continuous map $F : W \to X$ which restricts on the boundaries of W to f_1 and f_2. The *unoriented bordism group of* X is then defined to be

$$\mathcal{N}_n(X) \coloneqq \{\text{singular } n\text{-manifolds } M \to X\}/\text{bordism}.$$

Remark. We immediately see that the bordism groups $\mathcal{N}_n$ are given by $\mathcal{N}_n(*)$, i.e. $\mathcal{N}_n$ is the unoriented bordism group of the one point space.

Example. We saw earlier that $\mathcal{N}_1 = \mathcal{N}_1(*)$ was trivial. However, $\mathcal{N}_1(\mathbb{S}^1)$ is non-trivial. Consider the singular manifold $(\mathbb{S}^1, \mathrm{id})$. We show that this is not null-bordant, and hence does not belong to the trivial class in $\mathcal{N}_1(\mathbb{S}^1)$.

Suppose for a contradiction that there is a 2-manifold Σ with boundary $\mathbb{S}^1$, and a map $F : \Sigma \to \mathbb{S}^1$ which restricts to the indentity on $\partial\Sigma$. This is equivalent to a retraction $r : \Sigma \to \partial\Sigma$, but the existence of such a map is forbidden by the retraction theorem (a generalisation of Brouwer's fixed point theorem).

To turn bordism into a homology theory, we still need a notion of relative homology: $\mathcal{N}_n(X, A)$. In particular, the bordism group $\mathcal{N}_n(X)$ should be isomorphic to $\mathcal{N}_n(X, *)$.

Definition 4.1.4. Let (X, A) be a pair of spaces with $A \subset X$. Let M be a compact manifold. We say that (M, f) is a *singular manifold* if $f : M \to X$, and $f(\partial M) \subset A$.

Let $(M_1, f_1), (M_2, f_2)$ be singular n-manifolds. Since these might have boundary, a bordism W from M_1 to M_2 is now taken to be a compact $(n+1)$-manifold with boundary $M' \cup M_1 \cup M_2$, where M' can be thought of as a bordism from ∂M_1 to ∂M_2. We also require a map $F : W \to X$ which extends f_1 and f_2. Formally, a bordism is the data of (W, F) such that

- $\partial W = M' \cup M_1 \cup M_2$.
- $\partial M' = \partial M_1 \sqcup \partial M_2$, $M' \cap M_i = \partial M_i$.
- $F|_{M_i} = f_i$, where $F : W \to X$.
- $F(M') \subset A$.

The *relative unoriented bordism group* is then

$$\mathcal{N}_n(X, A) = \{\text{singular } n\text{-manifolds } M \to (X, A)\}/\text{bordism}.$$

In general the pair (W, F) is not a singular manifold! That would force $f(M_i) \subset A$, which makes things too trivial to be interesting.

Remark. Consider the pair $(X, *)$. Any singular manifold sends its boundary to $*$, so it factors through a closed singular manifold. Similarly any cobordism factors through a cobordism in the non-relative sense. Therefore $\mathcal{N}_n(X, *) = \mathcal{N}_n(X)$.

We are almost ready to turn this into a homology theory. It remains to describe some maps!

- Let $i : A \to X$ be the inclusion map. Then any singular manifold $f : M \to A$ determines a singular manifold $i_* f : M \to A \to X$. This describes the induced map on bordism:
$$i_* : \mathcal{N}_n(A) \to \mathcal{N}_n(X).$$

- Let $f : M \to X$ be a singular manifold. Then it is also a singular manifold of (X, A), since the boundary of M trivially maps into A (that is, the boundary of M is empty). This describes the induced map
$$j_* : \mathcal{N}_n(X) \to \mathcal{N}_n(X, A).$$
- Let $f : M \to (X, A)$ be a singular manifold. Then $f|_{\partial M} : \partial M \to A$ is a singular manifold of one less dimension. This describes the connecting homomorphism
$$d_* : \mathcal{N}_n(X, A) \to \mathcal{N}_{n-1}(A).$$

One can show that this gives a long exact sequence
$$\cdots \to \mathcal{N}_{n+1}(X, A) \to \mathcal{N}_n(A) \to \mathcal{N}_n(X) \to \mathcal{N}_n(X, A) \to \mathcal{N}_{n-1}(A) \to \cdots .$$

The first observation is that this is a *generalised homology theory* as it does not satisfy the dimension axiom. That is, the homology of a point is not concentrated in degree 0. We saw earlier that
$$\mathcal{N}_0(*) = \mathbb{Z}/2\mathbb{Z}, \mathcal{N}_1(*) = 0, \mathcal{N}_2(*) = \mathbb{Z}/2\mathbb{Z}.$$

In the next section we will formally encounter bordism groups with additional structure, but oriented bordism is easy to describe without the additional formalism. Therefore we introduce it here, and compute a few low dimensional examples.

Definition 4.1.5. Let M_1, M_2 be closed oriented n-manifolds. They are *oriented bordant* if there is an oriented compact $(n + 1)$-manifold W such that $\partial W = M_1 \sqcup (-M_2)$. The group of oriented cobordism classes of n-manifolds is denoted by Ω_n, or Ω_n^{SO}.

Analogously to the unoriented case, singular oriented manifolds give rise to oriented cobordism groups of spaces: $\Omega_n(X)$. These give rise to a homology theory.

Example. $\Omega_0 = \mathbb{Z}$.

Recall that 0-manifolds are classified by cardinality. Oriented 0-manifolds are further classified by a choice of sign for each component. Thus oriented 0-manifolds are in bijective correspondence with pairs (m, n) of non-negative integers (with m the number of positively signed points, and n the number of negatively signed points). A 0-manifold is null-cobordant if and only if $m - n = 0$, in which case each positively signed point is paired with a negatively signed point, and this gives an oriented 1-manifold. It follows that two oriented 0-manifolds are cobordant if and only if they have the same difference $m - n$. $\Omega_0 = \mathbb{Z}$.

Example. $\Omega_1 = 0$.

This is analogous to the unoriented case. Every oriented closed 1-manifold is a disjoint union of oriented circles, but these are all easily seen to be null-bordant (as they bound disks).

Example. $\Omega_2 = 0$.

Oriented closed 2-manifolds are exactly connected sums of tori and their disjoint unions. These all bound disjoint unions of boundary connected sums of solid tori, and are therefore null bordant.

Example. $\Omega_3 = 0$.

This follows from the surgery classification of oriented 3-manifolds (e.g. given at the end of my knot theory notes). Namely, every closed connected oriented 3-manifold is obtained by a finite sequence of 1-surgeries on the 3-sphere. (Explicitly, at each step a copy of $\mathbb{S}^1 \times D^2$ is removed and replaced with $D^2 \times \mathbb{S}^1$. But now consider a 4-ball D^4 bound by $\mathbb{S}^3$. With each surgery, replacing $\mathbb{S}^1 \times D^2$ with $D^2 \times \mathbb{S}^1$ corresponds to gluing a 2-handle to D^4. Therefore every connected oriented closed 3-manifold bounds a four dimensional handlebody. It follows that all closed oriented 3-manifolds bound an oriented 4-manifold (and are hence null bordant).

Example. $\Omega_4 = \mathbb{Z}$.

This is a little more difficult! We will not really compute it, but we will show that there is a surjection $\sigma : \Omega_4 \to \mathbb{Z}$. Recall that for an oriented closed 4-manifold, the cup product induces a symmetric unimodular bilinear form called the *intersection form*. The signature of this form is an invariant of the manifold, which we also call the signature and denote by $\sigma(M)$. We first claim that the signature of oriented 4-manifolds is and oriented bordism invariant.

Suppose that M_1 and M_2 are bordant oriented closed 4-manifolds. Then $M \coloneqq M_1 \sqcup (-M_2)$ is null bordant. If we can show that $\sigma(M) = 0$, it follows that $\sigma(M_1) = \sigma(M_2)$. Our proof outline is as follows:

1. Let W be the oriented compact 5-manifold with boundary M. Let $i : M \to W$ be the inclusion. We show that the restriction of the intersection form Q_M to $i^*(H^2(W;\mathbb{R}))$ is trivial.

2. Next we show that $\dim H^2(M) = 2 \dim i^*(H^2(W;\mathbb{R}))$.

3. Finally we write $H^2(M;\mathbb{R}) = P \oplus N$, where P is the subspace on which Q_M is positive definite, and N is the subspace on which Q_M is negative definite. Using the previous two results, we show that $\dim P = \dim N$, so the signature of M is 0.

