MS&E 246: Lecture 2
The basics

Ramesh Johari

January 16, 2007

Course overview

(Mainly) noncooperative game theory.

Noncooperative:

Focus on individual players' incentives (note these might lead to cooperation!)

Game theory:

Analyzing the behavior of rational, self interested players

What's in a game?

1. *Players:* Who?
2. *Strategies:* What actions are available?
3. *Rules:* How? When? What do they know?
4. *Outcomes:* What results?
5. *Payoffs:*

   How do players evaluate outcomes of the game?

Example: Chess

1. *Players:* Chess masters
2. *Strategies:* Moving a piece
3. *Rules:* How pieces are moved/removed
4. *Outcomes:* Victory or defeat
5. *Payoffs:*

   Thrill of victory,

   agony of defeat

Rationality

Players are *rational* and *self-interested:*

They will *always* choose actions that maximize their payoffs, given everything they know.

Static games

We first focus on *static games.*

(one-shot games, simultaneous-move games)

For any such game, the rules say:

All players must simultaneously pick a strategy.

This immediately determines an outcome, and hence their payoff.

Knowledge

- All players know
  the *structure of the game*:

  players, strategies, rules,
  outcomes, payoffs

Common knowledge

- All players know
  the structure of the game
  - All players know all players know
    the structure
    - All players know all players know all players know
      the structure

and so on… $\Rightarrow$

We say: the structure is *common knowledge*.

This is called *complete information*.

PART I: Static games of complete information

Representation

- N : # of players
- S_n : strategies available to player n
- Outcomes:
  Composite strategy vectors
- $\Pi_n(s_1, ..., s_N)$:
  payoff to player n when
  player i plays strategy s_i, i = 1, ..., N

Example: A routing game

MCI and AT&T:

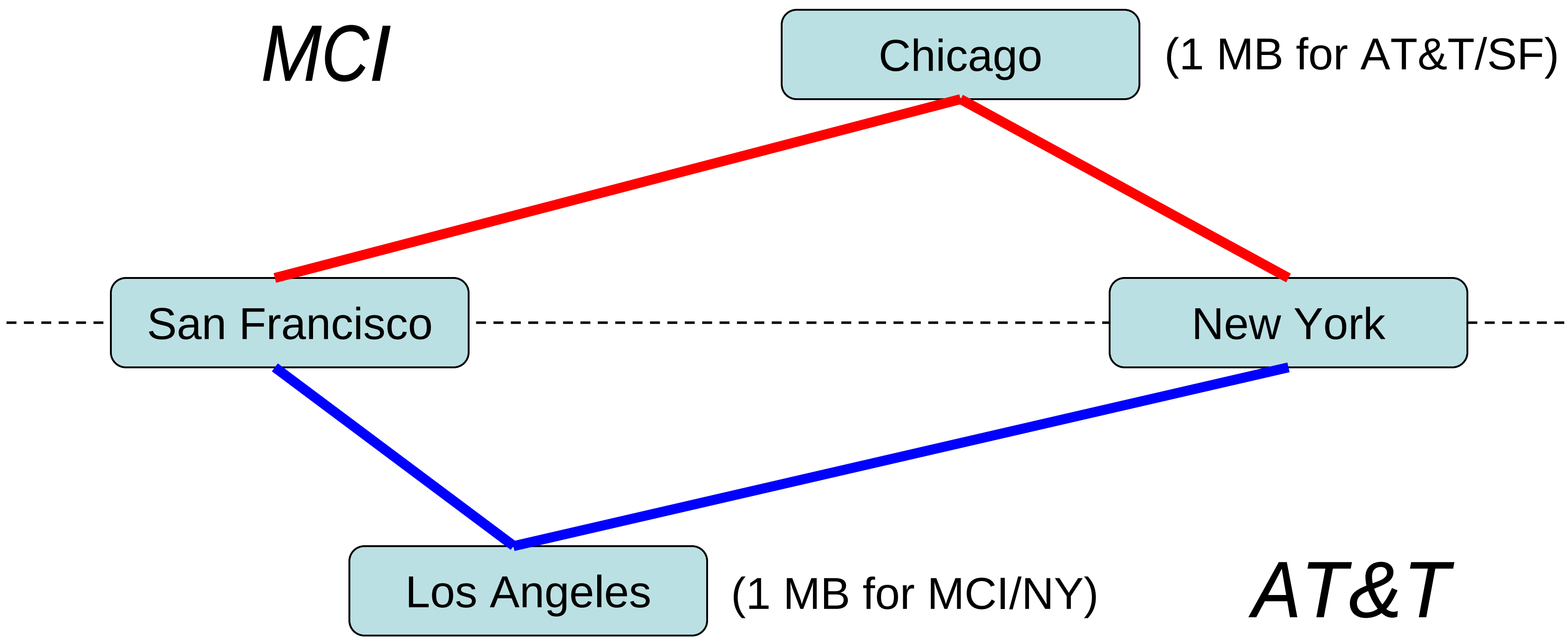
A Chicago customer of MCI wants to send 1 MB to an SF customer of AT&T.

