MS&E 246: Lecture 3
Pure strategy Nash equilibrium

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January 16, 2007

Outline

- Best response and pure strategy Nash equilibrium
- Relation to other equilibrium notions
- Examples
- Bertrand competition

Best response set

Best response set for player n to $\mathbf{s}_{-n}$:

$$R_n(\mathbf{s}_{-n}) = \text{arg max}_{s_n \in S_n} \Pi_n(s_n, \mathbf{s}_{-n})$$

[Note: arg max$_{x \in X}$ $f(x)$ is the *set* of x that maximize $f(x)$]

Nash equilibrium

Given: N-player game

A vector $\mathbf{s} = (s_1, \ldots, s_N)$ is a *(pure strategy) Nash equilibrium* if:

$$s_i \in R_i(\mathbf{s}_{-i})$$

for all players i.

Each *individual* plays a best response to the others.

Nash equilibrium

Pure strategy Nash equilibrium is robust to *unilateral deviations*

One of the hardest questions in game theory:

How do players know to play a Nash equilibrium?

Example: Prisoner's dilemma

Recall the routing game:

		AT&T	
		near	far
MCI	near	(-4,-4)	(-1,-5)
	far	(-5,-1)	(-2,-2)

Example: Prisoner's dilemma

Here (near,near) is the unique (pure strategy) NE:

		AT&T	
		near	far
MCI	near	(-4,-4)	(-1,-5)
	far	(-5,-1)	(-2,-2)

Summary of relationships

Given a game:

- Any DSE also survives ISD, and is a NE.

(DSE = dominant strategy equilibrium; ISD = iterated strict dominance)

Example: bidding game

Recall the bidding game from lecture 1:

Player 2's bid

Player 1's bid

	$0	***$1***	***$2***	***$3***	***$4***
$0	$4.00	$4.00	$4.00	$4.00	$4.00
$1	$11.00	$7.00	$5.67	$5.00	$4.60
$2	$10.00	$7.33	$6.00	$5.20	$4.67
$3	$9.00	$7.00	$5.80	$5.00	$4.43
$4	$8.00	$6.40	$5.33	$4.57	$4.00

Example: bidding game

Here (2,2) is the unique (pure strategy) NE:

Player 2's bid

Player 1's bid

	_$0_	**_$1_**	**_$2_**	**_$3_**	**_$4_**
_$0_	$4.00	$4.00	$4.00	$4.00	$4.00
_$1_	$11.00	$7.00	$5.67	$5.00	$4.60
_$2_	$10.00	$7.33	$6.00	$5.20	$4.67
_$3_	$9.00	$7.00	$5.80	$5.00	$4.43
_$4_	$8.00	$6.40	$5.33	$4.57	$4.00

Summary of relationships

Given a game:

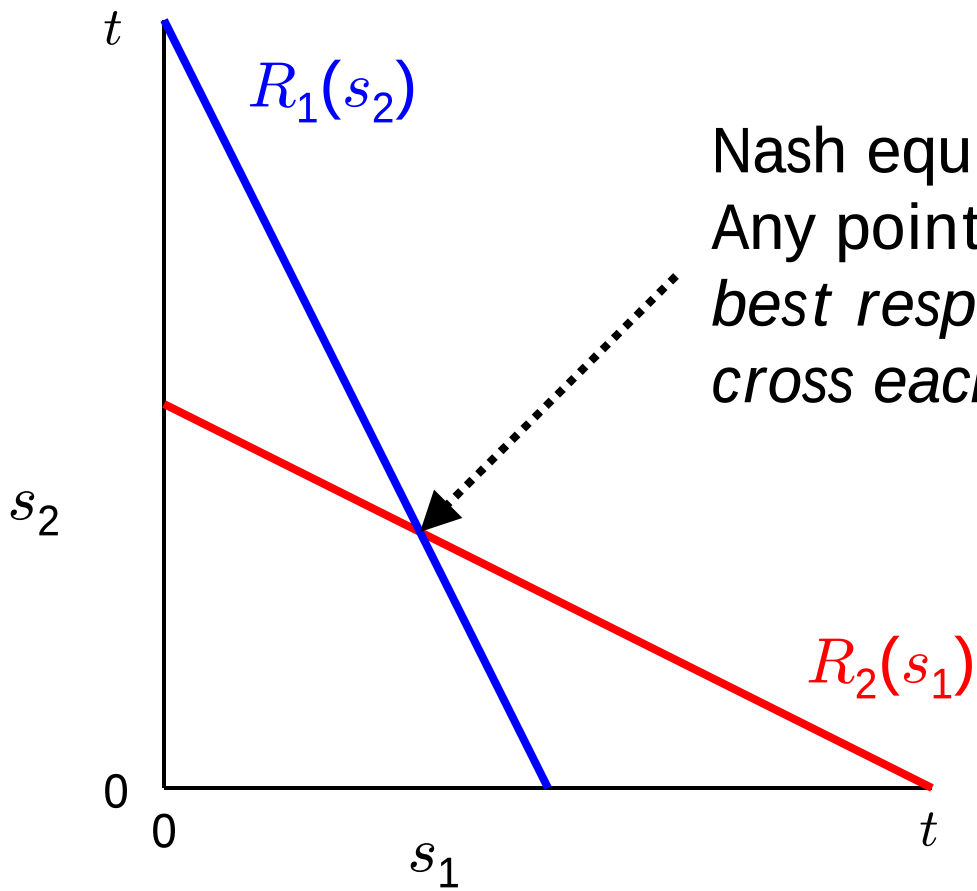
- Any DSE also survives ISD, and is a NE.
- If a game is dominance solvable, the resulting strategy vector is a NE

  Another example of this: the Cournot game.
- Any NE survives ISD (and is also rationalizable).

(DSE = dominant strategy equilibrium; ISD = iterated strict dominance)

Example: Cournot duopoly

Unique NE: (t/3 , t/3)



Nash equilibrium =
Any point *where the best response curves cross each other.*

Example: coordination game

Two players trying to *coordinate* their actions:

		Player 2	
		L	*R*
Player 1	*l*	(2,1)	(0,0)
	r	(0,0)	(1,2)

Example: coordination game

Best response of player 1:

$R_1(L) = \{ l \}$, $R_1(R) = \{ r \}$

		Player 2	
		L	*R*
Player 1	*l*	(<u>2</u>,1)	(0,0)
	r	(0,0)	(<u>1</u>,2)

Example: coordination game

Best response of player 2:

$R_2(l) = \{ L \}, R_2(r) = \{ R \}$

		Player 2	
		L	*R*
Player 1	*l*	(2,1)	(0,0)
	r	(0,0)	(1,2)

Example: coordination game

Two Nash equilibria: (l, L) and (r, R).

Moral: NE is not a *unique predictor of play!*

		Player 2	
		L	R
Player 1	l	(2,1)	(0,0)
	r	(0,0)	(1,2)

Example: matching pennies

No pure strategy NE for this game

Moral: Pure strategy NE may not exist.

		Player 2: H	Player 2: T
Player 1	H	(1,-1)	(-1,1)
	T	(-1,1)	(1,-1)

Example: Bertrand competition

- In *Cournot* competition, firms choose the *quantity* they will produce.

- In *Bertrand* competition, firms choose the *prices* they will charge.