1. Let i^*a, i^*b be arbitrary elements of $i^*(H^2(W;\mathbb{R}))$. To show that $Q_M(i^*a, i^*b) = 0$, it suffices to show that $\langle Q_M(i^*a, i^*b), [M]\rangle = 0$. Using naturality of fundamental classes and so on, a calculation gives

$$\begin{aligned} \langle Q_M(i^*a, i^*b), [M]\rangle &= \langle i^*(a \smile b), [M]\rangle \\ &= \langle i^*(a \smile b), \delta_*[W, M]\rangle \\ &= \langle \delta^* i^*(a \smile b), [W, M]\rangle = \langle 0, [W, M]\rangle. \end{aligned}$$

2. We know that $\dim H^2(M;\mathbb{R}) = \dim\ker(\delta^*) + \dim\ker(\delta^*)^\perp$. By exactness, we can write $\ker(\delta^*) = \mathrm{im}(i^*) = i^*(H^2(W;\mathbb{R}))$. On the other hand, by Poincaré duality, we have $\ker(\delta^*)^\perp \cong \ker(i_*)^\perp$. Now from the universal coefficient theorem, one can show that $\ker(i_*)^\perp \cong \mathrm{im}(i^*)$.

3. Finally write $H^2(M;\mathbb{R}) = P \oplus N$ as above. If $\dim P \neq \dim N$, then at least one of them (say P) has dimension greater than $i^*(H^2(W;\mathbb{R}))$. Then P intersects $i^*(H^2(W;\mathbb{R}))$ along a subspace of positive dimension, contradicting 1. Thus $\dim P = \dim N$ as required, so $\sigma(M) = 0$.

This shows that there is a well defined map $\sigma : \Omega_4 \to \mathbb{Z}$. To see that it is surjective, it suffices to notice that $\mathbb{CP}^2$ has signature 1.

4.2 G-Bordism and the Pontryagin-Thom construction

So far we have computed $\mathcal{N}_n$ for $n \leq 2$ and Ω_n for $n \leq 4$. To compute higher bordism groups, we use the Pontryagin-Thom construction. In this section we will also define the notion of a G-bordism, unifying oriented and unoriented bordism under a more general theory.

To define G-bordisms, we require the notion of stable normal bundles. We also describe some notation: a bundle in general will be denoted by a symbol such as ξ. The total space, base space, and projection map will usually be denoted by $E(\xi), B(\xi)$, and π_ξ.

Definition 4.2.1. Let M be an n manifold, and $i : M \to \mathbb{R}^{n+k}$ an embedding. The *normal bundle* of M is the quotient bundle $\nu = NM = i^*(T\mathbb{R}^{n+k})/TM$. The normal bundle depends on the embedding, but any two normal bundles are stably isomorphic. The *stable normal bundle* of M is the stable isomorphism class of a normal bundle of $M \to \mathbb{R}^{n+k}$.

When no additional structure is prescribed, the stable normal bundle is classified by a homotopy class in $[M, BO]$. An unoriented cobordism W can be embedded in $\mathbb{R}^{n+k}$, and its stable normal bundle is a class in $[W, BO]$. Thus unoriented cobordism can be considered to be orthogonal cobordism, which explains the notation Ω_*^O.

Definition 4.2.2. We now define the notion of a G-structure on a manifold. Let G_n be a sequence of groups with maps $G_n \to G_{n+1}$ and representations $G_n \to O(n)$. Suppose moreover that each square

$$\begin{tikzcd} G_n \arrow[r] \arrow[d] & G_{n+1} \arrow[d] \\ O(n) \arrow[r] & O(n+1) \end{tikzcd}$$

commutes. The stable normal bundle of a manifold M is classified by a homotopy class ν of maps $[M, BO]$. A G-structure on M is a lift of ν to a class $\widetilde{\nu}$ in $[M, BG]$ where $BG \to BO$ is the fibration induced by the $G_n \to O(n)$.

Example. An SO structure on a manifold is an orientation of its (stable) normal bundle. Therefore oriented bordism groups arise by considering exactly manifolds with SO-structures. In the following discussion we formalise what this means for general G.

Definition 4.2.3. Let $(M, \widetilde{\nu})$ be a G-manifold. $(M, \widetilde{\nu})$ is *null bordant* if there is a compact manifold W embedded in $\mathbb{R}^{n+k} \times \mathbb{R}_+$ such that

- $\partial W = W \cap (\mathbb{R}^{n+k} \times \{0\}) \cong M$.
- The above intersection is transverse.
- The classifying map of the normal bundle of W lifts as in the following diagram:

$$\begin{tikzcd} M \arrow[r, "\widetilde{\nu}"] \arrow[d, hook] & BG \arrow[d] \\ W \arrow[r, "f"] \arrow[ur, dashed, "\widetilde{f}"] & BO. \end{tikzcd}$$

A G-manifold $(M, \widetilde{\nu})$ has a negative, defined by embedding $M \times [0,1]$ in $\mathbb{R}^{n+k} \times \mathbb{R}_+$ as above, inducing a G-structure on $M \times [0,1]$, and restricting to $M \times \{0\}$.This turns bordisms of G-manifolds into a group, which we call the *G-bordism group*, and denote by Ω_*^G.

More generally the above construction can be repeated with singular G-manifolds $(M, f, \widetilde{\nu})$ with $f : M \to X$, which gives rise to G-bordism groups $\Omega_*^G(X)$.

Example. Unoriented bordism groups are exactly O-bordism groups, so

$$\mathcal{N}_*(X) = \Omega_*^O(X).$$

Oriented bordism groups are obtained by ensuring that all stable normal bundles are oriented, so they correspond to SO-bordism groups:

$$\Omega_*(X) = \Omega_*^{\mathrm{SO}}(X).$$

We can also restrict to $U(n) \hookrightarrow O(2n)$, which gives bordisms of almost complex manifolds called complex bordisms, or U-bordisms:

$$\Omega_*^U(X).$$

Finally we introduce the *framed bordism group.* This consists of bordisms with trivial normal bundle,

$$\Omega_*^{\text{framed}}(X) = \Omega_*^1(X).$$

The key theorem for this section is the *Pontryagin-Thom construction.*

Theorem 4.2.4. *Given some G, there is an isomorphism*

$$\Omega_*^G(X) \cong MG_*(X) := \pi_*(MG \wedge X_+)$$

of graded rings.

We do not prove the statement, but describe the construction of MG and the isomorphism in the forwards direction.

Recall the construction of the *Thom space* given a vector bundle: if $p : E \to B$ is a real vector bundle of rank n, each fibre is isomorphic to $\mathbb{R}^n$, and so can be one-point-compactified. This gives an n-sphere bundle $S(E) \to B$. $S(E)$ can be further quotiented so that all newly added points are identified - this is the *Thom space*, $T(E)$. Notice that if B is compact, then $T(E)$ is the one point compactification of E.

Generally given a vector bundle ξ, we write $T(\xi)$ to denote the Thom space (of the total space) of the vector bundle. The Thom construction is a covariant functor, and satisfies $T(\xi \times \zeta) = T(\xi) \wedge T(\zeta)$.

We now describe the Pontryagin-Thom construction. Let $(M, f, \widetilde{\nu})$ be a singular G-manifold in X. Fix an embedding $i : M \hookrightarrow \mathbb{R}^{n+k}$. Then $\widetilde{\nu} : M \to BG_k$ classifies the normal bundle. That is, we have the commuting diagram

$$\begin{tikzcd}
E(\nu) = \widetilde{\nu}^* EG_k \arrow[r] \arrow[d] & EG_k \arrow[d] \\
M \arrow[r, "\widetilde{\nu}"] & BG_k
\end{tikzcd}$$

We also have a map $f : M \to X$. The product of the above vector bundle with the 0-vector bundle over X gives a diagram

$$\begin{tikzcd}
E(\nu) \arrow[r] \arrow[d] & EG_k \times X \arrow[d] \\
M \arrow[r, "(\widetilde{\nu}, f)"] & BG_k \times X
\end{tikzcd}$$

Write ξ_k to denote the bundle $EG_k \to BG_k$. Then by functoriality, we have a map

$$T(\nu) \to T(\xi_k \times 0).$$

Notice that the 0-bundle over X has Thom space X_+, the one point compactification of X. On the other hand, there is an embedding from $E(\nu)$ into $\mathbb{R}^{n+k}$ (which can be thought of as a tubular neighbourhood). Since one point compactification of embeddings is a contravariant functor, this gives a map $\mathbb{S}^{n+k} \to T(\nu)$. In summary, we have a map

$$\mathbb{S}^{n+k} \to T(\xi_k) \wedge X_+.$$

In particular, we have a homotopy class in $\pi_{n+k}(MG_k \wedge X_+)$, we have written MG_k to denote $T(\xi_k)$. What is the effect of stabilisation? One can show that that extending the embedding to $\mathbb{R}^{n+k+1}$ gives a map $\mathbb{S}^{n+k+1} \to MG_{k+1} \wedge X_+$, which commutes with a map $\Sigma\mathbb{S}^{n+k} \to \Sigma MG_k \wedge X_+$. In summary, a singular G-manifold $(M, f, \widetilde{\nu})$ determines an element of

$$(MG)_n(X) \coloneqq \lim_k \pi_{n+k}(MG_k \wedge X_+) = \pi_n(MG \wedge X_+).$$

This is the *Pontryagin-Thom construction*; a map

$$\Omega_n^G(X) \to \pi_n(MG \wedge X_+).$$

What does this look like for certain examples of G? We begin by investigating the case when G is the trivial group; this should be the easiest example.

