In this draft we study a sequential entry game first studied by Kreps and Wilson [2] and Milgrom and Roberts [3]. A similar analysis was used by the same authors to demonstrate that cooperation can be sustained in the finitely repeated prisoner's dilemma [1], by introducing reputation effects through incomplete information.

1 The Game

There is an incumbent firm (denoted I), and N possible entrants. At each time step, an entrant faces the incumbent in a two stage game; assume that entrants are faced in reverse numerical order, and we number the stages correspondingly: thus stage 1 is the last stage, and entrant 1 is faced last. ALl players observe all actions taken by players in the past.

At stage k, first the entrant decides whether to exit (X) or enter (N). The incumbent decides whether to accommodate entry (A), or to fight (F). Payoffs are as follows:

$$(\Pi_k, \Pi_I) = \begin{cases} (0, a), & \text{after } X; \\ (b, 0), & \text{after } (N, A); \\ (-1, -1), & \text{after } (N, F). \end{cases}$$

We assume that $a > 1$ and $b > 0$.

In the absence of incomplete information, it is clear that this is a finite perfect information game that has a unique subgame perfect equilibrium, where the incumbent always accommodates after entry, and all entrants enter.

2 Incomplete Information

Now suppose that Nature chooses a type for the incumbent, either R ("rational") with probability $1 - p$, or A ("aggressive") with probability p. An aggressive incumbent always fights after entry, while a rational incumbent can either fight or accommodate (with the payoffs described above).

2.1 One Entrant

We first consider a simple version of this game where there is only one potential entrant ($N = 1$).

Incumbent: A type R incumbent always accommodates after entry.

Entrant: The entrant's expected payoff if he enters is $(-1)(p) + (b)(1 - p)$, and if he exits it is zero.

We conclude that in a sequential equilibrium, the incumbent always accommodates, and: the entrant enters if $p < b/(1+b)$; and the entrant exits if $p > b/(1+b)$. If $p = b/(1+b)$, the entrant is indifferent between entering and exiting.

2.2 Two Entrants

Now suppose $N = 2$. We will find a sequential equilibrium by a reasoning similar to backward induction. (Recall that the incumbent encounters the entrants in reverse order.)

Let $s_I(h;t)$ denote the incumbent's (mixed) action after a history h, given that he has type t; and let $s_k(h)$ denote the (mixed) action of entrant k after a history h. (When the action is truly mixed, we write, e.g., $s_k(h)(a)$ for the probability that is assigned to action a.) Also, let $\mu(h)$ denote the probability entrant 1 assigns to the incumbent being of type R, after the history h has been observed. Note that by definition, $s_I((h,N);A) = F$, i.e., the aggressive incumbent fights after any history where an entrant just entered.

Throughout this section, we assume $s_I(\cdot), s_1(\cdot), s_2(\cdot)$, and $\mu(\cdot)$ are a sequential equilibrium.

Incumbent at stage 1: We show that after any history where entrant 1 enters, a type R incumbent always accommodates. This follows by the same reasoning as in the one entrant case. Thus $s_I(X,N;R) = s_I(N,F,N;R) = s_I(N,A,N;R) = A$. [For clarification: N,F,N means entrant 2 entered; the incumbent fought; then entrant 2 entered.]

Entrant at stage 1: Given the belief $\mu(h)$, entrant 1 plays as he would in the one entrant game; we conclude that in a sequential equilibrium, $s_1(h) = N$ if $\mu(h) < b/(1+b)$, and $s_1(h) = X$ if $\mu(h) > b/(1+b)$. Entrant 1 is indifferent if $\mu(h) = b/(1+b)$, and in this case any mixed action is rational.

It is easy to compute the belief in two cases. First suppose that $h = (X)$, i.e., entrant 2 exited. Then no information is gained about the incumbent's type, so $\mu(X) = p$. If $h = (N,A)$, so entrant 2 enters and the incumbent accommodates, then $\mu(N,A) = 0$ — it is immediately known the incumbent is of type R.

The situation is more complex when the history is (N,F). In this case, entrant 1 uses the incumbent's strategy to form his beliefs using Bayes' rule. Suppose that $s_I(N;R)(F) = 1 - s_I(N;R)(A) = \gamma$; thus if entrant 2 enters, the type R incumbent fights with probability γ. The posterior estimate the incumbent is of type A is then formed using Bayes' rule, and is:

$$\mu(h) = \frac{p}{p + (1-p)\gamma}.$$

The numerator is the total probability that the incumbent is aggressive, and fights after entry; the denominator is the marginal probability that the incumbent fights after entry.

To summarize, for entrant 1:

$$\mu(X) = p; \ \ \mu(N,A) = 0; \ \ \mu(N,F) = \frac{p}{p + (1-p)s_I(N;R)(F)}.$$

And the entrant enters, exits, or is indifferent if $\mu(h)$ is $<$, $>$, or equal to $b/(1+b)$, respectively.

