MS&E 246: Lecture 5
Efficiency and fairness

Ramesh Johari

A digression

In this lecture:

We will use some of the insights of static game analysis to understand *efficiency and fairness.*

Basic setup

- N players
- S_n : strategy space of player n
- Z : space of outcomes
- $z(s_1, ..., s_N)$:
  outcome realized when $(s_1, ..., s_N)$ is played
- $\Pi_n(z)$:
  payoff to player n when outcome is z

(Pareto) Efficiency

An outcome z' *Pareto dominates* z if:

$\Pi_n(z') \geq \Pi_n(z)$ for all n,

and the inequality is strict for at least one n.

An outcome z is *Pareto efficient* if it is not Pareto dominated by any other $z' \in X$.

$\Rightarrow$ Can't make one player better off without making another worse off.

Are equilibria efficient?

Recall the Prisoner's dilemma:

Player 1

Player 2		defect	cooperate
	defect	(-4,-4)	(-1,-5)
	cooperate	(-5,-1)	(-2,-2)

Are equilibria efficient?

Recall the Prisoner's dilemma:

		Player 1	
		defect	cooperate
Player 2	defect	(-4,-4)	(-1,-5)
	cooperate	(-5,-1)	(-2,-2)

Unique dominant strategy eq.: (D, D).

Are equilibria efficient?

		Player 1	
		defect	cooperate
Player 2	defect	(-4,-4)	(-1,-5)
	cooperate	(-5,-1)	(-2,-2)

But (C, C) *Pareto dominates* (D, D).

Are equilibria efficient?

- Moral:
  Even when every player has a strict dominant strategy,
  the resulting equilibrium may be *inefficient*.

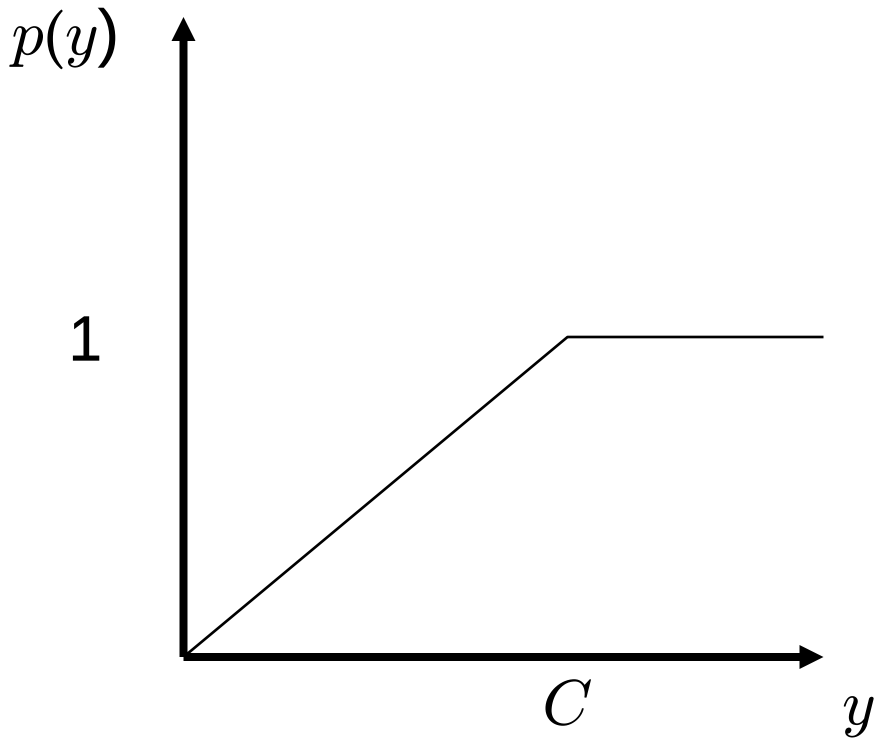
Resource sharing

- N users want to send data across a shared communication medium
- x_n : sending rate of user n (pkts/sec)
- $p(y)$: probability a packet is lost when total sending rate is y
- $\Pi_n(\mathbf{x})$ = *net throughput* of user n