Example. Fix G to be the trivial group. Then the universal bundle $EG \to BG$ is the trivial bundle of one point spaces, and $MG = T(EG)$ is the one point extension of EG. Since EG is already compact, this corresponds to gluing an additional isolated point, so MG is the two point space $\mathbb{S}^0$. By the Pontryagin-Thom construction, we have isomorphisms

$$\Omega_n^{\text{framed}}(X) \cong \pi_n^{\text{st}}(\mathbb{S}^0 \wedge X).$$

In particular, when X is trivial, this gives

$$\Omega_n^{\text{framed}} \cong \pi_n^{\text{st}}(\mathbb{S}^0) = \lim_k \pi_{n+k}(\mathbb{S}^k).$$

We can use this to calculate some stable homotopy groups of spheres! Let's try to compute $\pi_1^s \coloneqq \pi_1^{\text{st}}(\mathbb{S}^0) \cong \Omega_1^{\text{framed}}$. A framing of the stable normal bundle of a closed 1-manifold is equivalent to a framing of its tangent bundle. It suffices to understanding the cobordism class of each framing of $T\mathbb{S}^1$ embedded in $\mathbb{R}^2$. There are two possible framings (think of the tangent vectors being clockwise or anticlockwise). However, these are really the same as we can flip $\mathbb{R}^2$ upside down in $\mathbb{R}^3$. To show that Ω_1^{framed} is non-trivial, we must show that $\mathbb{S}^1$ is not null-bordant. In fact, by the previous argument, this also shows that $\Omega_1^{\text{framed}} \cong \mathbb{Z}/2\mathbb{Z}$. To see that $\mathbb{S}^1$ is not framed null-bordant, suppose for a contradiction that it is. The framing of its tangent space is then a restriction of the framing of the tangent space of some surface in $\mathbb{R}^3$ with boundary $\mathbb{S}^1$. But any such framing "cannot rotate" (to ensure global triviality), while the framing of $\mathbb{S}^1$ rotates around the boundary. This is of course just a visual sketch and not a proof! It follows that $\pi_1^s = \mathbb{Z}/2\mathbb{Z}$, and in particular $\pi_4(\mathbb{S}^3) = \mathbb{Z}/2\mathbb{Z}$.

As a final remark, we are also now equipped to define cobordism theory as a cohomology theory. By the Pontryagin-Thom construction, we have an isomorphism

$$\Omega_n^G(X) \cong \pi_n(MG \wedge X_+) = \lim_k \pi_{n+k}(MG_k \wedge X_+) = \lim_k [\mathbb{S}^{n+k}, MG_k \wedge X_+].$$

We can define a dual (contravariant) theory by moving the factor of X_+ in the right expression. Namely, the nth *Cobordism group* is given by

$$\Omega_G^n(X) = \lim_k [\mathbb{S}^{k-n} \wedge X_+, MG_k] = \lim_k [\Sigma^{k-n} X_+, MG_k].$$

Example. Consider X to be the one point space. Then $[\Sigma^{k-n} X_+, MG_k] = [\mathbb{S}^{k-n}, MG_k]$. On the other hand, $[\mathbb{S}^{n+k}, X_+ \wedge MG_k] = [\mathbb{S}^{n+k}, MG_k]$. This shows that $\Omega_*^G(X)$ and $\Omega_G^*(X)$ differ only by grading! Therefore bordism and cobordism are interchangable. (This is no longer true if X is not a one point space.)

Chapter 5

Equivariant cohomology

In this chapter we introduce *ordinary equivariant cohomology*, which is an equivariant cohomology that simultaneously generalises group cohomology and singular cohomology. This is also called *Borel cohomology*, or just *equivariant cohomology.* In particular, we discuss a case of the celebrated localisation theorem of equivariant cohomology. After this, we will explore another type of equivariant cohomology - namely equivariant K-theory. In the earlier section on K-theory we remarked that computing the K-theory of $\mathbb{RP}^n$ was actually beyond our capabilities! However, the machinery of equivariant K-theory will allow us to do this. In these notes the primary source for equivariant cohomology is [Bot98], while the primary source for equivariant K-theory is [Seg68].

5.1 Equivariant cohomology fundamentals

In this section we develop the definitions and machinery of equivariant cohomology and investigate some examples. In ordinary (singular) cohomology or homology, we learn information about a space based purely on its topology. However, many topological spaces have some symmetry (i.e. admit group actions). The idea of equivariant cohomology is to incorporate this symmetry into the cohomology theory in the hope that it will tell us more about the space. Moreover, it turns out that a sort of converse holds: knowing equivariant cohomological information about X says a lot about the possible group actions it admits. Without further ado, here are some definitions!

Definition 5.1.1. Let X be a space and G a group. An action of G is a group homomorphism $G \to \mathrm{Aut}(X)$. Equivalently, it is a map $X \times G \to X$ such that $x \cdot e = x$ for all x, and $x \cdot (gh) = (x \cdot g) \cdot h$ for all x.

The group action is said to be *free* if $x \cdot g = x$ implies that g is the identity. In this instance, the equivalence classes of the relation $x \sim y$ defined by $x \cdot g = y$ for some g inherit a lot of structure from X. This quotient space is denoted by X/G.

When X is a smooth manifold and G is a compact group acting freely on X, the quotient X/G is itself a smooth manifold, and the map $X \to X/G$ is a principal G-bundle over X/G.

What happens when the action of G is not free? The quotient space will no longer have the principal G-bundle over it, and quite frankly a lot of things break. For this reason we introduce the *homotopy quotient*.

If p is a single point, an action of any group G on p can be defined to be the trivial action. This is incredibly non-free! But EG (introduced back in the chapter on classifying spaces) has the same homotopy type as p, while also admiting a natural free action of G. Then $EG \to EG/G = BG$ is the "homotopy quotient" of the one point space. This extends more generally for any space X.

Definition 5.1.2. Let X be a space and G a group action on X. Then EG admits a free action of G, which further induces the free diagonal action on $X \times EG$. Notice that $X \times EG$ has the same homotopy type as X! The *homotopy quotient* of X, denoted $X//G$ or X_G, is the quotient

$$X \times EG \to (X \times EG)/G = X \times_G EG = X_G.$$

This is actually all we need to define before introducing equivariant cohomology!

Definition 5.1.3. The equivariant cohomology of a space X with a group action G is the singular cohomology of the homotopy quotient of X:

$$H^*_G(X) = H^*(X_G).$$

Remark. The homotopy quotient X_G comes equipped with a natural principal G-bundle as written above. This can alternatively be thought of as the pullback of the universal bundle $EG \to BG$ by the map $f : X_G \to BG$ defined by modding out by X.

Example. Suppose G is the trivial group. Then EG and BG are trivial, and X_G is just X! Therefore equivariant cohomology with G trivial returns ordinary cohomology.

Next suppose X is a trivial space. Then X_G is equal to BG, so $H^*_G(X) = H^*(BG)$. This is exactly the group cohomology of G! Therefore equivariant cohomology is a simultaneous generalisation of singular cohomology for a space and group cohomology.

Proposition 5.1.4. Suppose G acts freely on a space X. Then $H^*_G(X) = H^*(X/G)$.

This theorem justifies that the homotopy quotient is really a generalisation of the quotient of a free action to general actions.