Bertrand competition: model

- Two firms
- Each firm i chooses a price $p_i \geq 0$
- Each unit produced incurs a cost $c \geq 0$
- Consumers only buy from the producer offering the *lowest price*
- Demand is $D > 0$

Bertrand competition: model

- Two firms
- Each firm i chooses a price p_i
- Profit of firm i:

$$\Pi_i(p_1, p_2) = (p_i - c)D_i(p_1, p_2)$$

where

$$D_i(p_1, p_2) = \begin{cases} 0, & \text{if } p_i > p_{-i} \\ D, & \text{if } p_i < p_{-i} \\ \tfrac{1}{2} D, & \text{if } p_i = p_{-i} \end{cases}$$

Bertrand competition: analysis

Suppose firm 2 sets a price = $p_2 < c$.
What is the *best response set* of firm 1?

Firm 1 wants to price higher than p_2.

$$R_1(p_2) = (p_2, \infty)$$

Bertrand competition: analysis

Suppose firm 2 sets a price = $p_2 > c$.

What is the *best response set* of firm 1?

Firm 1 wants to price slightly lower than p_2

... but there is **no** *best* **response***!*

$$R_1(p_2) = \emptyset$$

Bertrand competition: analysis

Suppose firm 2 sets a price = p_2 = c.

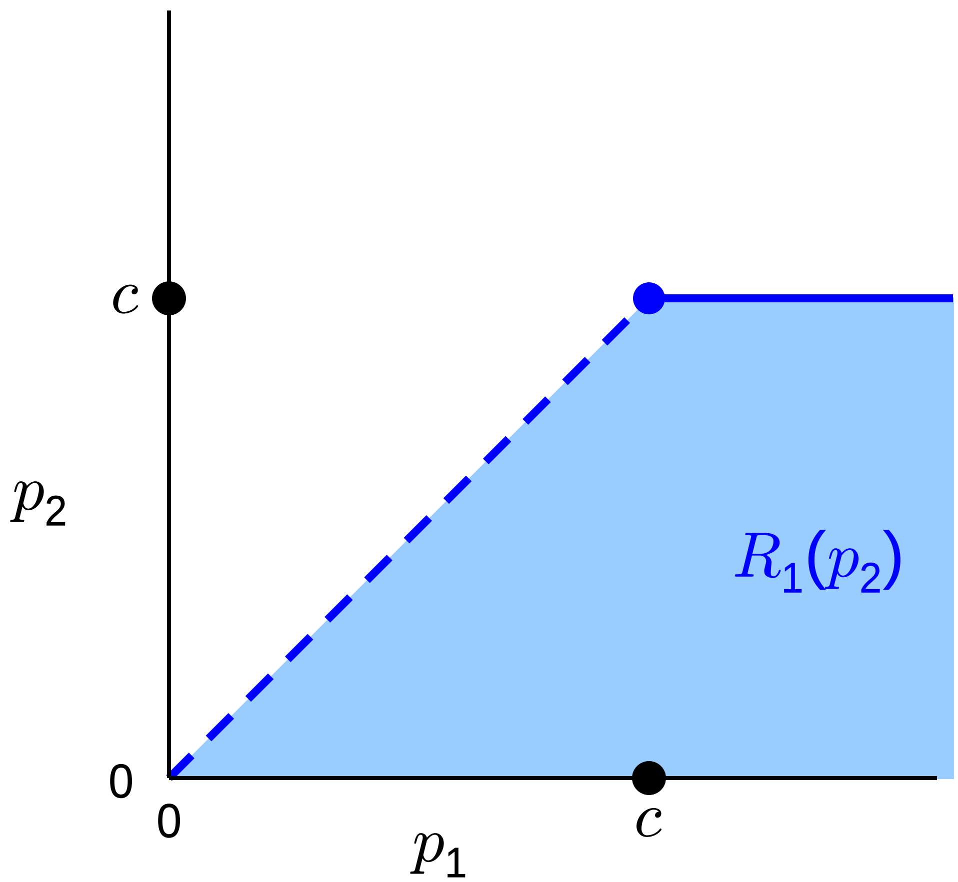
What is the *best response set* of firm 1?