Incumbent at stage 2: We now turn to the type R incumbent's action after entrant 2 has entered. This incumbent will have to use his conjecture of entrant 1's behavior to decide what to do. After (N, A), we have already established that entrant 1 will always enter. Suppose that further, the incumbent conjectures that $s_1(N, F)(N) = 1 - s_1(N, F)(X) = \lambda$, i.e., after (N, F) entrant 1 plays a mixed action with probability λ of entering.

If the incumbent accommodates after entrant 2 enters, his total payoff over two periods will be zero. If the incumbent fights after entrant 2 enters, then his payoff is:

$$-1 + (\lambda)(0) + (1 - \lambda)(a).$$

Thus if $\lambda > 1 - 1/a$, the type R incumbent will accommodate; if $\lambda < 1 - 1/a$, the type R incumbent will fight; and if $\lambda = 1 - 1/a$, the type R incumbent is indifferent.

We can combine these insights to reason about the equilibrium behavior of the type R incumbent, and subsequently, entrant 1. We first note that in equilibrium, we must have $s_I(N; R)(F) > 0$. If not, then entrant 1 always exits after seeing (N, F); but then fighting after entrant 2 enters is profitable for the incumbent, since the total payoff is $-1 + a$.

Now suppose that $p \geq b/(b+1)$. Note that regardless of the incumbent's strategy, in equilibrium $\mu(N, F) > p \geq b/(b + 1)$. But then with probability 1 in equilibrium, entrant 1 will exit, i.e., $s_1(N, F) = X$. The best response for the rational incumbent is to fight with probability 1, so $s_I(N; R) = F$.

Instead suppose that $p < b/(b + 1)$. We have already seen that $s_I(N; R)(F) > 0$. Suppose that $s_I(N; R) = F$ (i.e., the incumbent always fights). Then the belief of entrant 1 is $\mu(N, F) = p < b/(b + 1)$, so entrant 1 always enters after (N, F). But the best response for the type R incumbent to this is to accommodate; so in equilibrium, the incumbent must play a mixed action with $0 < s_I(N; R)(F) < 1$. This means the incumbent is indifferent between fighting and accommodating, which is only possible if entrant 1 randomizes as well. Thus, when $p < b/(b+1)$, we have $s_1(N, F)(N) = 1 - S_1(N, F)(X) = 1 - 1/a$, and the incumbent plays a mixed action with $0 < s_I(N; R)(F) < 1$ to ensure $\mu(N, F) = b/(1 + b)$ (so that entrant 1 is indifferent). Note that since the incumbent is indifferent between fighting and accommodating, his expected payoff is zero in this case if entrant 2 enters.

Entrant at stage 2: Finally, we consider entrant 2's decision. If entrant 2 exits, the payoff is zero. When $p \geq b/(b + 1)$, the incumbent fights with probability 1 after entry, so entrant 2 never enters in equilibrium; thus $s_2 = X$.

When $p < b/(b + 1)$, if entrant 2 enters, then the expected payoff is:

$$(-1)(p + (1 - p)\gamma) + b(1 - p)(1 - \gamma),$$

where $\gamma = s_I(N; R)(F)$. Note that from above, in equilibrium γ satisfies:

$$\frac{p}{p + (1 - p)\gamma} = \frac{b}{1 + b}.$$

Substituting and simplifying, entrant 2 will enter, exit, or be indifferent if p is $<$, $>$, or equal to $b^2/(1+b)^2$, respectively.

We thus have the complete sequential equilibrium characterization:

1. If $p \geq b/(b+1)$, then: entrant 2 never enters; the type R incumbent always fights in stage 2; entrant 1 enters if and only if the history was (N, A); and the type R incumbent always accommodates in stage 1. The expected payoff to the incumbent in equilibrium is $2a$.

2. If $b^2/(b+1)^2 < p < b/(b+1)$, then entrant 2 never enters; the type R incumbent plays a randomized strategy in stage 2; entrant 1 randomizes after the history (N, F), and always enters after the histories (X) or (N, A); and the type R incumbent always accommodates in stage 1. The expected payoff to the incumbent in equilibrium is a.

3. If $b^2/(b+1)^2 > p$, then entrant 2 always enters; and the remaining strategies are as in the previous case. The expected payoff to the incumbent in equilibrium is zero.

Note that if $p = b^2/(1+b)^2$, then entrant 2 is indifferent between entering and not entering, and can randomize in equilibrium. (As practice, you should write the equilibrium strategies formally, as a function of history, given the previous description; and also describe the equilibrium beliefs of entrant 1.)

2.3 Three Entrants

We will now develop an induction for an arbitrary number of entrants using the intuition from the previous section. In the previous section, we used two main steps. First, entrant 1 always played *as if* he was in the "one entrant" game, but with the initial probability of the incumbent being aggressive given by $\mu(h)$. Second, we first computed the behavior of the type R incumbent and entrant 1 together in the sequential equilibrium, then used this to deduce the behavior of entrant 2.