Proof. Consider the map $p : X_G \to X/G$ with fibre EG. Since EG is contractible, by a classical result, there is an isomorphism of cohomology $H^*(X_G) \cong H^*(X/G)$. The former is exactly the equivariant cohomology! $\square$

Proposition 5.1.5. Let G act on a space X. Suppose $K \lhd G$ acts *freely* on X. Then X/K admits a G/K action, and

$$H_G^*(X) \cong H_{G/K}^*(X/K).$$

This theorem hints at the fact that equivariant cohomology is concentrated at fixed points of an action.

Proof. The space $EG \times E(G/K)$ is contractible and admits a free G action (namely the diagonal action). Therefore we have $X_G = X \times_G (EG \times E(G/K))$. But now we have a fibre bundle

$$X \times_G (EG \times E(G/K)) \to X \times_G E(G/K)$$

with fibre EG. By the same result used in the previous theorem, it follows that $H^*(X_G) = H^*(X \times_G E(G/K))$. But on the right hand side, we have $X \times_G E(G/K) = (X/K) \times_{G/K} E(X/G) = (X/K)_{G/K}$. This completes the proof that

$$H_G^*(X) = H^*(X_G) = H^*((X/K)_{G/K}) = H_{G/K}^*(X/K).$$

□

For G the trivial group, we showed earlier that $H_G^*(X) = H^*(X)$. But what if G is a non-trivial group that acts trivially on X?

Proposition 5.1.6. Let G act trivially on X. Then $H_G^*(X) \cong H^*(X) \otimes H^*(BG)$. In particular, $H_G^*(X)$ is a free $H^*(BG)$ module.

Proof. If G acts trivially, then $X_G = X \times_G EG = X \times BG$, so the result follows from the Künneth formula. □

Proposition 5.1.7. Suppose $K < G$ acts on Y. Define $X \coloneqq G \times_K Y$. This admits a canonical G action, and $H_G^*(X) = H_K^*(Y)$.

Proof. Notice that $G \times_K Y = (G \times Y)/K$. We define the G action to be trivial on the second component. Now

$$X_G = (G \times_K Y) \times_G EG = Y \times_K EG = Y \times_K EK = Y_K.$$

The central equality is because K acts freely on EG, and EG is contractible. Therefore "EG is an EK". □

Example. We now compute the equivariant cohomology of several spheres with $\mathbb{S}^1$-actions. First, we consider the $\mathbb{S}^1$ action on $\mathbb{S}^1$ itself. This is free, so the equivariant cohomology is given by

$$H_{\mathbb{S}^1}^*(\mathbb{S}^1) \cong H^*(\mathbb{S}^1/\mathbb{S}^1) = H^*(\mathrm{pt}).$$

Thus $\mathbb{S}^1$ has trivial equivariant cohomology.

$\mathbb{S}^1$ also admits an action on $\mathbb{S}^2$, by rotation. This is immediately less tractable as the action is not free. We do not compute this for now, as it is beyond our capabilities. However, we will later use the *Leray spectral sequence* to attempt the calculation.

Finally we look at the $\mathbb{S}^1$ action on $\mathbb{S}^3$ defining the Hopf fibration. This is again free, and $\mathbb{S}^3/\mathbb{S}^1 = \mathbb{S}^2$. Therefore $H^*_{\mathbb{S}^1}(\mathbb{S}^3) = H^*(\mathbb{S}^2)$.

Loosely speaking, the localisation theorem states that the equivariant cohomology is concentrated near the fixed points of a group action. Observe that the free actions above had easily computable equivariant cohomology, but the action of $\mathbb{S}^1$ on $\mathbb{S}^2$ (which has two fixed points) was elusive. We can carry out the calculation using the localisation theorem. The localisation theorem is stated in terms of de Rham theory, so before stating and proving this result, we must develop equivariant de Rham theory. (Of course, de Rham cohomology of a smooth manifold is isomorphic to singular cohomology of the manifold with real coefficients, so this is really an alternative model for the same theory.)

We give a brief recapitulation of de Rham theory. Let M be a smooth manifold. Then TM denotes its tangent bundle, and T^kM denotes the k-fold tensor product bundle of TM with itself. This contains the symmetric and antisymmetric tensor bundles S^kM and Λ^kM as subbundles. Sections $\tau \in \Gamma(T^kM)$ are called *k-tensor fields*, or just *tensors*, and a particularly important class of tensor fields are precisely the antisymmetric ones: $\omega \in \Gamma(\Lambda^kM) = \Omega^k(M)$ are called *differential k-forms*.

There is a notion of differentiation $d : \Omega^k(M) \to \Omega^{k+1}(M)$ called the *exterior derivative*. This defines the de Rham complex

$$C^\infty(M) = \Omega^0(M) \to \Omega^1(M) \to \cdots \to \Omega^n(M),$$

where $\dim M = n$. The homology of this sequence is the *de Rham cohomology*.

One of the main reason differential forms are important is because they are exactly what we can integrate. For M compact and oriented, there is a natural map

$$H^n(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega.$$

This can be interpreted as a map $H^n(M) \to H^0(\mathrm{pt})$, and this interpretation generalises to integration over fibres of a bundle $E \to B$.

Let $\pi : E \to B$ be a fibre bundle. Then a homomorphism $\pi_* : H^*(E) \to H^{*-f}(B)$ is induced which "integrates over the fibres". Here f is the dimension of the fibre. This is not generally a ring homomorphism, although it is additive. We can consider $H^*(E)$ as a module over $H^*(B)$, by defining $u \cdot v := u \smile \pi^*(v)$. Then π_* is a module homomorphism.

Now we consider a special case. We have a natural map $M_G \to BG$ which defines a fibration. Integration of the fibres gives a map $\pi_* : H^*_G(M) \to H^*(BG)$. This is the Weil-Cartan model of equivariant cohomology.

We now make things a little more specific! We are generally interested in the case where G is a compact connected Lie group. In this instance, the following theorem shows that the cohomology of a G is captured entirely by its Lie algebra $\mathfrak{g}$:

Theorem 5.1.8. *For G a compact and connected Lie group there is an isomorphism*

$$H^*(\Omega^*(\mathfrak{g}^*)) \to H^*(\Omega^* G) = H^*(G).$$

We will soon use this result to further develop the Cartan model of equivariant cohomology.

Proof. Here we have written $H^*(\Omega^*(\mathfrak{g}^*))$ and $H^*(\Omega^* G)$ to denote the homolologies of the complexes $\Omega^*(\mathfrak{g}^*)$ and $\Omega^* G$. $\Omega^*(\mathfrak{g}^*)$ consists of the left invariant differential forms on G. There is an inclusion $i : \Omega^*(\mathfrak{g}^*) \to \Omega^* G$ which induces the forwards map. On the other hand, we can construct a map $\Omega^* G \to \Omega^*(\mathfrak{g}^*)$ by averaging forms over G (by compactness). This map is a left inverse of i. To see that it also a right inverse, it remains to show that all classes in $H^*(G)$ are left-invariant. This follows from connectedness. $\square$

This result hints that maybe $EG \to BG$ can also be understood infinitesimally! This is indeed the case. Writing $[-]^G$ to denote the G-invariant subspace, we can define

$$\Omega^i_G(M) = \bigoplus_j [S^j(\mathfrak{g}^*) \otimes \Omega^{i-2j}(M)]^G.$$

These form a complex with $d_G : \Omega^i_G(M) \to \Omega^{i+1}_G(M)$ defined as follows: fix a basis $\{\xi_a\}$ of $\mathfrak{g}$, and let $\{f_a\}$ be its dual basis. Then

$$d_G(F \otimes \sigma) \coloneqq F \otimes d\sigma - \sum_a f_a F \otimes \iota_{\xi_a} \sigma.$$

This definition is inspired by the Cartan formula for the Lie derivative. We are now ready to describe the final form of the Cartan model for equivariant cohomology.

Theorem 5.1.9. *The equivariant cohomology $H^*_G(M)$ is the cohomology of the complex*

$$0 \to \Omega^0_G(M) \xrightarrow{d_G} \Omega^1_G(M) \xrightarrow{d_G} \Omega^2_G(M) \to \cdots .$$

Proof. We give only a proof sketch. First, one observes that the de Rham complex of $M_G = M \times_G EG = (M \times EG)/G$ is embedded in the de Rham complex of $M \times EG$ as G-invariant subspaces. Therefore we find the de Rham complex of $M \times EG$, and restrict to the invariant part. The de Rham complex of the first factor is just $\Omega^*(M)$, but we must determine $\Omega^*(EG)$ in simpler terms. One can show (e.g. in Bott's notes cited at the start of the chapter) that $\Omega^*(EG)$ is the *Koszul complex* $W \coloneqq S^*(\mathfrak{g}^*) \otimes \Lambda^*(\mathfrak{g}^*)$. This gives

$$H^*_G(M) = H^*(M \times_G EG) \cong H^*([\Omega(M) \otimes W]^G).$$

The desired isomorphism now follows from the underlying algebra. $\square$

5.2 Leray spectral sequence

The Leray spectral sequence is one of the original spectral sequences, invented (or discovered, if you prefer) before we even knew how to formalise spectral sequences. In modern times, the Leray spectral sequence is considered a special case of the Grothendieck spectral sequence. None the less, we will not discuss the Leray spectral sequence in that generality, and simply state the result here. After developing this machinery we will apply it in a calculation of the equivariant cohomology of the 2-sphere.

