CS154

Lecture 7:
Finish up Communication,
Start up Turing Machines
Communication Complexity

A theoretical model of distributed computing

- **Function** $f : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}$
  - Two inputs, $x \in \{0, 1\}^*$ and $y \in \{0, 1\}^*$
  - We assume $|x| = |y| = n$. Think of $n$ as HUGE

- **Two computers**: Alice and Bob
  - Alice *only* knows $x$, Bob *only* knows $y$

- **Goal**: Compute $f(x, y)$ by communicating as few bits as possible between Alice and Bob

*We do not count computation cost.* We only care about the number of bits communicated.
Def. A *protocol* for a function \( f \) is a pair of functions \( A, B : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1, \text{STOP}\} \) with the semantics:

On input \((x, y)\), let \( r := 0, b_0 = \varepsilon \).

While \((b_r \neq \text{STOP})\),

\[

r + +

\]

If \( r \) is odd, Alice sends \( b_r = A(x, b_1 \cdots b_{r-1}) \)
else Bob sends \( b_r = B(y, b_1 \cdots b_{r-1}) \)

Output \( b_{r-1} \). Number of rounds \( = r - 1 \)
Def. The cost of a protocol $P$ for $f$ on $n$-bit strings is
$$\max_{x,y \in \{0,1\}^n} \text{number of rounds in } P \text{ to compute } f(x, y)$$

The communication complexity of $f$ on $n$-bit strings is the minimum cost over all protocols for $f$ on $n$-bit strings = the minimum number of rounds used in any protocol for computing $f(x, y)$ over all $n$-bit $x, y$
Example. Let $f : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$ be arbitrary.

There is always a "trivial" protocol:

- **Alice** sends bits of $x$ in odd rounds
- **Bob** sends bits of $y$ in even rounds

After $2n$ rounds, they both know each other’s input!

*The communication complexity of every $f$ is at most $2n$*
Example. $\text{EQUALS}(x, y) = 1 \iff x = y$

What’s a good protocol for computing $\text{EQUALS}$?

$\text{Communication complexity of EQUALS is at most } 2n$
Connection to Streaming and DFAs

Let $L \subseteq \{0,1\}^*$

Def. $f_L: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$ for $x, y$ with $|x| = |y|$ as:

$$f_L(x, y) = 1 \iff xy \in L$$

Examples:

$L = \{ x \mid x \text{ has an odd number of 1s}\}$
$$\Rightarrow f_L(x, y) = \text{PARITY}(x,y) = \sum_i x_i + \sum_i y_i \mod 2$$

$L = \{ x \mid x \text{ has more 1s than 0s}\}$
$$\Rightarrow f_L(x, y) = \text{MAJORITY}(x,y)$$

$L = \{ xx \mid x \in \{0,1\}^*\}$
$$\Rightarrow f_L(x, y) = \text{EQUALS}(x,y)$$
Connection to Streaming and DFAs

Let $L \subseteq \{0,1\}^*$
Def. $f_L : \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$
for $x, y$ with $|x| = |y|$ as:
$f_L(x, y) = 1 \iff xy \in L$

Theorem: If $L$ has a streaming algorithm using $\leq s$ space, then the comm. complexity of $f_L$ is at most $4s + 5$.

Proof: Alice runs streaming algorithm A on $x$.
Sends the memory content of A: this is $s$ bits of space
Bob starts up A with that memory content, runs A on $y$.
Gets an output bit, sends to Alice.
(...why $4s+5$ rounds? Can you do better?)
Connection to Streaming and DFAs

Let $L \subseteq \{0,1\}^*$  
Def. $f_L(x, y) = 1 \Leftrightarrow xy \in L$

**Theorem:** If $L$ has a streaming algorithm using $\leq s$ space, then the comm. complexity of $f_L$ is at most $4s + 5$.

**Corollary:** For every regular language $L$, the comm. complexity of $f_L$ is $O(1)$.

**Example:** Comm. Complexity of PARITY is $O(1)$

**Corollary:** Comm. Complexity of MAJORITY is $O(\log n)$

What about the Comm. Complexity of EQUALS?
**Theorem:** The comm. complexity of EQUALS is $\Theta(n)$. In particular, every protocol for EQUALS needs $\geq n$ bits of communication.

*No communication protocol can do much better than “send your entire input”!* 

**Corollary:** $L = \{ww \mid w \in \{0,1\}^*\}$ is not regular. Moreover, every streaming algorithm for $L$ needs $c n$ bits of memory, for some constant $c > 0$!
Theorem: The comm. complexity of EQUALS is $\Theta(n)$. In particular, every protocol for EQUALS needs $\geq n$ bits of communication.

Idea: Consider all possible ways they can communicate.

Definition: The communication pattern of a protocol on $(x, y)$ is the sequence of bits that Alice & Bob send.

Pattern: 0110
Communication complexity of EQUALS

**Theorem:** The comm. complexity of EQUALS is Θ(n). In particular, every protocol for EQUALS needs ≥ n bits of communication.

**Proof:** By contradiction. Suppose CC of EQUALS is ≤ n − 1. Then there are ≤ 2^n − 1 possible communication patterns of that protocol, over all pairs of inputs (x, y).

**Claim:** There are x ≠ y such that on (x, x) and on (y, y), the protocol uses the same pattern P.

What is the communication pattern on (x, y)?

This pattern is also P! *(WHY?)*

So Alice & Bob output the same bit on (x, y) and (x, x). But EQUALS(x, y) = 0 and EQUALS(x, x) = 1. *Contradiction!*
Randomized Protocols Help!

EQUALS needs $cn$ bits of communication, but...