Suppose now that $N = 3$. Using the same approach as in the previous section, we first consider the behavior of the type R incumbent after entrant 3 has entered. As before, it is straightforward to establish that $s_I(N; R)(F) > 0$; i.e., if entrant 3 enters, the incumbent fights with positive probability. Let $\mu(h)$ again denote the belief of later entrants given the history h; in particular, $\mu(N, F)$ denotes the belief of entrants 1 and 2 given that entrant 3 entered, and the incumbent fought. As above, we have:

$$\mu(N, F) = \frac{p}{p + (1-p)s_I(N; R)(F)}.$$

First assume that $p \geq b^2/(1+b)^2$. Then regardless of the actual value of $s_I(N; R)(F)$, $\mu(N, F) > p \geq b^2/(1+b)^2$. After stage 3, the players play a sequential equilibrium of the two entrant game, but starting with the belief $\mu(N, F)$. Looking at that equilibrium, we conclude that entrant 2 will exit with probability 1. In this case, the strict best response for the type R incumbent is to fight with probability 1 after entry by entrant 3 (since this yields him a positive payoff, while accommodating yields a zero payoff). We conclude that $s_I(N; R) = F$.

Assume instead that $p < b^2/(1+b)^2$. We have already seen that $s_I(N;R)(F) > 0$. Reasoning as before, suppose instead that $s_I(N;R) = F$. Then the beliefs of future entrants after the history (N,F) are $\mu(N,F) = p$. In this case future play follows the sequential equilibrium of the two entrant game, with initial belief p. This is the third case above, so the total expected payoff is -1 (expected payoff from future stages is zero). This is not optimal (accommodating in stage 3 yields zero total payoff), so in equilibrium the type R incumbent must randomize after entrant 3 enters. This means the type R incumbent must be indifferent between accommodating and fighting; looking at the expected payoffs in cases 1, 2, and 3 of the two entrant game above, this can only happen if entrant 2 randomizes after the history (N,F). The incumbent will only be indifferent if entrant 2 enters with probability $1-1/a$, and exits otherwise: $s_2(N,F)(N) = 1 - s_2(N,F)(X) = 1 - 1/a$. And since entrant 2 is indifferent, the type R incumbent chooses $s_I(N;R)(F)$ so that $\mu(N,F) = b^2/(1+b)^2$.

It should now be clear that entrant 3 will enter, exit, or be indifferent if p is $<$, $>$, or equal to $b^3/(1+b)^3$, respectively, using the same reasoning as in the two entrant case.

2.4 The General Result

Suppose there are N stages. At the beginning of stage k (i.e., after the first $N-k$ entrants have already played against the incumbent), suppose the history was h, and let $\mu(h)$ be the equilibrium belief of the future entrants that the incumbent is aggressive, after history h. We initialize $\mu(\emptyset) = p$. In equilibrium, the strategies at stage k are:

- Entrant k enters if $\mu(h) < (b/(1+b))^k$; exits if $\mu(h) > (b/(1+b))^k$; and randomizes if $\mu(h) = (b/(1+b))^k$, with probability $1-1/a$ of entering, and probability $1/a$ of exiting.
- After entry, the type R incumbent fights if $\mu(h) \geq (b/(1+b))^{k-1}$. If $\mu(h) < (b/(1+b))^{k-1}$, then the type R incumbent randomizes so that:

$$\left(\frac{b}{1+b}\right)^{k-1} = \frac{\mu(h)}{\mu(h) + (1-\mu(h))s_I(h,N;R)(F)}.$$

  If $k = 1$, the type R incumbent accommodates.

Beliefs are updated following stage k as follows:

- $\mu(h,X) = \mu(h)$.
- $\mu(h,N,A) = 0$.
- $\mu(h,N,F) = \max\{\mu(h), (b/(1+b))^{k-1}\}$.

Comments:

1. The equilibrium has the property that the belief is a sufficient statistic for the entire past history of play; i.e., strategies after the history h depend only on the belief $\mu(h)$. This is an instance of what is called a *Markov perfect equilibrium*: we can view $\mu(h)$ as a "state" that

evolves forward, and in equilibrium the strategies of the agents depend only on the state. It is *not* generally true that the belief is a sufficient statistic for the past history of the game: typically strategies will depend on both history and beliefs.

2. Note that as soon as we introduce incomplete information, regardless of the belief, *there is positive probability the type R incumbent fights after any history where he has not accommodated in the past*. This is already a significant difference from the complete information case, where the incumbent always accommodates. It is also the main insight of the model: introducing incomplete information immediately creates an incentive for the rational type to create a "reputation" as an aggressive incumbent. Note that this is true no matter how small p is.

3. In equilibrium, observe that for a fixed p, as long as $k > \log p/(\log b - \log(1+b))$, prospective entrants will not enter; and the first time that $k \leq \log p/(\log b - \log(1+b))$, entrant k will enter. As the number of prospective entrants N increases, the value of p needed to deter entry at least once becomes smaller and smaller.

References

[1] D. M. Kreps, P. Milgrom, J. Roberts, and R. Wilson. Rational cooperation in the finitely repeated prisoners' dilemma. *Journal of Economic Theory*, 27:245–252, 1982.

[2] D. M. Kreps and R. Wilson. Reputation and imperfect information. *Journal of Economic Theory*, 27:253–279, 1982.

[3] P. Milgrom and J. Roberts. Predation, reputation, and entry deterrence. *Journal of Economic Theory*, 27:280–312, 1982.