MS&E 246: Lecture 9
Sequential bargaining

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Nash bargaining solution

Recall Nash's approach to bargaining:

The planner is *given* the set of achievable payoffs and status quo point.

Implicitly:

The *process* of bargaining does not matter.

Dynamics of bargaining

In this lecture:

We use a dynamic game of perfect information to model the *process* of bargaining.

An interference model

Recall the interference model:

- Two devices
- Device 1 given channel a fraction q of the time
- For efficiency:
  When device n has control,
  it transmits at full power P

An interference model

- When timesharing is used, the set of Pareto efficient payoffs becomes:
  $\{ (\Pi_1, \Pi_2) : \Pi_1 = q R_1, \Pi_2 = (1 - q) R_2 \}$

- We now assume the devices bargain through a sequence of *alternating offers*.

Alternating offers

- At time 0:
  - Stage 0A:
    Device 1 *proposes* a choice of q (denoted q_1)
  - Stage 0B:
    Device 2 decides to *accept* or *reject* device 1's offer

Two period model

Assumption 1:

If device 2 *rejects* at stage 0B, then predetermined choice $Q \in [0, 1]$ is implemented at time 1

Discounting

Assumption 2:

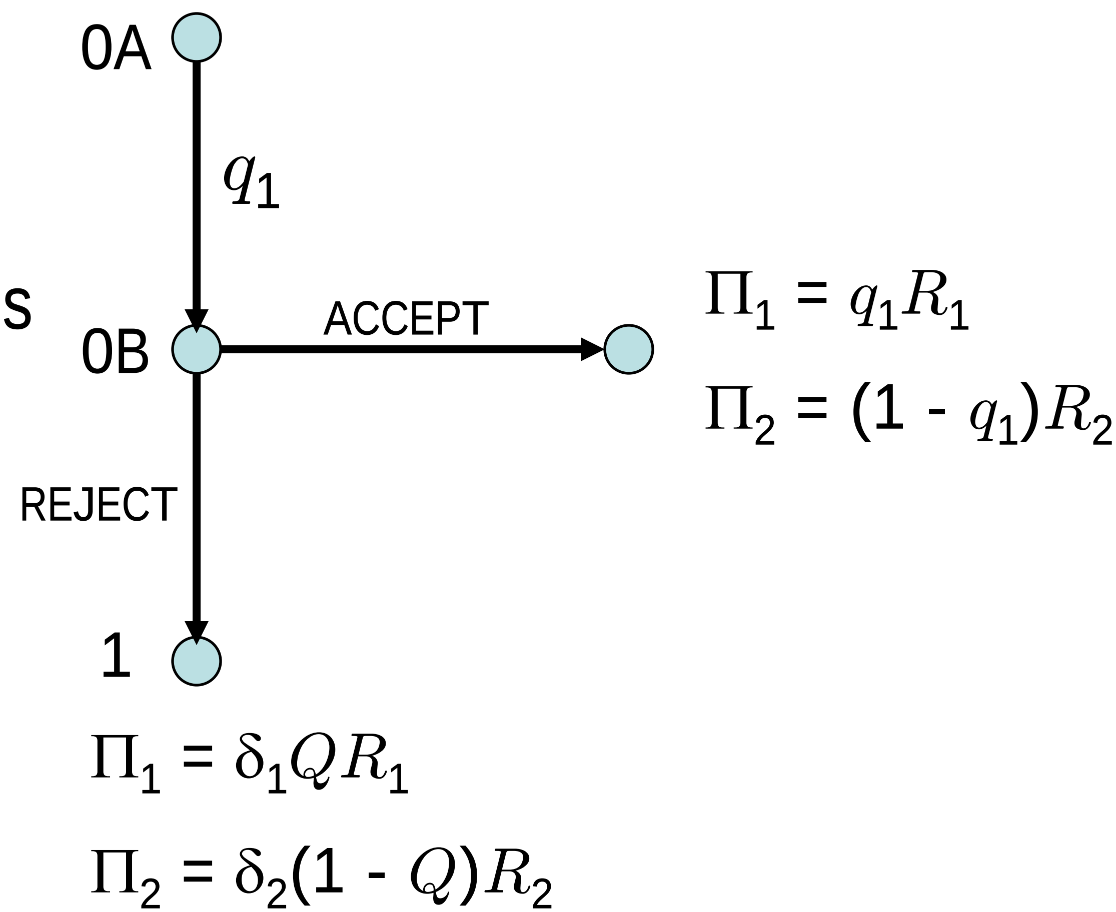
Devices care about delay:
Any payoff received by device i at time k is *discounted* by δ_i^k.