Definition 5.2.1. Fix an abelian category (such as the category of modules over a ring). A *cohomological spectral sequence* is a collection of three sequences:

- For each $r \geq 0$, an object E_r called the *rth page*.
- Maps $d_r : E_r \to E_r$ satisfying $d_r^2 = 0$, called *boundary maps*.
- Isomorphisms of E_{r+1} with the homology $H(E_r)$ of E_r with respect to d_r.

Generally the isomorphism is implicit, with $E_{r+1} \coloneqq H(E_r)$.

We will further assume that each E_r is bigraded:

$$E_r = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q}.$$

The boundary maps are taken to have bidegree $(r, -r+1)$. That is, each d_r restricts to maps

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

If we further assume $E_r^{p,q} = 0$ for $p, q < 0$, we obtain a *first quadrant spectral sequence*.

Proposition 5.2.2. A first quadrant spectral sequence as above eventually stabilises. That is, for any (p,q), there exists r such that $E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} \cdots$. This stationary object is denoted by $E_\infty^{p,q}$.

Proof. Suppose $r \geq \max(p+1, q+2)$. Recall that d_r has bidegree $(r, -r+1)$. But now $E_{r+1}^{p,q}$ is the quotient

$$\frac{\ker d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}}{\operatorname{im} d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}}.$$

By our choice of r, $q - r + 1 \leq q - (q+2) + 1 \leq -1$. Therefore $E_r^{p+r,q-r+1}$ is trivial, so the kernel (in the numerator of the quotient) is all of $E_r^{p,q}$. Similarly the image (in the denominator) can be shown to be trivial, so $E_{r+1}^{p,q} = E_r^{p,q}$. $\square$

To state Leray's theorem on spectral sequences, we introduce two filtrations of the spectral sequence.

Definition 5.2.3. A *filtration* on an module M is a decreasing sequence of submodules $M = M_0 \supset M_1 \supset M_2 \supset \cdots$. The *associated graded module*, denoted by GM, is the direct sum of succcessive quotients

$$GM = M_0/M_1 \oplus M_1/M_2 \oplus M_2/M_3 \oplus \cdots .$$

Notice that a spectral sequence $E = \bigoplus_{p,q} E^{p,q}$ has two natural filtrations - namely by p and by q. Formally, the *filtration by p* is

$$F_p = \bigoplus_{i \geq p} \bigoplus_{q} E^{i,q}.$$

The *filtration by q* is

$$F'_q = \bigoplus_{i \geq q} \bigoplus_{p} E^{p,i}.$$

This means that $F_0 = E$, but F_1 is missing the first "column" of each page, F_2 the first two "columns" and so on. The associated graded module is obtained by direct sums of each column. We are now ready to state Leray's theorem. The proof is not included, but can be found in Weibel's classic text on homological algebra (as the Grothendieck spectral sequence).

Theorem 5.2.4. *Let $\pi : E \to B$ be a fibre bundle, with fibre F. Suppose B is simply connected. Assume (for each n) that $H^n(F)$ is free of finite rank. Then there exists a spectral sequence E satisfying the following:*

1. $E_2^{p,q} = H^p(B) \otimes H^q(F)$.
2. $E_2^{p,q} \Rightarrow H^{p+q}(E)$. *More precisely, the filtration D_p by p of the second page induces a filtration $D_p \cap H^n(E)$ on $H^n(E)$ whose successive quotients are $E_\infty^{p,n-p}$.*

As an application, we will use this to compute some equivariant cohomology!

Example. Let $\mathbb{S}^1$ act on $\mathbb{S}^2$ by rotations. We will determine $H^*_{\mathbb{S}^1}(\mathbb{S}^2)$. There is a fibration

$$\mathbb{S}^2 \to \mathbb{S}^2 \times_{\mathbb{S}^1} E\mathbb{S}^1 \to E\mathbb{S}^1/\mathbb{S}^1 = B\mathbb{S}^1.$$

By definition we know that $H^*_{\mathbb{S}^1}(\mathbb{S}^2) = H^*(\mathbb{S}^2 \times_{\mathbb{S}^1} E\mathbb{S}^1)$, and in chapter 1 we showed that $B\mathbb{S}^1 = \mathbb{CP}^\infty$. The corresponding Leray spectral sequence is a spectral sequence (E, d) with

- $E_2^{p,q} = H^p(\mathbb{S}^2) \otimes H^q(\mathbb{CP}^\infty) = \frac{\mathbb{Z}[x]}{(x^2)} \otimes \mathbb{Z}[y]$, where x and y have degree 2.
- $E_2^{p,q} \Rightarrow E_\infty^{p,q}$, and $H^n_{\mathbb{S}^1}(\mathbb{S}^2) = \bigoplus_{p+q=n} E_\infty^{p,q}$.

$$
\begin{array}{c|cccccccccc}
\vdots & & & & & & & & & & \\
3 & & & & & & & & & & \\
2 & x & & xy & & xy^2 & & xy^3 & & xy^4 & \cdots \\
1 & & & & & & & & & & \\
0 & 1 & & y & & y^2 & & y^3 & & y^4 & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots
\end{array}
$$

Figure 5.1: Second page of the Leray spectral sequence.

To make the next argument clearer, we draw the second page of the spectral sequence: Recall that on the rth page, the boundary maps d_r have bi-degree $(r, -r+1)$. In particular the two components of the bi-degrees have different parities for each r. Therefore any d_r mapping in or out of a non-zero term in the second page must be trivial (as all non-zero terms occur with both components having even degree). It follows that $E_r^{p,q} = H(E_r) = E_{r-1}$. In particular, $E_\infty^{p,q} = E_2^{p,q}$. But now $H^{2n}_{\mathbb{S}^1}(\mathbb{S}^2) = E_2^{2n,0} \oplus E_2^{2n-2,2} = xy^{n-1}\mathbb{Z} \oplus y^n\mathbb{Z}$, and the equivariant cohomology of $\mathbb{S}^2$ in odd degrees vanish. Overall we have

$$H^*_{\mathbb{S}^1}(\mathbb{S}^2) = \mathbb{Z}[y] \oplus x\mathbb{Z}[y].$$

5.3 Localisation theorem

The most important result in equivariant cohomology is the localisation theorem. We will state a general form along with a "Cartan model form". First we must recall what localisation even means! The general form is taken from Tom Dieck's *Transformation groups.*

Definition 5.3.1. Let R be an arbitrary unital ring. A subset S of R is said to be *multiplicative* if S is closed under multiplication and contains unity. The *localisation* of R at S, denoted $S^{-1}R$, is the ring obtained by inverting all elements of S. Formally, it is the ring satisfying the following universal property: there is a canonical map ψ, and given any $f : R \to T$ sending all elements of S to units, there is a unique lift g as in the diagram.

$$
\begin{tikzcd}
R \arrow[r, "\psi"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dotted, "g"] \\
 & T
\end{tikzcd}
$$

Intuitively, the localisation can be thought of as zooming into the parts of R in S, which kills information. (E.g. zooming into the surface of a sphere, things look flat - killing the curvature information). As a result the information remaining in $\mathbb{S}^{-1}R$ is the information of R less the information of S.

Definition 5.3.2. Let M be an R-module, and $S \subset R$ multiplicative. Then the localisation of M at S, $S^{-1}M$, is $M \otimes_R S^{-1}R$.

We now list some elementary properties of localisations.

Proposition 5.3.3.

- Localisation $R \mapsto S^{-1}R$ is a functor.
- Infact, localisation is an exact functor: if $A \to B \to C$ is exact, then $S^{-1}A \to S^{-1}B \to S^{-1}C$ is exact.
- Localisation commutes with colimits (also known as direct limits).