  $= x_n\,(1 - p(\sum_i x_i)\,)$

Resource sharing

- Suppose: $p(y) = \min(y/C, 1)$



Resource sharing

- Suppose: $p(y)$ = min(y/C, 1)
- Given $\mathbf{x}$:
  Define $Y = \sum_i x_i$ and $Y_{-n} = \sum_{i \neq n} x_i$
- Thus, given $\mathbf{x}_{-n}$,

$$\Pi_n(x_n, \mathbf{x}_{-n}) = x_n \left(1 - \frac{x_n + Y_{-n}}{C}\right)$$

if $x_n + Y_{-n} \leq C$, and zero otherwise

Pure strategy Nash equilibrium

- We only search for NE s.t. $\sum_i x_i \leq C$ (Why?)
- In this region, first order conditions are:

$$1 - Y_{-n}/C - 2x_n/C = 0, \quad \text{for all } n$$

Pure strategy Nash equilibrium

- We only search for NE s.t. $\sum_i x_i \leq C$ (Why?)
- In this region, first order conditions are:

$$1 - Y/C = x_n / C, \quad \text{for all } n$$

- If we sum over n and solve for Y, we find:

$$Y^{\text{NE}} = N\, C \,/ (N + 1)$$

- So: $x_n^{\text{NE}} = C \,/\, (N + 1)$, and

$$\Pi_n(\mathbf{x}^{\text{NE}}) = C \,/ (N + 1)^2$$

Maximum throughput

- Note that total throughput $= \sum_n \Pi_n(\mathbf{x}) = Y(1 - p(Y)) = Y(1 - Y/C)$
- This is maximized at $Y^{\mathrm{MAX}} = C / 2$
- Define $x_n^{\mathrm{MAX}} = Y^{\mathrm{MAX}} / N = C / 2N$
- Then (if $N > 1$):

$$\Pi_n(\mathbf{x}^{\mathrm{MAX}}) = C / 4N > C / (N+1)^2 = \Pi_n(\mathbf{x}^{\mathrm{NE}})$$

So: $\mathbf{x}^{\mathrm{NE}}$ *is not efficient.*

Resource sharing: summary

- At NE, users' rates are *too high.* Why?
- When user n maximizes Π_n, he ignores reduction in throughput he causes for other players (the *negative externality*)
- AKA: Tragedy of the Commons
- If externality is *positive*,
  then NE strategies are *too low*

An interference model

- N = 2 wireless devices want to send data
- Strategy = transmit power
  $S_1 = S_2 = \{ 0, P \}$
- Each device sees the other's transmission as *interference*

An interference model

Payoff matrix ($0 < \varepsilon << R_2 < R_1$):

		Device 2	
		0	P
Device 1	0	(0, 0)	$(0, R_2)$
	P	$(R_1, 0)$	$(\varepsilon, \varepsilon)$

An interference model

- (P, P) is unique strict dominant strategy equilibrium (and hence unique NE)
- Note that (P, P) is not Pareto dominated by any pure strategy pair
- But...the mixed strategy pair $(\mathbf{p}_1, \mathbf{p}_2)$ with $p_1(0) = p_2(0) = p_1(P) = p_2(P) = 1/2$ Pareto dominates (P, P) if $R_n >> \varepsilon$

  (Payoffs: $\Pi_n(\mathbf{p}_1, \mathbf{p}_2) = R_n/4 + \varepsilon/4$)

An interference model

- How can coordination improve throughput?

- *Idea:*
  Suppose both devices agree to a protocol that decides when each device is allowed to transmit.

An interference model

- *Cooperative timesharing:*

  Device 1 is allowed to transmit a fraction q of the time.