$0 < \delta_i < 1$: *discount factor* of device i

Game tree

Device 1 makes initial offer

Device 2 accepts or rejects



Game tree

- This is a *dynamic game of perfect information.*

- We solve it using *backward induction.*

Backward induction

1. Given q_1, at Stage 0B:

   - Device 2 *rejects* if:
     $\delta_2 (1 - Q) \quad > \quad (1 - q_1)$

Backward induction

1. Given q_1, at Stage 0B:

- Device 2 *rejects* ($s_2(q_1)$ = R) if:
  $q_1 > 1 - \delta_2 (1 - Q)$
- Device 2 *accepts* ($s_2(q_1)$ = A) if:
  $q_1 < 1 - \delta_2 (1 - Q)$
- Device 2 is *indifferent*
  ($s_2(q_1) \in$ { A, R }) if
  $q_1 = 1 - \delta_2 (1 - Q)$

Backward induction

2. At Stage 0A:

- Device 1 maximizes $\Pi_1(q_1, s_2(q_1))$ over offers ($0 \leq q_1 \leq 1$)
- *Claim:* Maximum value of Π_1 is

$$(1 - \delta_2 (1 - Q)) R_1$$

Backward induction

2. At Stage 0A:

- *Claim:* Maximum value of Π_1 is

  $\Pi_1^{MAX} = (1 - \delta_2 (1 - Q)) R_1$

- *Proof:*

  (a) Maximum is achievable:

  If q_1 increases to $1 - \delta_2(1 - Q)$,

  then Π_1 increases to Π_1^{MAX}

Backward induction

2. At Stage 0A:

- *Claim:* Maximum value of Π_1 is

  $\Pi_1^{\text{MAX}} = (1 - \delta_2 (1 - Q)) R_1$

- *Proof:*

  (b) If $q_1 > 1 - \delta_2(1 - Q)$, then $\Pi_1 < \Pi_1^{\text{MAX}}$:

  Device 2 rejects $\Rightarrow$ $\Pi_1 = \delta_1 Q R_1$

Backward induction

2. At Stage 0A:

- *Claim:* Maximum value of Π_1 is

  $\Pi_1^{MAX} = (1 - \delta_2 (1 - Q)) R_1$

- *Proof:*

  (b) If $q_1 > 1 - \delta_2(1 - Q)$, then $\Pi_1 < \Pi_1^{MAX}$:

  But note that: $\delta_1 Q + \delta_2 (1 - Q) < 1$

Backward induction

2. At Stage 0A:

- *Claim:* Maximum value of Π_1 is

  $\Pi_1^{MAX} = (1 - \delta_2 (1 - Q)) R_1$

- *Proof:*

  (b) If $q_1 > 1 - \delta_2(1 - Q)$, then $\Pi_1 < \Pi_1^{MAX}$:

  But note that: $\delta_1 Q < 1 - \delta_2 (1 - Q)$

Backward induction

2. At Stage 0A:

- *Claim:* Maximum value of Π_1 is

  $\Pi_1^{MAX} = (1 - \delta_2 (1 - Q)) R_1$

- *Proof:*

  (b) If $q_1 > 1 - \delta_2(1 - Q)$, then $\Pi_1 < \Pi_1^{MAX}$:

  So $\delta_1 Q R_1 < (1 - \delta_2 (1 - Q)) R_1$

Backward induction

2. At Stage 0A:

- *Best responses* for device 1 :

  All choices of q_1 that achieve Π_1^{MAX}

  The only possibility:

$$q_1^* = 1 - \delta_2 (1 - Q)$$

Backward induction

2. At Stage 0A:

- If $s_2(q_1{*})$ = reject,
  no best response exists for device 1!

- If $s_2(q_1{*})$ = accept,
  best response for device 1 is $q_1 = q_1{*}$

Unique SPNE

What is the unique SPNE?

- Must give *strategies* for both players!

Unique SPNE

What is the unique SPNE?

- Device 1:

  At Stage 0A, offer $q_1 = q_1$*

- Device 2:

  At Stage 0B,

  accept if $q_1 \leq q_1$*, *reject* if $q_1 > q_1$*

Payoffs at unique SPNE

- So the offer of device 1 is *accepted immediately* by device 2.