Firm 1 wants to price at or higher than c.

$$R_1(p_2) = [c, \infty)$$

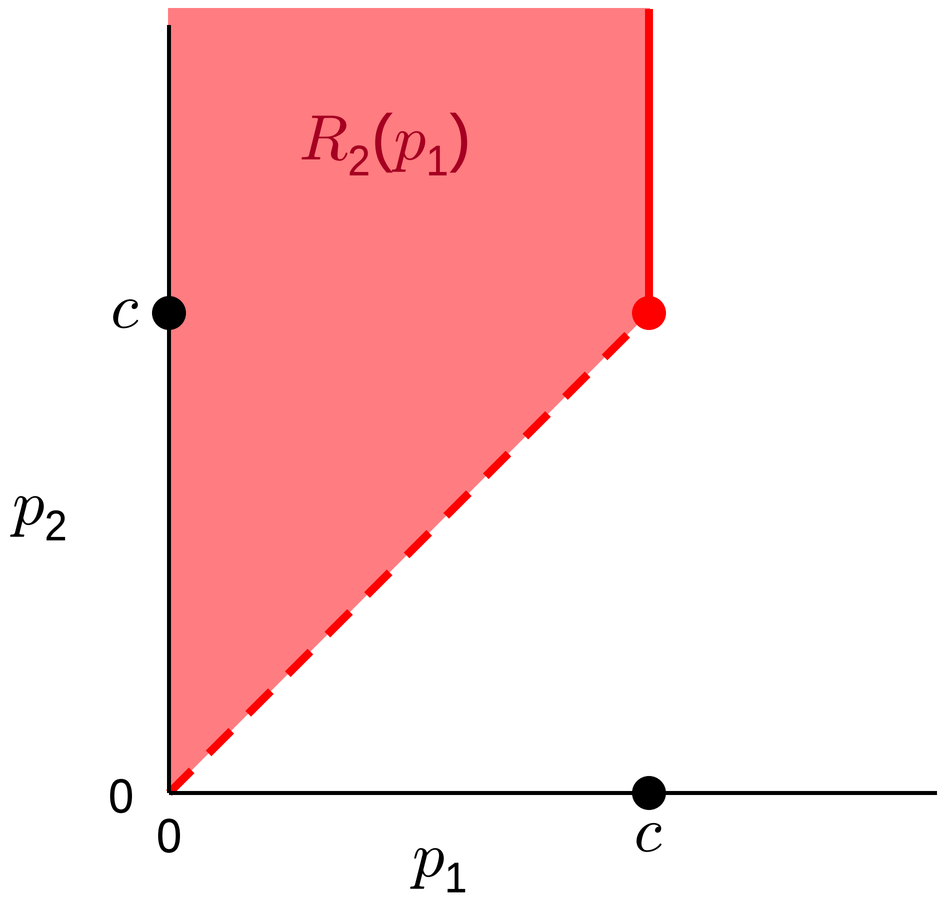
Bertrand competition: analysis

Best response of firm 1:



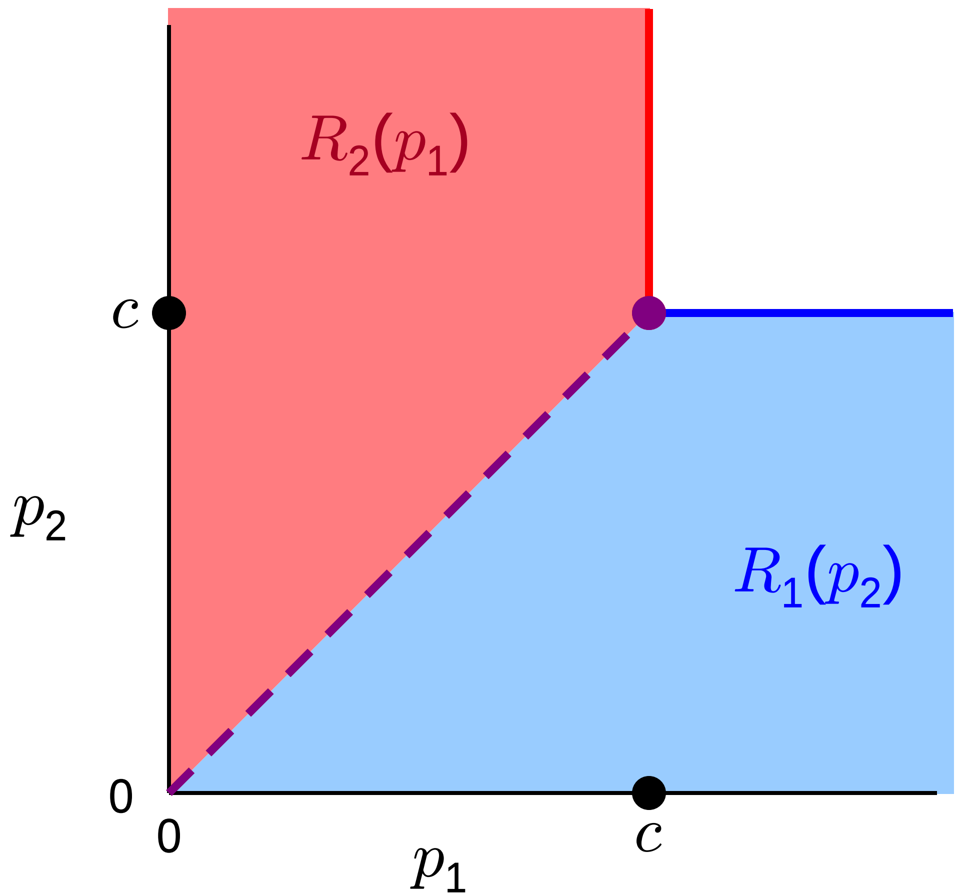
Bertrand competition: analysis

Best response of firm 2:



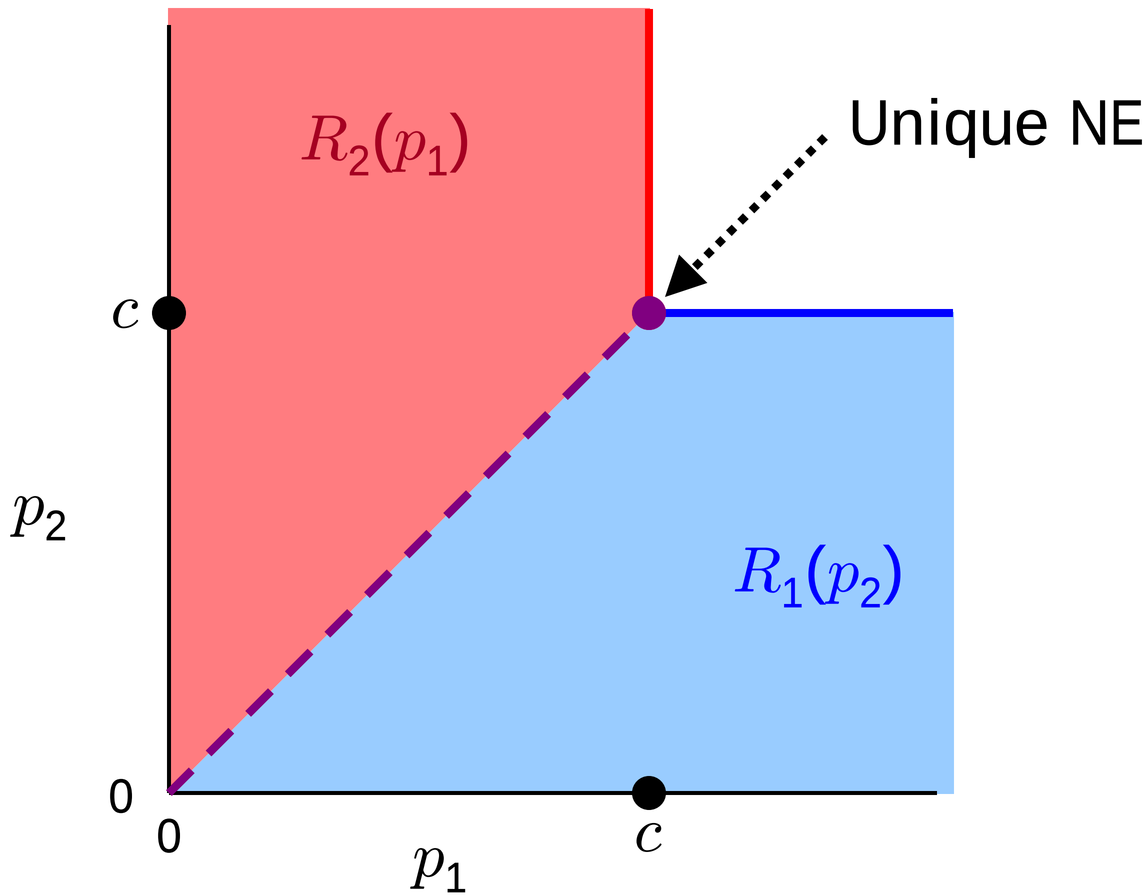
Bertrand competition: analysis

Where do they “cross”?



Bertrand competition: analysis

Thus the unique NE is where $p_1 = c$, $p_2 = c$.



Bertrand competition

Straightforward to show:

The same result holds if demand depends on price, i.e., if the demand at price p is $D(p) > 0$.

Proof technique:

(1) Show $p_i < c$ is never played in a NE.

(2) Show if $c < p_1 < p_2$, then firm 2 prefers to lower p_2.

(3) Show if $c < p_1 = p_2$, then firm 2 prefers to lower p_2

Bertrand competition

What happens if $c_1 < c_2$?

No pure NE exists; however, an ε-NE exists:

Each player is happy as long as they are within ε of their optimal payoff.

ε-NE : $p_2 = c_2$, $p_1 = c_2 - \delta$

(where δ is infinitesimal)

Bertrand vs. Cournot

Assume demand is D(p) = a - p.

Interpretation: D(p) denotes the total number of consumers willing to pay *at least* p for the good.

Then the *inverse demand* is

$$P(Q) = a - Q.$$

This is the market-clearing price at which Q total units of supply would be sold.

Bertrand vs. Cournot

Assume demand is D(p) = a - p.