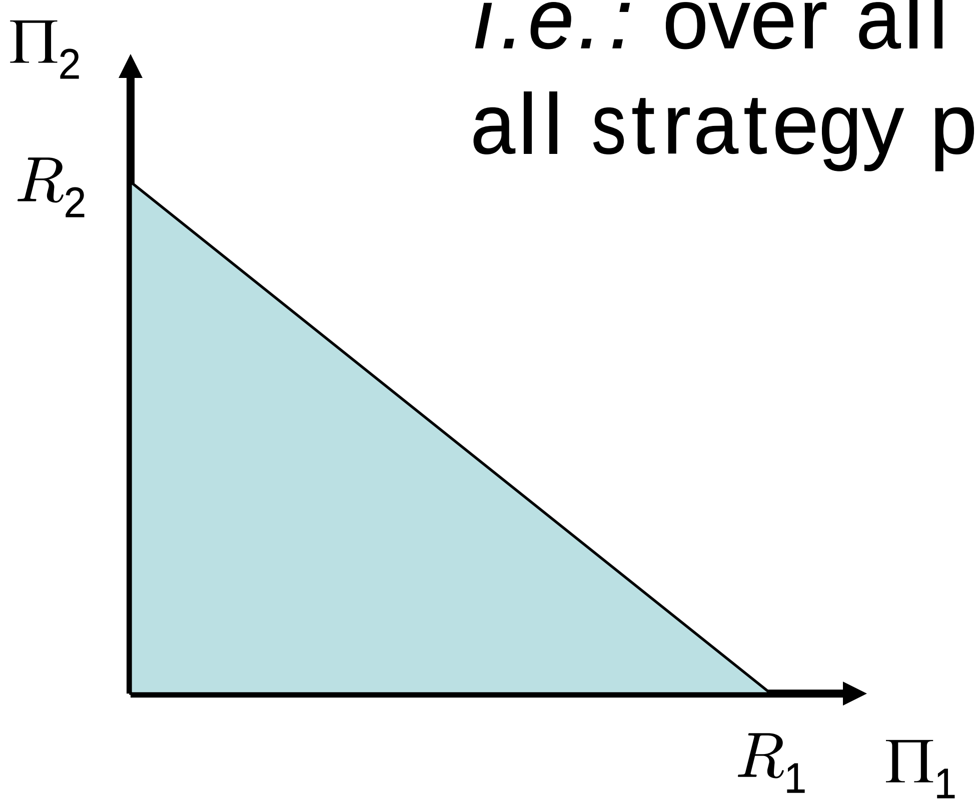
  Device 2 is allowed to transmit a fraction 1 - q of the time.

  Devices can use any mixed strategy when they control the channel.

An interference model

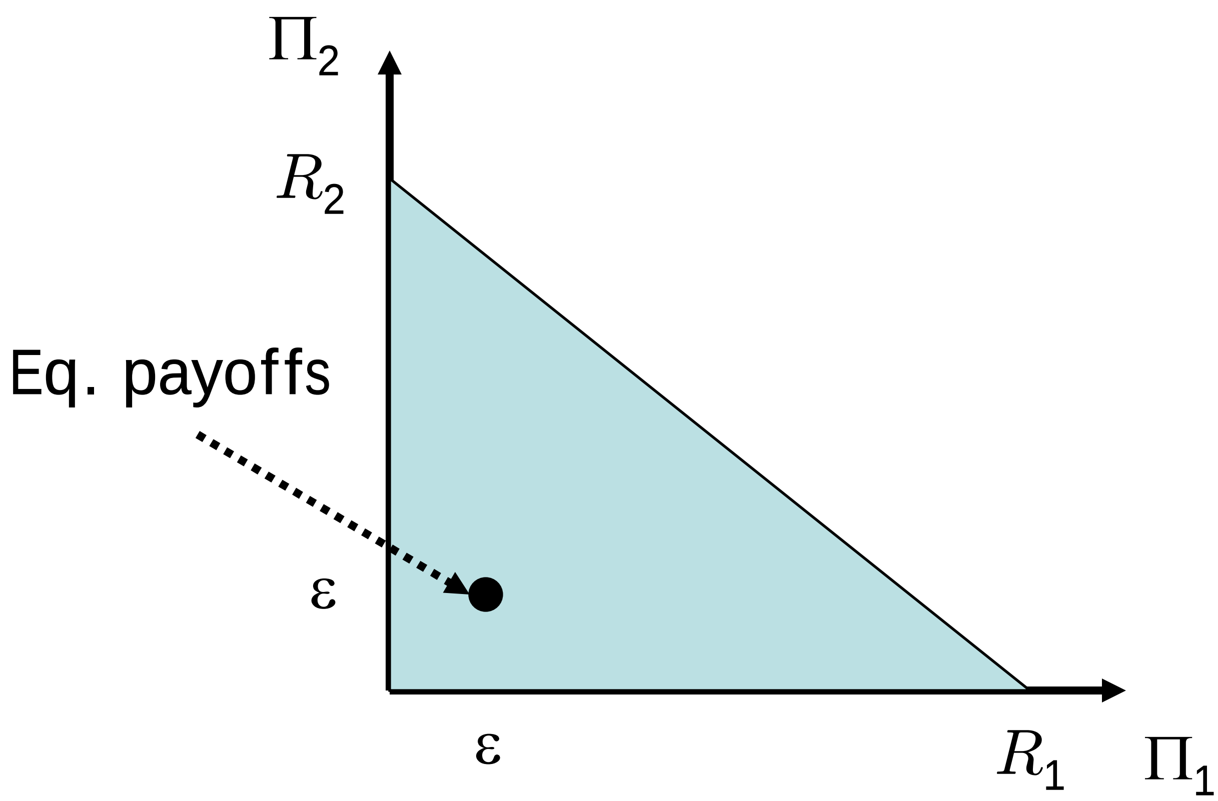
Achievable payoffs via timesharing:

i.e.: over all q and all strategy pairs (σ_1, σ_2)



An interference model

Achievable payoffs via timesharing:



An interference model

- So when timesharing is used, the set of Pareto efficient payoffs becomes:
  $\{ (\Pi_1, \Pi_2) : \Pi_1 = q\, R_1, \Pi_2 = (1 - q)\, R_2 \}$

- For efficiency:
  When device n has control,
  it transmits at power P

An interference model

- So when timesharing is used, the set of Pareto efficient payoffs becomes:
  $\{ (\Pi_1, \Pi_2) : \Pi_1 = q\, R_1, \Pi_2 = (1 - q)\, R_2 \}$

- For efficiency:
  When device n has control,
  it transmits at power P

An interference model

- So when timesharing is used, the set of Pareto efficient payoffs becomes:
  $\{ (\Pi_1, \Pi_2) : \Pi_1 = q\, R_1, \Pi_2 = (1 - q)\, R_2 \}$

  (*Note:* in general, the set of achievable payoffs is the *convex hull* of entries in the payoff matrix)

Choosing an efficient point

Which q should the protocol choose?

- Choice 1: *Utilitarian solution*

  $\Rightarrow$ Maximize total throughput

  $\max_q q R_1 + (1 - q) R_2 \quad \Rightarrow \quad q = 1$

  $\Pi_1 = R_1$, $\Pi_2 = 0$

  Is this "fair"?

Choosing an efficient point

Which q should the protocol choose?

- Choice 2: *Max-min fair solution*

  $\Rightarrow$ Maximize smallest Π_n

  $\max_q \min \{ q\, R_1, (1 - q)\, R_2 \}$

  $\Rightarrow q\, R_1 = (1 - q)\, R_2$,

  so $\Pi_1 = \Pi_2$ (i.e., equalize rates)

Fairness

Fairness corresponds to a rule for choosing between multiple efficient outcomes.

Unlike efficiency, there is no universally accepted definition of "fair."

Nash bargaining solution (NBS)

- Fix desirable properties of a “fair” outcome

- Show there exists a unique outcome satisfying those properties

NBS: Framework

- $T = \{ (\Pi_1, \Pi_2) : (\Pi_1, \Pi_2)$ is achievable $\}$
  - assumed closed, bounded, and convex

- $\Pi^* = (\Pi_1^*, \Pi_2^*)$: *status quo* point
  - each n can guarantee Π_n^* for himself through unilateral action

NBS: Framework

- $f(T, \Pi^*) = (f_1(T, \Pi^*),\ f_2(T, \Pi^*)) \in T$:
  a *"bargaining solution"*,
  i.e., a rule for choosing a payoff pair

- What properties (*axioms*) should f satisfy?

Axioms

Axiom 1: Pareto efficiency

The payoff pair $f(T, \Pi^*)$ must be Pareto efficient in T.