Suppose $A \subset X$ is a subspace, with G acting on both A and X. The inclusion $A \to X$ induces a homomorphism $H_G^*(X) \to H_G^*(A)$. The $H_G^*(X), H_G^*(A)$ can be thought of as $H_G^*(\mathrm{pt}) = H^*(BG)$-modules. Therefore given any $S \subset H^*(BG)$ multiplicative, we have an induced map

$$S^{-1}H_G^*(X) \to S^{-1}H_G^*(A).$$

The localisation theorem concerns when this induced map is an isomorphism. Finish this section later.

5.4 Equivariant K-theory

The equivariant cohomology theory developed above for ordinary cohomology also applies to K-theory! We outline the construction now.

To define equivariant K-theory, we must define a notion of equivariant vector bundles over a G-space. (By a G-space, we mean a space X equipped with a group action by G.)

Definition 5.4.1. Let G act on X (so that X is a *G-space*). By a *G-vector bundle* on X, we mean a map $\pi : E \to X$ such that

- E is a G-space, and π is a G-map. (That is, $g(\pi(\xi)) = \pi(g\xi)$.)
- $\pi : E \to X$ is a complex vector bundle.
- For any g, the action $g : E_x \to E_{gx}$ is a linear map.

Example. Suppose X is equipped with the trivial g action. Then $\pi(\xi) = \pi(g\xi)$ for all g and all ξ. Thus each g defines an element of (E_x). Equivalently, a G-vector bundle is a family of representations $G \to (E_x)$ where E_x is parametrised by $x \in X$ and vary smoothly.

More specifically, suppose X is the one point space. Then $K_G(X)$ is the completion of the space of all complex representations of G! This is called the *representation ring* of G, denoted by $R(G)$.

Example. Suppose X is a smooth manifold, and a Lie group G acts on X. Then $T_X \otimes C$ is a G-vector bundle (by acting trivially on the second component and canonically on the first).

For simplicitly we assume hereafter that all groups G are compact. We are now already ready to define equivariant K-theory:

Definition 5.4.2. The space of all G-vector bundles over a G-space X forms a monoid. The Gröthendieck completion of this monoid is denoted by $K_G(X)$, and is called the *equivariant K-theory* of X. As with usual K-theory, this can be intuitively thought of as consisting of formal differences $E_G - F_G$ of G-vector bundles over X.

It is not yet clear why this defines a theory analogous to ordinary equivariant cohomology. Can we interpret $K_G(X)$ as $K(X \times_G EG)$ or something similar? It turns out that we almost can but not quite!

Proposition 5.4.3. Define $K_G^{\mathrm{Bor}}(X) \coloneqq K(X_G)$, where $X_G = X \times_G EG$. Then there is a canonical map

$$K_G(X) \to K_G^{\mathrm{Bor}}(X),$$

but this is not generally an isomorphism.

Proof. The projection map $X \times EG \to X$ defines a map $K_G(X) \to K_G(X \times EG)$ by pullback. But G acts freely on $X \times EG$, so $K_G(X \times EG) \cong K((X \times EG)/G)$ (as we will very shortly prove!) This defines the map $K_G(X) \to K(X \times_G EG) = K_G^{\mathrm{Bor}}(X)$.

To see that the above map is not generally an isomorphism, fix X to be the one point space. Then $K_G(X) = R(G)$, while $K_G^{\mathrm{Bor}}(X) = K(BG)$. These are not isomorphic whenever G is a non-trivial group. Why? Let's ask Ciprian! $\square$

In the above proof, we made use of a fact that we like to believe is true without verification! We state and prove this fact now.

Proposition 5.4.4. Let G act freely on a space X. Then $K_G(X) = K(X/G)$. More generally, if N is a normal subgroup of G acting freely on X, then $K_G(X) = K_{G/N}(X/N)$.

Proof. Consider the projection $X \to X/G$. This induces a map $p^* : K(X/G) \to K(X)$. In fact, pulling back a vector bundle over X/G to a bundle over X automatically turns the bundle into a G-bundle! So the induced map can be further refined to $p^* : K(X/G) \to K_G(X)$.

Now suppose G acts freely on X. Then if $E \to X$ is a G-bundle, there is an induced bundle $E/G \to X/G$. In this case, the map $E \mapsto E/G$ is inverse to p^*! Therefore p^* is an isomorphism. $\square$

Proposition 5.4.5. In the case where G acts trivially on X, there is a natural isomorphism

$$R(G) \otimes K(X) \to K_G(X).$$

This is reminiscent of the correspond theorem for ordinary equivariant cohomology, in which $H_G^*(X) \cong H^*(BG) \otimes H^*(X)$ for trivial G actions.

Next we develop some properties of equivariant K-theory that are reminiscent of the usual K-theory introduced in an earlier chapter. Namely, equivariant K-theory is also a cohomology theory! To this end we also introduce reduced equivariant K-theory.

Definition 5.4.6. Let G act on a space X. The stable isomorphism classes of G-vector bundles over X form a ring called the *reduced equivariant K-theory* of X, denoted $\widetilde{K}_G(X)$.

Following the same construction as usual K-theory, the sequence of inclusions of G-spaces

$$A \to X \to X \cup CA$$

induces an exact sequence of reduced equivariant K-theories

$$\widetilde{K}_G(X \cup CA) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A).$$

Moreover, taking suspensions, we have a long exact sequence

$$\cdots \to \widetilde{K}_G(\Sigma(X \cup CA)) \to \widetilde{K}_G(\Sigma X) \to \widetilde{K}_G(\Sigma A) \to \widetilde{K}_G(X \cup CA) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A).$$

We introduce the notation $\widetilde{K}_G^{-q}(X) = \widetilde{K}_G(\Sigma^q X)$, and $\widetilde{K}_G^{-q}(X, A) = \widetilde{K}_G(\Sigma^q(X \cup CA))$. This gives a cohomology theory $\widetilde{K}_G^*(X)$.

As with the usual K-theory, the above reduced cohomology theory has an unreduced version. This is K_G^* defined by

$$K_G^n(X) = \widetilde{K}_G^n(X_+), \quad K_G^n(X, A) = \widetilde{K}_G^n(X_+, A_+),$$

where X_+ and A_+ are the spaces $X \cup \{*\}$ and $Y \cup \{*\}$.

Again, analogously to usual K-theory, if A is a closed G-contractible subspace of a G-space X, then $K_G(X/A)$ is isomorphic to $K_G(X)$. This implies the following result:

Proposition 5.4.7. If A is a closed G-subspace of a locally compact G-space X, then the natural map

$$K_G^n(X - A) \to K_G^n(X, A)$$

is an isomorphism.

Next we state the equivariant periodicity theorem.

Theorem 5.4.8. *$K_G^{-q}(X)$ is naturally isomorphic to $K_G^{-q-2}(X)$. The map is given by multiplication by a certain element of $K_G^{-2}(*)$. In particular, by defining $K_G^1 := K_G^{-1}$, the long exact sequence of equivariant K-theory reads*

$$\begin{array}{ccccc} K_G^0(X,A) & \longrightarrow & K_G^0(X) & \longrightarrow & K_G^0(A) \\ \uparrow & & & & \downarrow \\ K_G^1(A) & \longleftarrow & K_G^1(X) & \longleftarrow & K_G^1(X,A). \end{array}$$

So far we have just been establishing results that are analogous to results form usual K-theory of equivariant (ordinary) cohomology. We finish with one of the most important results from equivariant K-theory; one which has yet to be mentioned in a more specific form!

Theorem 5.4.9. *Let X be a locally compact space, and $E \to X$ a G-vector bundle. There is an isomorphism $K_G^*(X) \to K_G^*(E)$ called the* Thom isomorphism.

Notice that E is not compact! $K_G^*(E)$ is defined to be the equivariant cohomology of vector bundles with compact support. Equivalently, it can be defined as the equivariant cohomology of the Thom space $T(E)$. We will not prove this theorem, but loosely speaking the isomorphism can be considered multiplication by $(1-L)^n$, where n is the rank of E and L is the tautological bundle over the projectivisation of E.

This concludes our general study of equivariant K-theory. We finish with an example - namely, by using equivariant K-theory, we compute the usual K-theory of $\mathbb{RP}^n$.