Then the *inverse demand* is

$$P(Q) = a - Q.$$

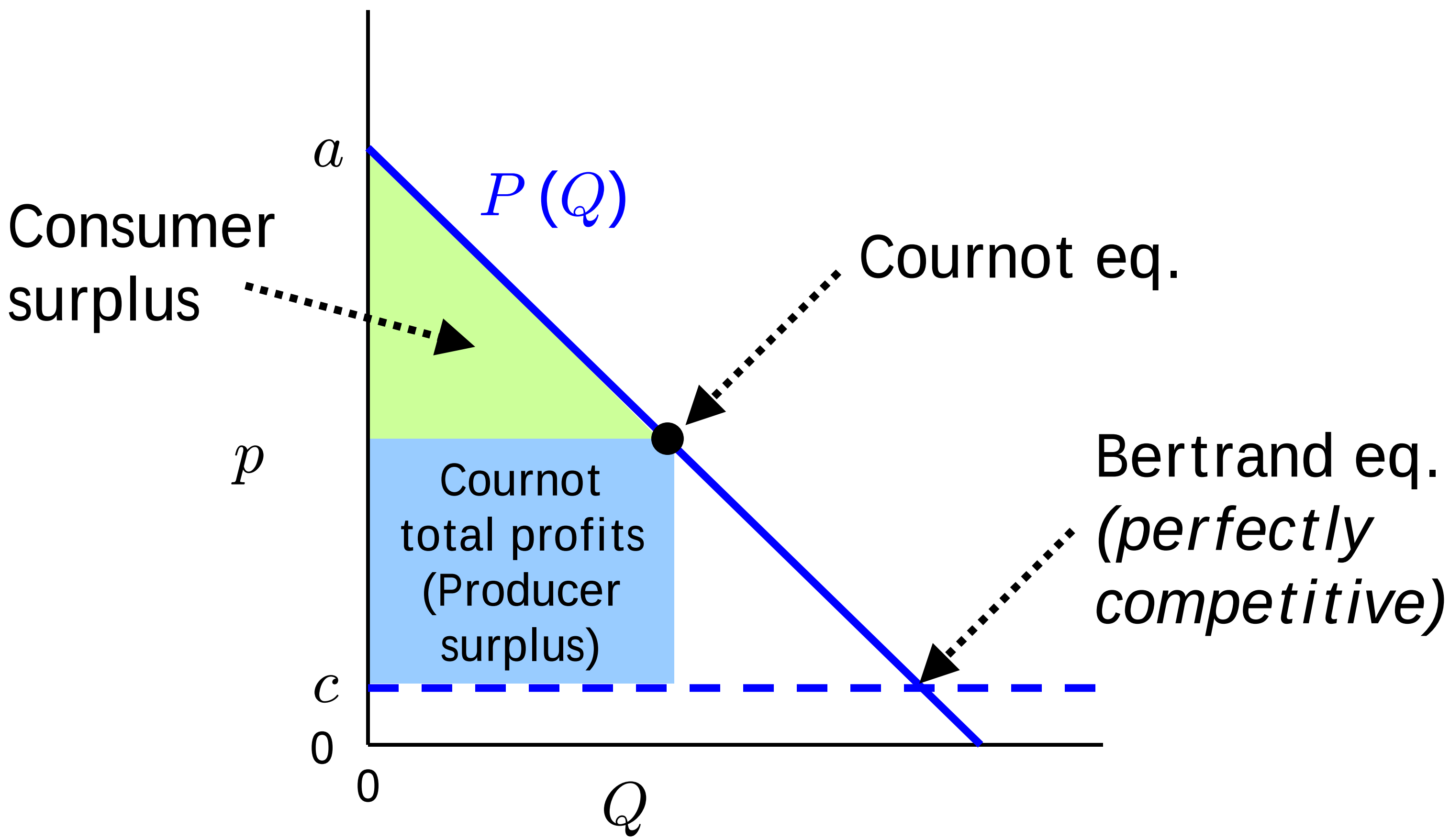
Assume $c < a$.

