FM LASER OSCILLATION – THEORY[1]

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(internal phase perturbation; T)

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FM oscillation of the He-Ne laser has recently been demonstrated by Harris and Targ.[2] To achieve such oscillation, an intracavity phase perturbation is driven at a frequency which is approximately but not exactly that of the axial mode spacing. The laser modes then oscillate with Bessel function amplitudes and with FM phases, and thereby comprise the sidebands of a frequency modulated light signal. This Letter gives a first order theory of such oscillation. The basis of the calculation will be the self-consistency equations of Lamb,[3] which are as follows:

$$(\nu_n + \dot{\phi}_n - \Omega_n)E_n = -1/2\,(\nu/\epsilon_0)\,C_n(t) \qquad (1a)$$

and

$$\dot{E}_n + 1/2\,(\nu/Q_n)\,E_n = -1/2\,(\nu/\epsilon_0)\,S_n(t)\,. \qquad (1b)$$

In deriving these equations, Lamb assumed a cavity electromagnetic field of the form

$$E(z,t) = \sum_n E_n(t)\cos\left[\nu_n t + \phi_n(t)\right] U_n(z)\,, \quad (2a)$$

where $U_n(z) = \sin(n\pi z/L)$ and a polarization driving the nth mode given by

$$P_n(t) = C_n(t)\cos\left[\nu_n t + \phi_n(t)\right] + S_n(t)\sin\left[\nu_n t + \phi_n(t)\right]\,. \quad (2b)$$

Thus E_n, ν_n, and ϕ_n are the amplitude, frequency,[4] and phase respectively of the nth mode; and $C_n(t)$ and $S_n(t)$ are the in-phase and quadrature components of its polarization. Other symbols are defined as follows: Ω_n = frequency of the nth mode in the absence of a driving polarization; $\Delta\Omega$ = frequency interval between axial modes in the absence of a driving polarization ($\Delta\Omega = \pi c/L$) ; $Q_n = Q$ of the nth mode; ν = average optical frequency.

We assume a dielectric perturbation having a time-varying susceptibility denoted by $\Delta\chi' \cos \nu_m t$, to occupy a length a of the laser cavity, and assume that a is small compared with the total cavity length L. The contribution of the dielectric perturbation to the polarization of the nth cavity mode is then:

$$P_n(t) = \frac{2\epsilon_0 \Delta\chi' \cos \nu_m t}{L} \sum_q E_q(t)\cos\left[\nu_q t + \phi_q(t)\right] \int_0^a U_q(z)\,U_n(z)\,dz\,. \quad (3)$$

We assume that the driving frequency ν_m is approximately equal to $\Delta\Omega$, and that the cavity Q's are sufficiently high that only the contributions of immediately adjacent modes need be retained. We then have:

$$P_n(t) = \frac{a\epsilon_0\Delta\chi'}{2L}\left[E_{n+1}\cos(\nu_n t + \phi_{n+1}) + E_{n-1}\cos(\nu_n t + \phi_{n-1})\right]. \quad (4)$$

We assume the laser medium to possess a linear atomic susceptibility, χ_n'', and thus to contribute the term $\epsilon_0 E_n \chi_n''$ to the quadrature component of the polarization of the nth mode. We neglect the contribution of the atomic media to the in-phase component of the polarization, and thereby neglect mode pulling effects. We next do the following: (1) Resolve $P_n(t)$ of Eq. (3) into in-phase and quadrature components of the form of Eq. (2b). (2) Add the atomic polarizability term to the quadrature component. (3) Substitute the resulting $C_n(t)$ and $S_n(t)$ into Eqs. (1a) and (1b) respectively. (4) Assume that the modulation frequency ν_m differs from the mode spacing frequency by a detuning $\Delta\nu$ such that $\Omega_0 + n\nu_m - \Omega_n = n\Delta\nu$. Thus $n = 0$ denotes the only mode which is exactly on resonance. (5) Define $\theta_n = \phi_n - \phi_{n-1}$. Equations (1a) and (1b) then become:

$$(\dot{\phi}_n + n\Delta\nu)E_n = -\frac{\nu a \Delta\chi'}{4L}\left[E_{n+1}\cos\theta_{n+1} + E_{n-1}\cos\theta_n\right], \quad (5a)$$

and

$$\dot{E}_n + \frac{\nu}{2}\left(\frac{1}{Q_n} + \chi_n''\right)E_n = -\frac{\nu a\Delta\chi'}{4L}\left[-E_{n+1}\sin\theta_{n+1} + E_{n-1}\sin\theta_n\right]. \quad (5b)$$