Example. Let $\mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{S}^n$ in the usual way. (That is, the non-trivial element swaps each point in $\mathbb{S}^n$ with its antipode.) This a free action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{S}^n$, and

$$\mathbb{S}^n/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{RP}^n.$$

By the general theory, we now know that $K_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) = K(\mathbb{RP}^n)$. We will use the long exact sequence of equivariant K-theory to compute $K_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n)$.

$$\begin{array}{ccccc} K^0_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1},\mathbb{S}^n) & \longrightarrow & K^0_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1}) & \longrightarrow & K^0_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) \\ \uparrow & & & & \downarrow \\ K^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) & \longleftarrow & K^1_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1}) & \longleftarrow & K^1_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1},\mathbb{S}^n). \end{array}$$

Suppose n is odd. Since D is G-contractible, we have $K^i_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1}) \cong K^i_{\mathbb{Z}/2\mathbb{Z}}(*)$. For $i=0$, we have $K^i_{\mathbb{Z}/2\mathbb{Z}}(*) = K_{\mathbb{Z}/2\mathbb{Z}}(*) \cong R(\mathbb{Z}/2\mathbb{Z}) \otimes K(*) \cong R(\mathbb{Z}/2\mathbb{Z})$. Now $R(\mathbb{Z}/2\mathbb{Z})$ is determined by the images of the generator of $\mathbb{Z}/2\mathbb{Z}$. The only constraint is that it has order 2, so $R(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}[x]/(x^2-1)$.

More formally, consider the map $\mathbb{Z}[x] \to R(\mathbb{Z}/2\mathbb{Z})$ defined by sending x to the sign representation $\mathbb{Z}/2\mathbb{Z} \to (\mathbb{C})$, and $n \in \mathbb{Z}$ to the trivial representation $\mathbb{Z}/2\mathbb{Z} \to (\mathbb{C}^n)$. This extends uniquely to a ring homomorphism $\mathbb{Z}[x] \to R(\mathbb{Z}/2\mathbb{Z})$ whose kernel is (x^2-1).

Thus we have shown that $K^0_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1}) = \mathbb{Z}[x]/(x^2-1)$. Next we study $K^1_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1})$. Again this is isomorphic to $K^1_{\mathbb{Z}/2\mathbb{Z}}(*) \cong \widetilde{K}^1_{\mathbb{Z}/2\mathbb{Z}}(*) \cong \widetilde{K}_{\mathbb{Z}/2\mathbb{Z}}(I)$. Here I is the interval, which is the suspension of a point. The $\mathbb{Z}/2\mathbb{Z}$ action is the unique non-trivial symmetry of I. This has a G-contraction to a point. Therefore

$$K^1_{\mathbb{Z}/2\mathbb{Z}}(*) \cong \widetilde{K}_{\mathbb{Z}/2\mathbb{Z}}(*) \cong 0.$$

The last isomorphism comes from the fact that all $\mathbb{Z}/2\mathbb{Z}$-vector bundles over a point are stably isomorphic, corresponding to $x^2 = 1$ in the previous (non-reduced) calculation.

Next we investigate the two relative K-theories. Notice that $K_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1}, \mathbb{S}^n)$ consists of formal differences of $\mathbb{Z}/2\mathbb{Z}$-vector bundles over D^{n+1} which are trivial on $\mathbb{S}^n$. But $K_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^{n+1})$ consists of formal differences of $\mathbb{Z}/2\mathbb{Z}$-vector bundles which are compactly supported. Therefore by appropriate radial scaling, we can identify $K_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1}, \mathbb{S}^n) \cong K_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{R}^{n+1})$. Moreover, $\mathbb{R}^{n+1}$ is isomorphic to $\mathbb{C}^m$ for some m, with the isomorphism preserving the $\mathbb{Z}/2\mathbb{Z}$ structure (since n was chosen to be odd). This is just a $\mathbb{Z}/2\mathbb{Z}$-vector bundle over a point! By the Thom isomorphism, this gives $K_{\mathbb{Z}/2\mathbb{Z}}(D^{n+1}, \mathbb{S}^n) \cong K_{\mathbb{Z}/2\mathbb{Z}}(*)$. Applying this calculation to the previous two calculations for $K^i_{\mathbb{Z}/2\mathbb{Z}}(*)$, the long exact rectangle evaluates to

$$\begin{tikzcd}
\mathbb{Z}[x]/(x^2-1) \arrow[r] & \mathbb{Z}[x]/(x^2-1) \arrow[r] & K^0_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) \arrow[d] \\
K^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) \arrow[u] & 0 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

To determine $K^0_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n)$, we must understand the map $\mathbb{Z}[x]/(x^2-1) \to \mathbb{Z}[x]/(x^2-1)$. This consists of several maps being composed, but the one non-trivial map comes from the Thom isomorphism $K_{\mathbb{Z}/2\mathbb{Z}}(*) \to K_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{C}^m)$. This map is multiplication by $(1-x)^m$. Therefore the image of this map is the ideal $(1-x)^m$. By exactness, this is the kernel of the subsequent map $\mathbb{Z}[x]/(x^2-1) \to K_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n)$. This map is necessarily surjective, so the first isomorphism theorem gives

$$K_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) \cong \frac{\mathbb{Z}[x]}{(x^2-1, (1-x)^m)}.$$

We can re-write this as $\mathbb{Z}[y]/(y^2-2y, y^m)$. Elements of this ring are of the form $ay+b$. We now right this as an abelian group (ignoring the ring structure). The coefficient b is not constrained. However, we also have

$$0 = y^m = 2y^{m-1} = 2^2 y^{m-2} = \cdots = 2^{m-1} y.$$

Therefore a is constrained modulo 2^{m-1}. As a group, we have

$$K^0(\mathbb{RP}^n) = K^0_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) = \mathbb{Z} \oplus \mathbb{Z}/2^{m-1}\mathbb{Z}.$$

Next we calculate $K^1(\mathbb{RP}^n) = K^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n)$. This time the map $K^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^n) \to \mathbb{Z}[x]/(x^2-1)$ is injective, so the domain is isomorphic to its image. By exactness, this is isomorphic to the kernel of $\mathbb{Z}[x]/(x^2-1) \to \mathbb{Z}[x]/(x^2-1)$ given by multiplication by $(1-x)^m$. A similar calculation to above shows that the kernel is isomorphic to $\mathbb{Z}$. Therefore

$$K^1(\mathbb{RP}^n) = \mathbb{Z}.$$

In summary for n odd, $K^0(\mathbb{RP}^n) = \mathbb{Z} \oplus \mathbb{Z}/2^{m-1}\mathbb{Z}$, and $K^1(\mathbb{RP}^n) = \mathbb{Z}$.

For n even, we would like to say that $K^0(\mathbb{RP}^{2m})K^0_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^{2m}) = K^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{S}^{2m-1}) = K^1(\mathbb{RP}^{2m-1})$ and bootstrap off the odd dimensional calculations. However, this does not hold because the suspension of $\mathbb{S}^{2m-1}$ does not give $\mathbb{S}^{2m}$ with the antipodal action; but rather an action of $\mathbb{Z}/2\mathbb{Z}$ with two fixed points. The computation for even dimensions again requires the long exact sequence.

Chapter 6

Atiyah-Hirzebruch spectral sequence

In this final section, we introduce an important tool used in the calculation of generalised (co)homology theories, namely the Atiyah-Hirzebruch spectral sequence. Will will use this to compute examples of K-theories and bordism groups (which are the two generalised (co)homology theories we have encountered in these notes).

6.1 AHSS in general

We start by recalling the Leray spectral sequence from the previous chapter:

Theorem 6.1.1. *Let $\pi : E \to B$ be a fibre bundle, with fibre F. Suppose B is simply connected, and each $H^n(F)$ is free of finite rank. Then there is a spectral sequence E satisfying*

- $E_2^{p,q} = H^p(B) \otimes H^q(F)$.
- $E_2^{p,q} \Rightarrow H^{p+q}(E)$.

The maps d_r on the rth page are taken to have bidegree $(r, -r+1)$.

This theorem has rather strict premises - for one, it applies only to fibre bundles. The more standard generality of this theorem is the *Serre spectral sequence.*

Theorem 6.1.2. *Let $F \to E \to B$ be a Serre fibration. (That is, it has the homotopy lifting property with respect to CW complexes.) There is a* Serre spectral sequence *E satisfying*

- $E_2^{p,q} = H^p(B; H^q(F))$.
- $E_2^{p,q} \Rightarrow H^{p+q}(E)$.

Technically we must be a little careful: in general $H^p(B; H^q(F))$ denotes the cohomology of B with coefficients in the *local system* $H^q(F)$. If B is simply connected, the local system $H^q(F)$ is guaranteed to be constant.

Finally, the *Atiyah-Hirzebruch spectral sequence* (AHSS) is a further generalisation of the Serre spectral sequence, applying to not only to ordinary cohomology but also to generalised cohomology theories.