Bertrand eq.: $p_1 = p_2 = c$

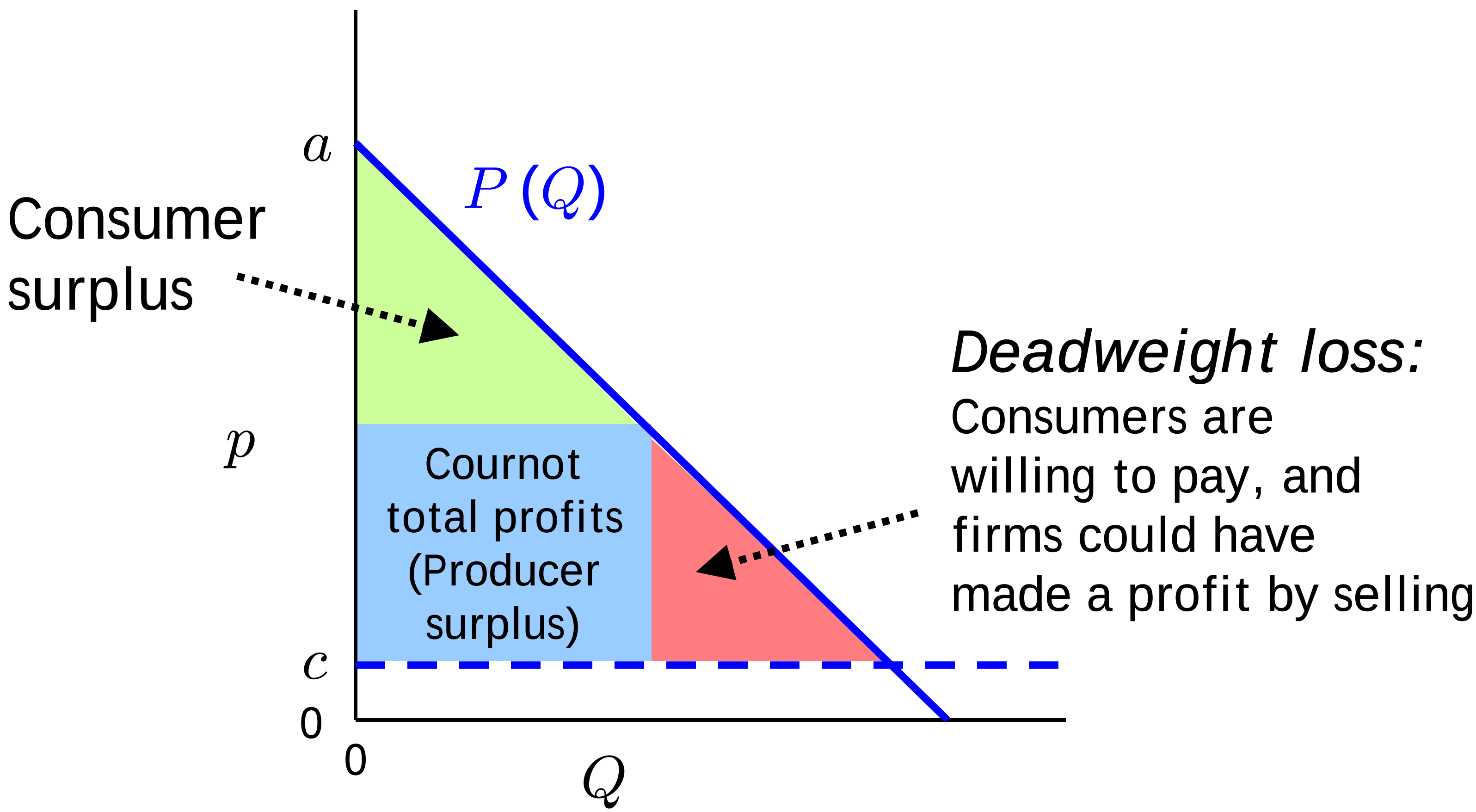
Cournot eq: $q_1 = q_2 = (a - c)/3$

$\Rightarrow$ Cournot price = $a/3 + 2c/3 > c$

Bertrand vs. Cournot



Bertrand vs. Cournot



Bertrand vs. Cournot

- Cournot eq. price > Bertrand eq. price
- Bertrand price =
  marginal cost of production
- In Cournot eq., there is positive deadweight loss.

  This is because firms have *market power: they anticipate their effect on prices.*

Questions to think about

- Can a *weakly dominated* strategy be played in a Nash equilibrium?
- Can a *strictly dominated* strategy be played in a Nash equilibrium?
- Why is any NE rationalizable?
- What are real-world examples of Bertrand competition?
  Cournot competition?

Summary: Finding NE

Finding NE is typically a matter of checking the definition.

Two basic approaches...

Finding NE: Approach 1

First approach to finding NE:

(1) Compute the complete best response mapping for each player.

(2) Find where they intersect each other (graphically or otherwise).

Finding NE: Approach 2

Second approach to finding NE:

Fix a strategy vector (s_1, ..., s_N).

Check if any player has a *profitable deviation.*

If so, it cannot be a NE.

If not, it is an NE.