A LA customer of AT&T wants to send 1 MB to an NY customer of MCI.

Providers minimize their own cost.

Key: MCI and AT&T only exchange traffic ("peer") in NY and SF.

Example: A routing game



Costs (per MB): Long links = 2; Short links = 1

Example: A routing game

Players: MCI and AT&T (N = 2)

Strategies: Choice of traffic exit

$S_1 = S_2 =$ { nearest exit, furthest exit }

Payoffs:

Both choose furthest exit: $\Pi_{MCI} = \Pi_{AT\&T} = -2$

Both choose nearest exit: $\Pi_{MCI} = \Pi_{AT\&T} = -4$

MCI chooses near, AT&T chooses far:

$\Pi_{MCI} = -1$, $\Pi_{AT\&T} = -5$

Example: A routing game

AT&T

		near	far
MCI	near	(-4,-4)	(-1,-5)
	far	(-5,-1)	(-2,-2)

Games with $N = 2$, S_n finite for each n are called *bimatrix games*.

Example: Matching pennies

		Player 2	
		H	T
Player 1	H	(1,-1)	(-1,1)
	T	(-1,1)	(1,-1)

This is a *zero-sum matrix game.*

Dominance

$s_n \in S_n$ is a *(weakly) dominated strategy* if

there exists $s_n{*} \in S_n$ such that

$\Pi_n(s_n{*}, \mathbf{s}_{-n}) \geq \Pi_n(s_n, \mathbf{s}_{-n})$,

for any choice of $\mathbf{s}_{-n}$, with
strict ineq. for at least one choice of $\mathbf{s}_{-n}$

If the ineq. is always strict, then s_n is a *strictly dominated strategy*.

Dominance

s_n* $\in S_n$ is a *weak dominant strategy* if

s_n* weakly dominates all other $s_n \in S_n$.

s_n* $\in S_n$ is a *strict dominant strategy* if

s_n* strictly dominates all other $s_n \in S_n$.

(Note: dominant strategies are unique!)

Dominant strategy equilibrium

$\mathbf{s} \in S_1 \times \cdots \times S_N$ is a

strict (or weak) dominant strategy equilibrium

if s_n is a strict (or weak) dominant strategy for each n.

Back to the routing game

		AT&T	
		near	far
MCI	near	(-4,-4)	(-1,-5)
	far	(-5,-1)	(-2,-2)

Nearest exit is strict dominant strategy for MCI.

Back to the routing game

		AT&T	
		near	far
MCI	near	(-4,-4)	(-1,-5)
	far	(-5,-1)	(-2,-2)

Nearest exit is strict dominant strategy for AT&T.

Back to the routing game

		AT&T	
		near	far
MCI	near	(-4,-4)	(-1,-5)
	far	(-5,-1)	(-2,-2)

Both choosing nearest exit is a strict dominant strategy equilibrium.

Example: Second price auction

- N bidders
- *Strategies:* S_n = [0, ∞); s_n = "bid"
- *Rules & outcomes:*

  High bidder wins, pays *second highest* bid
- *Payoffs:*
  - Zero if a player loses
  - If player n wins and pays t_n , then

    $\Pi_n = v_n - t_n$
  - v_n : *valuation* of player n

Example: Second price auction

- *Claim: Truthful bidding* ($s_n = v_n$) is a weak dominant strategy for player n.

- *Proof:*

  If player n considers a bid > v_n :

  Payoff may be lower when n wins, and the same (zero) when n loses

Example: Second price auction

- *Claim: Truthful bidding* ($s_n = v_n$) is a weak dominant strategy for player n.

- *Proof:*

  If player n considers a bid < v_n :

  Payoff will be same when n wins, but may be worse when n loses

Example: Second price auction

- We conclude:

  Truthful bidding is a (weak) dominant strategy equilibrium for the second price auction.

Example: Matching pennies

		Player 2	
		H	T
Player 1	H	(1,-1)	(-1,1)
	T	(-1,1)	(1,-1)

No dominant/dominated strategy exists!

Moral: *Dominant strategy eq. may not exist.*

Iterated strict dominance

Given a game:

- Construct a new game by *removing* a strictly dominated strategy from one of the strategy spaces S_n .
- Repeat this procedure until no strictly dominated strategies remain.

If this results in a unique strategy profile, the game is called *dominance solvable.*

Iterated strict dominance

- Note that the bidding game in Lecture 1 was dominance solvable.

- There the unique resulting strategy profile was (6,6).