If the peak single pass phase retardation of the dielectric perturbation is denoted by δ, then $\Delta\chi' = \delta\lambda/\pi a$. We also define

$$\Gamma = \frac{c}{L\Delta\nu}\,\delta = \frac{1}{\pi}\frac{\Delta\Omega}{\Delta\nu}\,\delta\ , \qquad (6a)$$

and

$$\rho_n = \frac{2c\delta}{L\nu\left(\frac{1}{Q_n} + \chi_n''\right)} = \frac{1}{\pi}\frac{\Delta\Omega}{\Delta\nu_1}\delta\ , \qquad (6b)$$

where $\Delta\nu_1$ is the oscillation half width of a single laser mode in the absence of the perturbation. For a lasing media, χ_n'' will be negative and sufficiently large that ρ_n will be almost infinite. We now seek a steady state solution and thus set $\dot{E}_n = \dot{\phi}_n = 0$. Equations (5) then become:

$$\frac{2n}{\Gamma}E_n = -\left[E_{n+1}\cos\theta_{n+1} + E_{n-1}\cos\theta_n\right] \quad (7a)$$

$$\frac{2}{\rho_n}E_n = -\left[-E_{n+1}\sin\theta_{n+1} + E_{n-1}\sin\theta_n\right]\ . \quad (7b)$$

The form of the solution of Eqs. (7) depends on the relative magnitude of Γ and ρ_n. To obtain FM laser oscillation, the modulator is detuned by an amount many times greater than $\Delta\nu_1$; and therefore $\rho_n \gg \Gamma$.

By noting the Bessel function identity[5]

$$\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z)\ , \qquad (8)$$

it is seen that Eqs. (7) are satisfied by the steady state solution $E_n = J_n(\Gamma)$ and $\theta_n = \theta_{n+1} = \pi$. The solution is thus a frequency modulated light signal having a modulation depth of Γ. To obtain this result it was necessary to assume that the infinity of laser modes see a sufficiently negative χ''. This is equivalent to the statement that a pure FM signal must have an infinity of sidebands. In practice, as observed by Harris and Targ,[2] the finite atomic linewidth limits the obtainable Γ.

From Eq. (6), it is seen that Γ varies inversely as the modulator detuning. However, if the detuning becomes sufficiently small that $\Gamma \gg \rho$, *i.e.*, if the modulator frequency is essentially equal to $\Delta\Omega$, then it appears that a stable solution to Eqs. (7) no longer exists. By noting the recurrence formula for the derivatives of Bessel functions, it is seen that in this case Eqs. (5) have the solution

$$E_n(t) = J_n\left(\frac{\delta\Delta\Omega}{\pi}t\right);\ \theta_{n+1} = \theta_n = -\frac{\pi}{2}$$

That is, their solution is an FM signal whose modulation depth is an infinitely growing function of time. The existence of this unstable on-frequency solution has been previously noted by Yariv.[6] The important contribution of the present paper is the existence of the steady-state off-frequency FM solution. Though the present analysis neglects the question of saturation, it appears to explain many of the observations of Harris and Targ.[2] Targ has recently performed an experiment with results in excellent agreement with Eq. (6*a*).[7] The problem of the effect of perturbations in an optical resonator has also been considered analytically by Gordon and Rigden,[8] and by DiDomenico.[9]

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[2] S. E. Harris and R. Targ, *Appl. Phys. Letters* **5**, 202 (1964).

[3] W. E. Lamb, Jr., *Phys. Rev.* **134**, A1429 (1964).

[4] Following Lamb's notation, we adopt the convention that all symbols for frequencies shall denote circular frequencies.

[5] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Cambridge, England, 1962), p 17.

[6] A. Yariv, Seminar at Stanford University and private communications.

[7] R. Targ (private communications).

[8] E. I. Gordon and J. D. Rigden, *Bell System Tech. J.* **XLII**, 1, 155 (1963).

[9] M. J. DiDomenico, Jr., *J. Appl. Phys.* (to be published, Oct. 1964).