Axiom 2: Individual rationality

For all n, $f_n(T, \Pi^*) \geq \Pi_n^*$.

Axioms

Given $\mathbf{v} = (v_1, v_2)$, let

$$T + \mathbf{v} = \{\, (\Pi_1 + v_1,\ \Pi_2 + v_2) : (\Pi_1, \Pi_2) \in T \,\}$$

(i.e., a change of origin)

Axiom 3: Independence of utility origins

Given any $\mathbf{v} = (v_1, v_2)$,

$$f(T + \mathbf{v}, \Pi^* + \mathbf{v}) = f(T, \Pi^*) + \mathbf{v}$$

Axioms

Given $\beta = (\beta_1, \beta_2)$, let

$\beta \cdot T = \{ (\beta_1 \Pi_1, \beta_2 \Pi_2) : (\Pi_1, \Pi_2) \in T \}$

(i.e., a change of utility units)

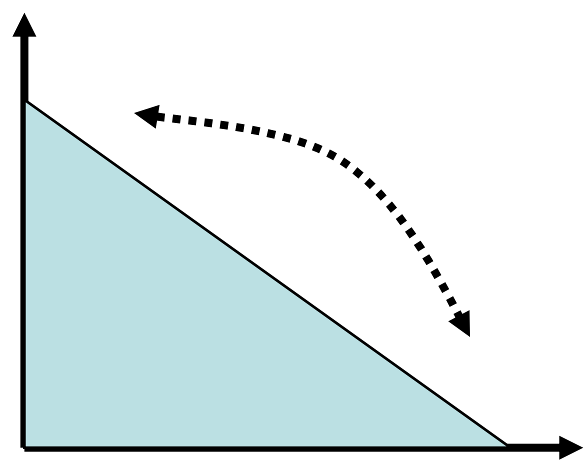
Axiom 4: Independence of utility units

Given any $\beta = (\beta_1, \beta_2)$, for each n we have

$$f_n(\beta \cdot T, (\beta_1 \Pi_1^*, \beta_2 \Pi_2^*)) = \beta_n f_n(T, \Pi^*)$$

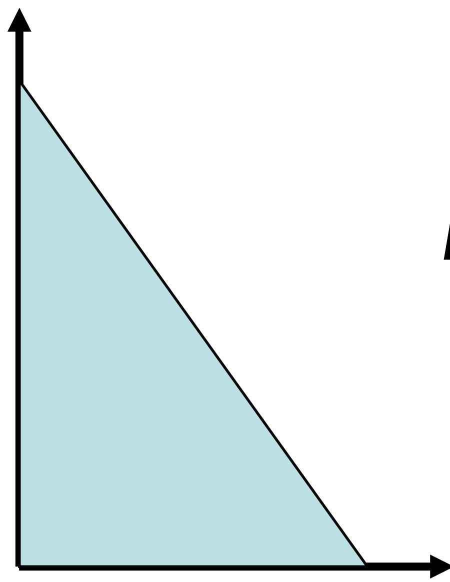
Axioms

The set T is *symmetric* if it looks the same when the Π_1-Π_2 axes are swapped:



Axioms

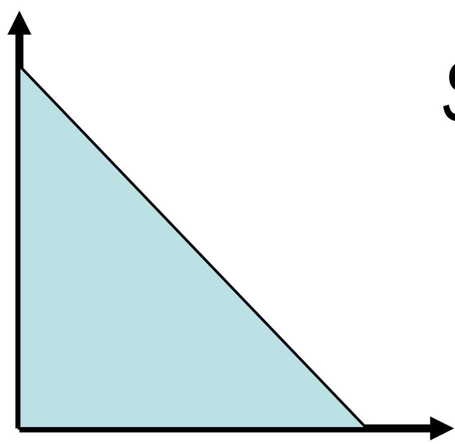
The set T is *symmetric* if it looks the same when the Π_1-Π_2 axes are swapped:



Not symmetric

Axioms

The set T is *symmetric* if it looks the same when the Π_1-Π_2 axes are swapped:



Symmetric

Axioms

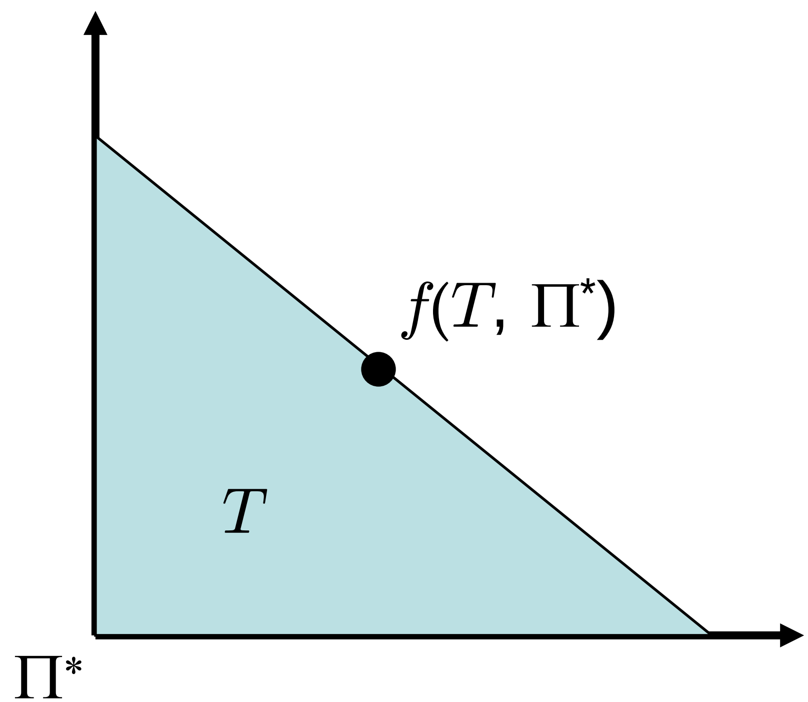
The set T is *symmetric* if it looks the same when the Π_1-Π_2 axes are swapped.

Axiom 5: Symmetry

If T is symmetric and $\Pi_1^* = \Pi_2^*$, then $f_1(T, \Pi^*) = f_2(T, \Pi^*)$.