- Device 1 gets: $\Pi_1 = (1 - \delta_2(1 - Q)) R_1$

- Device 2 gets: $\Pi_2 = \delta_2(1 - q_0) R_2$

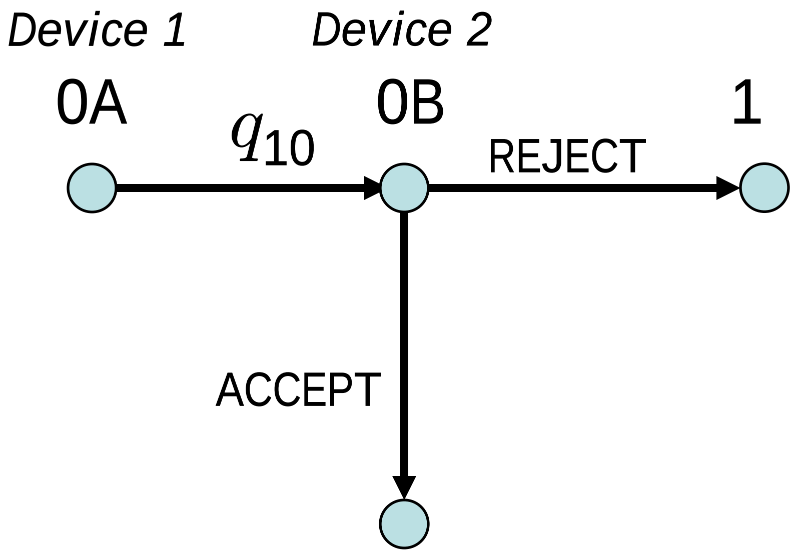
Infinite horizon

More realistic model:

Devices alternate offers indefinitely.

For simplicity: assume $\delta_1 = \delta_2 = \delta$

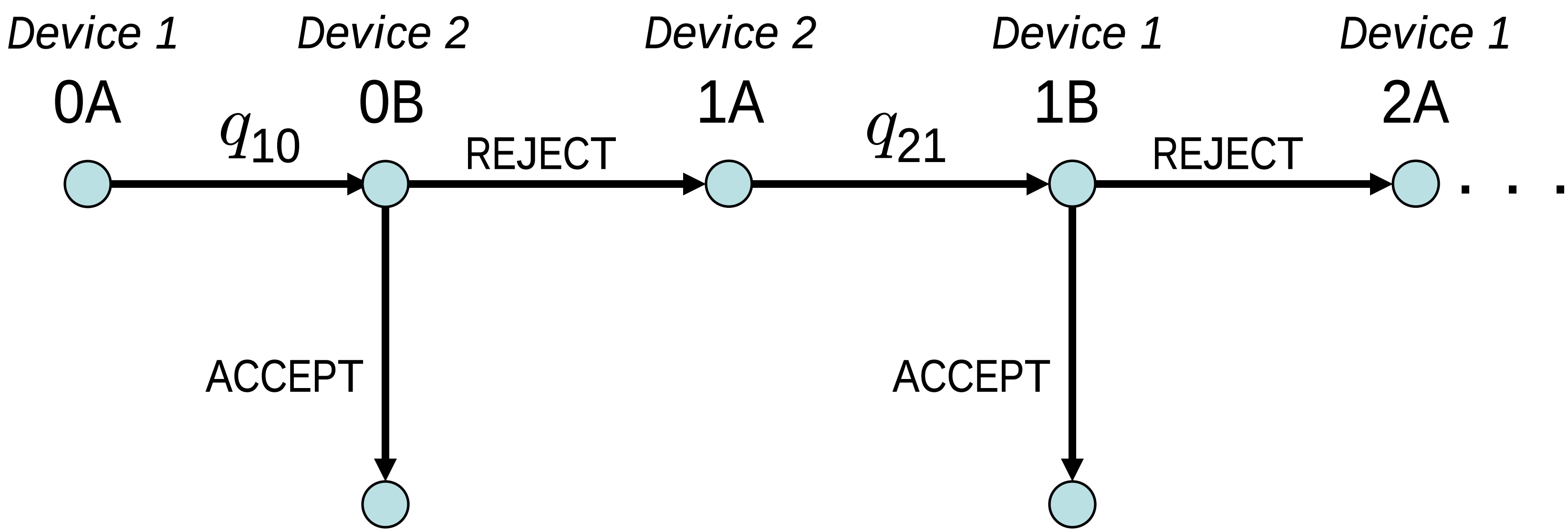
Finite horizon



$\Pi_1 = q_{10} R_1$

$\Pi_2 = (1 - q_{10}) R_2$

Infinite horizon



$\Pi_1 = q_{10} R_1$

$\Pi_2 = (1 - q_{10}) R_2$

$\Pi_1 = \delta q_{21} R_1$

$\Pi_2 = \delta(1 - q_{21}) R_2$

Infinite horizon: formal model

- Device 1 offers q_{1k} at stage kA, for k even
- Device 2 offers q_{2k} at stage kA, for k odd

- Device 2 accepts/rejects stage kA offer at stage kB, for k even
- Device 1 accepts/rejects stage kA offer at stage kB, for k odd

Infinite horizon: formal model

- Payoffs:

  $\Pi_1 = \Pi_2 = 0$ if no offer ever accepted (similar to status quo in NBS)

Infinite horizon: formal model

- Payoffs:

If offer made at stage kA by player i accepted at stage kB :

$$\Pi_1 = \delta^k \, q_{ik} \, R_1$$

$$\Pi_2 = \delta^k \, (1 - q_{ik}) \, R_2$$

Infinite horizon

- Can't use backward induction!
- Use *stationarity:*

  Subgame rooted at 1A is
  the same as the original game,
  with roles of 1 and 2 reversed.