Theorem 6.1.3. *Let $F \to E \to B$ be a Serre fibration of finite type (i.e. the spaces are CW complexes with finitely many cells). Let K be a generalised cohomology theory. Then there is an* Atiyah-Hirzebruch spectral sequence *E satisfying*

- $E_2^{p,q} = H^p(B; K^q(F))$.
- $E_2^{p,q} \Rightarrow K^{p+q}(E)$.

We do not prove this result (that is, we do not give a construction of the spectral sequence). However, we will investigate two examples! The most significant application is that every space X trivially has a fibration $\mathrm{pt} \to X \to X$, and the Atiyah-Hirzebruch spectral sequence then gives the generalised cohomology of X in terms of the ordinary cohomology of X, with coefficients the generalised cohomology of a point.

6.2 AHSS applied to K-theory and cobordism theory

In the previous chapter, we computed the K-theory of $\mathbb{CP}^n$ and $\mathbb{RP}^{2n+1}$, the former following directly from the long exact sequence of K-theory and the latter using equivariant K-theory. Next we will give calculations using the AHSS. The first step is to find appropriate fibrations.

Consider $\mathrm{pt} \to \mathbb{CP}^n \to \mathbb{CP}^n$. Then the second page of the corresponding Atiyah-Hirzebruch spectral sequence is $H^p(\mathbb{CP}^n; K^q(\mathrm{pt})$. The K theory of a point is given by

$$K^q(\mathrm{pt}) = \begin{cases} \mathbb{Z} & q \text{ even} \\ 0 & q \text{ odd.} \end{cases}.$$

Therefore for q odd, the entries on the second page vanish. For q even, we have the integral cohomology of $\mathbb{CP}^n$, namely

$$H^p(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & 0 \le p \le n,\ p \text{ even} \\ 0 & \text{else.} \end{cases}.$$

Overall this gives the second page as in the following figure: The differential on the rth page has bidegree $(r, 1-r)$, so by the same argument used with the Leray spectral sequence in the previous chapter, we find that $E_2^{p,q} = E_\infty^{p,q}$. By inspection, it follows that $K^0(\mathbb{CP}^n) = \mathbb{Z}^{n+1}$,

$\vdots$	$\vdots$		$\vdots$				$\vdots$			
2	$\mathbb{Z}$		$\mathbb{Z}$		$\cdots$		$\mathbb{Z}$			
1										
0	$\mathbb{Z}$		$\mathbb{Z}$		$\cdots$		$\mathbb{Z}$			
-1										
-2	$\mathbb{Z}$		$\mathbb{Z}$		$\cdots$		$\mathbb{Z}$			
$\vdots$	$\vdots$		$\vdots$				$\vdots$			
	0	1	2	3	$\cdots$	$2n-1$	$2n$	$2n+1$	$2n+2$	$\cdots$

Figure 6.1: Second page of the Atiyah-Hirzebruch spectral sequence for $\mathbb{CP}^n$.

while $K^1(\mathbb{CP}^n) = 0$. Notice that the spectral sequence is periodic in the vertical direction. This demonstrates Bott periodicity.

Next we wish to study the K-theory of $\mathbb{RP}^n$. There are two cases to consider - n odd and n even. Since we focused on the odd case earlier, we will again repeat the odd case now to create a better contrast. We use the fibration $\mathrm{pt} \to \mathbb{RP}^{2k+1} \to \mathbb{RP}^{2k+1}$. We know the integral cohomology of $\mathbb{RP}^{2k+1}$: $\mathbb{Z}$ in the bottom and top degrees, $\mathbb{Z}/2\mathbb{Z}$ for even middle degrees, and 0 elsewhere. The corresponding second page of the Atiyah-Hirzebruch spectral sequence is given in the following diagram. In this example it is slightly less immediate that

$\vdots$	$\vdots$		$\vdots$				$\vdots$	$\vdots$		
2	$\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$		$\cdots$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$		
1										
0	$\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$		$\cdots$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$		
-1										
-2	$\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$		$\cdots$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$		
$\vdots$	$\vdots$		$\vdots$				$\vdots$	$\vdots$		
	0	1	2	3	$\cdots$	$2k-1$	$2k$	$2k+1$	$2k+2$	$\cdots$

Figure 6.2: Second page of the Atiyah-Hirzebruch spectral sequence for $\mathbb{RP}^{2k+1}$.

all differentials vanish, but again they do! The maps with non-trivial domain and codomain are exactly those that map from some $\mathbb{Z}/2\mathbb{Z}$ into $\mathbb{Z}$ - but any such map is automatically trivial. It follows that the second page again collapses to the last page. However, we now meet the extension problem for spectral sequences! The cohomology in degree n must have successive quotients $E_\infty^{p,q}$ where $p+q=n$. But then, for example, a first quadrant spectral sequence whose last page has $E^{1,0}=E^{0,1}=\mathbb{Z}/2\mathbb{Z}$ could have first cohomology $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Each of these are distinct as abelian groups, but are both extensions of $\mathbb{Z}/2\mathbb{Z}$ by itself.

In our specific example, we can conclude that $K^0(\mathbb{RP}^{2k+1})$ and $K^1(\mathbb{RP}^{2k+1})$ both have rank 1, but K^1 is torsion free while K^0 has torsion of size 2^{k-1}. While we do not know whether the torsion is $\mathbb{Z}/2^{k-1}\mathbb{Z}$, $(\mathbb{Z}/2\mathbb{Z})^{k-1}$, or something else, (meaning we have objectively less information than our previous calculation in which it was shown that the torsion is $\mathbb{Z}/2^{k-1}\mathbb{Z}$), the work was a lot easier!

Finally we look at an example in cobordism theory. Recall that $\mathcal{N}_i(X)$ is the ith unoriented bordism group of a space X. For X trivial, we computed these spaces for small i. As soon as X is non-trivial, the calculations get a little more difficult! Consider X to be the circle. We will compute $\mathcal{N}_1(\mathbb{S}^1)$. To this end, we use the Atiyah-Hirzebruch spectral sequence:

$$E^2_{p,q}=H_p(\mathbb{S}^1;\mathcal{N}_q(\mathrm{pt})),\quad E^\infty_{p,q}=\mathcal{N}_{p+q}(\mathbb{S}^1).$$

Recall from earlier that $\mathcal{N}_q(\mathrm{pt})$ was calulcated for small q:

$$\mathcal{N}_0(\mathrm{pt})=\mathbb{Z}/2\mathbb{Z},\mathcal{N}_1(\mathrm{pt})=0,\mathcal{N}_2(\mathrm{pt})=\mathbb{Z}/2\mathbb{Z},\cdots.$$

Since the integral cohomology of $\mathbb{S}^1$ is known, these together give an Atiyah-Hirzebruch spectral sequence with the second page as in the following diagram: Therefore by "adding

$$\begin{array}{c|ccccc}
\vdots & \vdots & \vdots & & & \\
2 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & & \\
1 & 0 & 0 & 0 & & \\
0 & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & & \\
\hline
 & 0 & 1 & 2 & 3 & \cdots
\end{array}$$

Figure 6.3: Second page of the Atiyah-Hirzebruch spectral sequence for $\mathcal{N}(\mathbb{S}^1)$.

up the diagonals with bidegree summing to 1" we find that $\mathcal{N}_1(\mathbb{S}^1)=\mathbb{Z}/2\mathbb{Z}$. This is a little surprising! Why is a double cover $f:\mathbb{S}^1\to\mathbb{S}^1$ null bordant? This is because it bounds a mobius strip! This geometrically shows that the map of degree 0 and the map of degree 1 should determine all bordism classes.

References

[Bot98] Raoul Bott. An introduction to equivariant cohomology. *Quantum Field Theory: Perspective and Prospective*, 530:35–57, 1998.

[Coh98] Ralph Cohen. *The Topology of Fiber Bundles Lecture Notes*. `http://math.stanford.edu/~ralph/fiber.pdf`, 01 1998.

[Hat03] A. Hatcher. *Vector Bundles and K-Theory*. `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`, 2003.

[Kot12] Chris Kottke. *Bundles, classifying spaces, and characteristic classes*. `https://ckottke.ncf.edu/docs/bundles.pdf`, 2012.

[Mil01] Haynes Miller. *Notes on Cobordism*. `https:www-math.mit.edu/~hrm/papers/cobordism.pdf`, 2001.

[Mit06] Stephen A. Mitchell. *Notes on principal bundles and classifying spaces*. `http://www-math.mit.edu/~mbehrens/18.906/prin.pdf`, 2006.

[Seg68] Graeme Segal. Equivariant K-theory. *Publications Mathématiques de l'IHÉS*, 34:129–151, 1968.