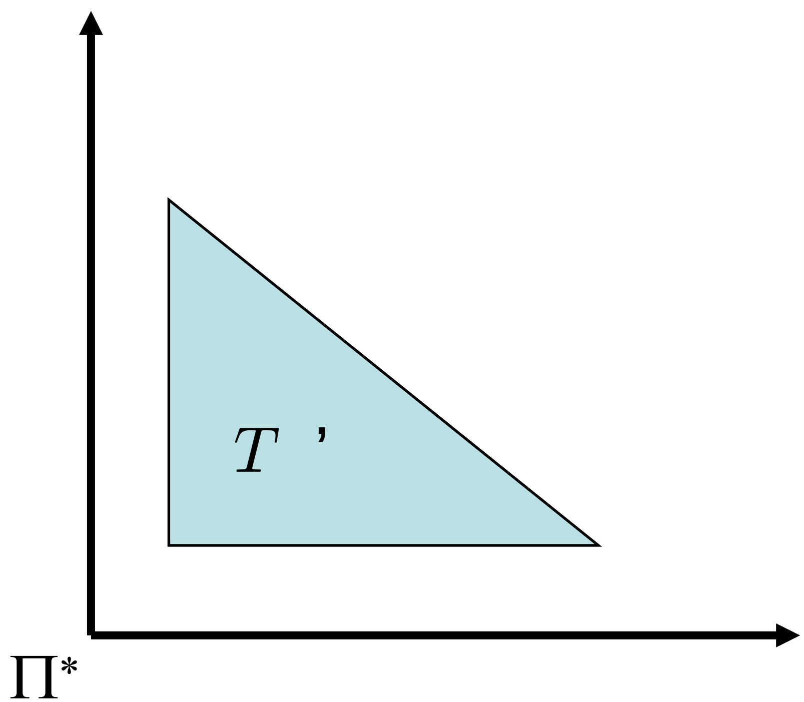
Axioms

Axiom 6: Independence of irrelevant alternatives



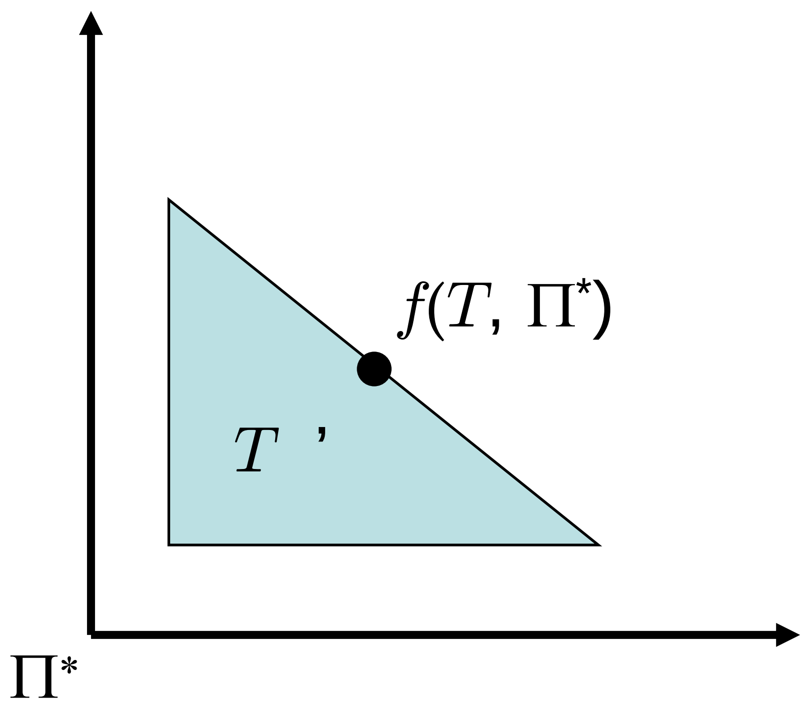
Axioms

Axiom 6: Independence of irrelevant alternatives



Axioms

Axiom 6: Independence of irrelevant alternatives



Axioms

Axiom 6: Independence of irrelevant alternatives

If $T' \subset T$ and $f(T, \Pi^*) \in T'$, then $f(T, \Pi^*) = f(T', \Pi^*)$.

Nash bargaining solution

Theorem (Nash):

There exists a unique f satisfying Axioms 1-6, and it is given by:

$f(T, \Pi^*)$

$= \arg\max_{\Pi \in T \,:\, \Pi \geq \Pi^*} (\Pi_1 - \Pi_1^*)(\Pi_2 - \Pi_2^*)$

$= \arg\max_{\Pi \in T \,:\, \Pi \geq \Pi^*} \sum_{n = 1,2} \log (\Pi_n - \Pi_n^*)$

(Sometimes called proportional fairness.)

Nash bargaining solution

- The proof relies on *all* the axioms

- The utilitarian solution and
  the max-min fair solution do not satisfy
  independence of utility units

- See course website for excerpt from MWG

Back to the interference model

- $T = \{ (\Pi_1, \Pi_2) \geq 0$:
  $\Pi_1 \leq q\, R_1, \Pi_2 \leq (1 - q)\, R_2$, $0 \leq q \leq 1\}$
- $\Pi^* = (\varepsilon, \varepsilon)$
- NBS: $\max_q \log(q\, R_1 - \varepsilon) + \log((1 - q)\, R_2 - \varepsilon)$

Solution: $q = 1/2 + (\varepsilon/2)(1/R_1 - 1/R_2)$

e.g., when $\varepsilon = 0$,

$\Pi_1^{NBS} = R_1/2, \;\; \Pi_2^{NBS} = R_2/2$

Comparisons

Assume $\varepsilon = 0$, $R_1 > R_2$

	q	Π_1	Π_2
Utilitarian	1	R_1	0
NBS	1/2	$R_1/2$	$R_2/2$
Max-min fair	$\frac{R_2}{R_1 + R_2}$	$\frac{R_1 R_2}{R_1 + R_2}$	$\frac{R_1 R_2}{R_1 + R_2}$

Summary

- When we say "efficient", we mean *Pareto efficient*.
- When we say "fair", we must make clear what we mean!
- Typically, Nash equilibria are not efficient
- The Nash bargaining solution is one axiomatic approach to fairness