SPNE

Define V and v:

$V\,R_1$ = highest time 0 payoff to device 1 among *all* SPNE

$v\,R_1$ = lowest time 0 payoff to device 1 among *all* SPNE

SPNE

Then if device 2 rejects at 0B:

$V R_2$= highest time 1 payoff to device 2 among *all* SPNE

$v R_2$= lowest time 1 payoff to device 2 among *all* SPNE

SPNE: Two inequalities

- $v\ R_1 \geq (1 - \delta\ V)\ R_1$

At Stage 0B:

Device 2 will accept any $q_{10} < 1 - \delta\ V$

So at Stage 0A:

Device 1 must earn at least $(1 - \delta\ V)\ R_1$

SPNE: Two inequalities

- $V R_1 \leq (1 - \delta\, v)\, R_1$

  If offer q_{10} is *accepted* at stage 0B, device 2 must get a timeshare
  of at least $\delta\, v$

  $\Rightarrow \quad q_{10} \leq 1 - \delta\, v$

SPNE: Two inequalities

- $V R_1 \leq (1 - \delta v) R_1$

  If offer q_{10} is *rejected* at stage 0B, device 1 earns at most $\delta (1 - v) R_1$ since device 2 earns at least $\delta v R_2$

  $\Rightarrow \Pi_1 \leq \delta (1 - v) R_1 \leq (1 - \delta v) R_1$

Combining inequalities

- $v \leq V$
- $v \geq 1 - \delta V$
- $V \leq 1 - \delta v$

Combining inequalities

- $v \leq V$
- $v + \delta V \geq 1$
- $V + \delta v \leq 1$

So: $$V + \delta v \leq v + \delta V$$

$$\Rightarrow \quad (1 - \delta) V \leq (1 - \delta) v$$

$$\Rightarrow \quad V = v$$

Unique SPNE

- So $V = 1 - \delta V \Rightarrow$

$$V = \frac{1}{1+\delta}$$

- SPNE strategies for device 1:

  At Stage kA , k even:

  Offer $q_{1k} = 1 - \delta V$

  At Stage kB , k odd:

  Accept if $q_{2k} \geq \delta V$

Unique SPNE

- So $V = 1 - \delta V \Rightarrow$

$$V = \frac{1}{1+\delta}$$

- SPNE strategies for device 2:

At Stage kA , k odd:

Offer $q_{2k} = \delta V$

At Stage kB , k even:

Accept if $q_{1k} \leq 1 - \delta V$

Unique SPNE: Payoffs

Stage 0A offer by device 1 is accepted in Stage 0B by device 2.

$$\Pi_1^{\mathsf{SPNE}} = \frac{R_1}{1+\delta}, \quad \Pi_2^{\mathsf{SPNE}} = \frac{\delta R_2}{1+\delta}$$

Infinite horizon: Discussion

- Outcome is *efficient:*
  No "lost utility" due to discounting
- *Stationary* SPNE strategies:
  Actions do not depend on time k
- *First mover* advantage:
  $\Pi_1^{\text{SPNE}} > \Pi_2^{\text{SPNE}}$

Shortening time periods

Shorten each time step to length $t < 1$ ...

... Same as changing discount factor to δ^t

$$\Pi_1^{\text{SPNE}} = \frac{R_1}{1+\delta^t}, \quad \Pi_2^{\text{SPNE}} = \frac{\delta^t R_2}{1+\delta^t}$$

As $t \to 0$, note that $\Pi_i^{\text{SPNE}} \to R_i/2$.

Nash bargaining solution!

In general

If $\delta_1 \neq \delta_2$:

Find SPNE using two period model:

- Note that Q must be SPNE payoff when device 2 offers first

Can show (for an appropriate limit) that

- *weighted NBS* obtained as $t \to 0$:
- More patient player weighted higher

Summary

- Alternating offers: finite horizon

  Backward induction solution
- Alternating offers: infinite horizon

  Unique SPNE

  Relation to Nash bargaining solution