**Theorem:** For all $(x, y)$ of $n$ bits each, there is a *randomized* protocol for $\text{EQUALS}(x, y)$ using only $O(\log n)$ bits of communication, which works with probability **99.9%**!
Turing Machines
Turing Machine

$q_1$

INPUT

INFINITE TAPE
ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO
THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

The “computable” numbers may be described briefly as the real
numbers whose expressions as a decimal are calculable by finite means.
Although the subject of this paper is ostensibly the computable numbers,
it is almost equally easy to define and investigate computable functions
of an integral variable or a real or computable variable, computable
predicates, and so forth. The fundamental problems involved are,
however, the same in each case, and I have chosen the computable numbers
for explicit treatment as involving the least cumbrous technique. I hope
shortly to give an account of the relations of the computable numbers,
function of one computable number. This will include development...
Great, a warehouse filled with miles and miles of rewritable tape! What are we ever going to do with this, Alan?

And thus the Turing Machine was born.
Turing Machines versus DFAs

TM can both write to and read from the tape

The head can move left and right

The input doesn’t have to be read entirely, and the computation can continue further (even, forever) after all input has been read

Accept and Reject take immediate effect
This Turing machine \textit{decides} the language \{0\}
This Turing machine *recognizes* the language \{0\}
Deciding the language \( L = \{ w#w \mid w \in \{0,1\}^* \} \)

1. If there’s no \# on the tape (or more than one \#), reject.
2. While there is a bit to the left of \#, Replace the first bit with X, and check if the first bit \( b \) to the right of the \# is identical. (If not, reject.) Replace that bit \( b \) with an X too.
3. If there’s a bit to the right of \#, then reject else accept
**Definition:** A Turing Machine is a 7-tuple \( T = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}}) \), where:

- \( Q \) is a finite set of states
- \( \Sigma \) is the input alphabet, where \( \square \notin \Sigma \)
- \( \Gamma \) is the tape alphabet, where \( \square \in \Gamma \) and \( \Sigma \subseteq \Gamma \)
- \( \delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\} \)
- \( q_0 \in Q \) is the start state
- \( q_{\text{accept}} \in Q \) is the accept state
- \( q_{\text{reject}} \in Q \) is the reject state, and \( q_{\text{reject}} \neq q_{\text{accept}} \)
Turing Machine Configurations

$q_7$

\[ \begin{array}{cccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
\end{array} \]

\[ q_7 \]

\[ \text{corresponds to the configuration:} \]

\[ 11010q_700110 \in \{Q \cup \Gamma\}^* \]
Defining Acceptance and Rejection for TMs

Let $C_1$ and $C_2$ be configurations of a TM $M$

**Definition.** $C_1$ *yields* $C_2$ if $M$ is in configuration $C_2$

after running $M$ in configuration $C_1$ for one step

Suppose $\delta(q_1, b) = (q_2, c, L)$
Then $a a q_1 b b$ yields $a q_2 a c b$

Suppose $\delta(q_1, a) = (q_2, c, R)$
Then $c a b q_1 a$ yields $c a b c q_2$

Let $w \in \Sigma^*$ and $M$ be a Turing machine

$M$ *accepts* $w$ if there are configs $C_0$, $C_1$, ..., $C_k$, s.t.

- $C_0 = q_0 w$
- $C_i$ yields $C_{i+1}$ for $i = 0$, ..., $k-1$, and
- $C_k$ contains the accept state $q_{\text{accept}}$
A TM $M$ **recognizes** a language $L$ if $M$ **accepts** exactly those strings in $L$.

A language $L$ is **recognizable** *(a.k.a. recursively enumerable)* if some TM recognizes $L$.

A TM $M$ **decides** a language $L$ if $M$ **accepts** all strings in $L$ and **rejects** all strings not in $L$.

A language $L$ is **decidable** *(a.k.a. recursive)* if some TM decides $L$.
\[ w \in \Sigma^* \]

**TM**

**w \in L ?**

- **yes** → **accept**
- **no** → **reject**

\[ L \text{ is decidable (recursive)} \]

**TM**

**w \in L ?**

- **yes** → **accept**
- **no** → **reject or loop**

\[ L \text{ is recognizable (recursively enumerable)} \]
A Turing machine for deciding \( \{ 0^{2^n} | n \geq 0 \} \)

**Turing Machine PSEUDOCODE:**

1. Sweep from left to right, cross out every other 0
2. If in step 1, the tape had only one 0, *accept*
3. If in step 1, the tape had an odd number of 0’s, *reject*
4. Move the head back to the first input symbol.
5. Go to step 1.

*Why does this work?*

*Idea:* Every time we return to stage 1, the number of 0’s on the tape has been halved.
\{ 0^{2n} \mid n \geq 0 \}

Step 1

Step 2

Step 3

Step 4

$q_0 \rightarrow q_1$

$q_1 \rightarrow q_2$

$q_2 \rightarrow q_3$

$q_3 \rightarrow q_4$

$q_4 \rightarrow q_0$

$q_0 \rightarrow q_{reject}$

$q_1 \rightarrow q_{accept}$

$x \rightarrow x, R$

$0 \rightarrow x, R$

$0 \rightarrow 0, R$

$0 \rightarrow 0, L$

$x \rightarrow x, R$

$x \rightarrow x, L$

$\square \rightarrow \square, R$

$\square \rightarrow \square, L$

$\square \rightarrow \square, R$

$\square \rightarrow \square, R$

$x \rightarrow x, R$

$x \rightarrow x, R$

$x \rightarrow x, R$

$x \rightarrow x, R$

$x \rightarrow x, R$
Multitape Turing Machines

\[ \delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L,R\}^k \]
Theorem: Every Multitape Turing Machine can be transformed into a single tape Turing Machine.
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Theorem: $L$ is decidable iff both $L$ and $\neg L$ are recognizable (recursively enumerable).