Example

Player 2

		Left	Middle	Right
Player 1	Up	(1,0)	(1,2)	(0,1)
	Down	(0,3)	(0,1)	(2,0)

Example

		Player 2		
		Left	Middle	Right
Player 1	Up	(1,0)	(1,2)	(0,1)
	Down	(0,3)	(0,1)	(2,0)

Example

Player 2

		Left	Middle	Right
Player 1	Up	(1,0)	(1,2)	(0,1)
	Down	(0,3)	(0,1)	(2,0)

Example

		Player 2		
		Left	Middle	Right
Player 1	Up	(1,0)	(1,2)	(0,1)
	Down	(0,3)	(0,1)	(2,0)

Thus the game is dominance solvable.

Example: Cournot duopoly

- Two firms (N = 2)
- *Cournot competition*:
  each firm chooses a quantity $s_n \geq 0$
- Cost of producing s_n : $c\ s_n$
- *Demand curve*:
  Price = $P(s_1 + s_2) = a - b\ (s_1 + s_2)$
- Payoffs:
  Profit = $\Pi_n(s_1,\ s_2) = P(s_1 + s_2)\ s_n - c\ s_n$

Example: Cournot duopoly

- *Claim:*

  The Cournot duopoly is dominance solvable.

- *Proof technique*:

  First construct the *best response* for each player.

Best response

Best response set for player n to $\mathbf{s}_{-n}$:

$$R_n(\mathbf{s}_{-n}) = \arg\max_{s_n \in S_n} \Pi_n(s_n, \mathbf{s}_{-n})$$

[Note: $\arg\max_{x \in X} f(x)$ is the *set* of x that maximize $f(x)$]

Example: Cournot duopoly

Calculating the best response given $s_{\text{-}n}$:

$$\max_{s_n \geq 0} [(a - bs_n - bs_{-n})s_n - cs_n] \implies$$

Differentiate and solve:

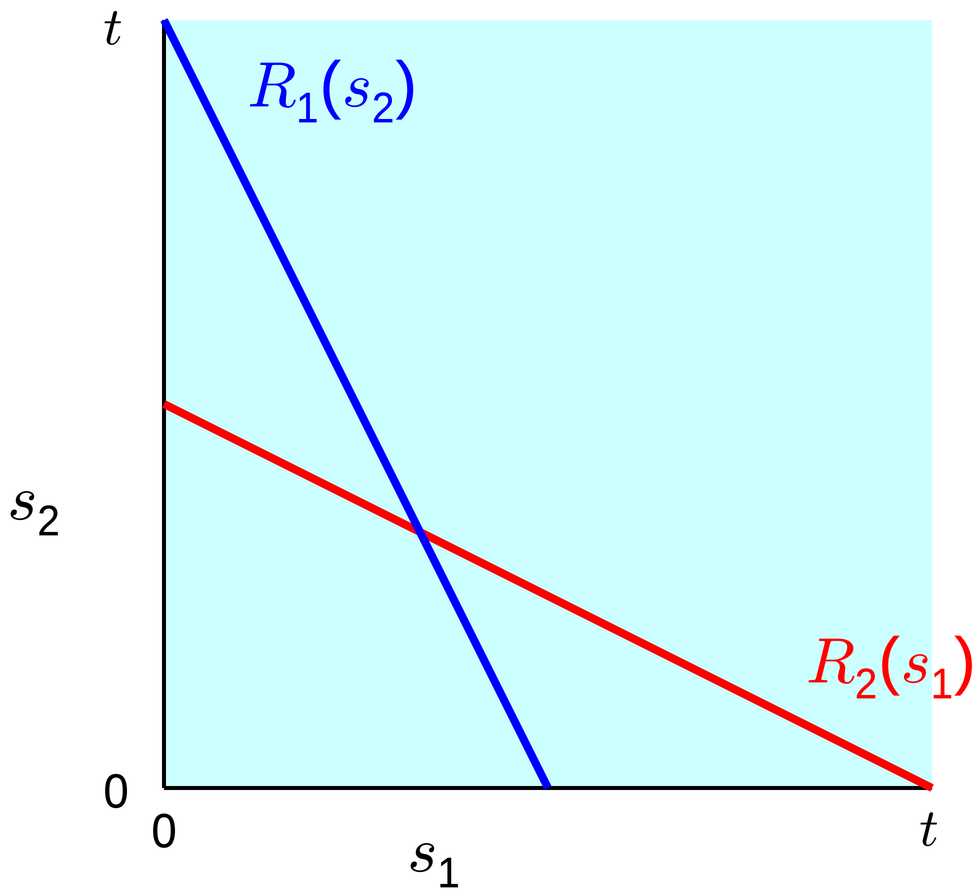
$$a - c - bs_{-n} - 2bs_n = 0$$

So:

$$R_n(s_{-n}) = \left[\frac{a-c}{2b} - \frac{s_{-n}}{2}\right]^+$$

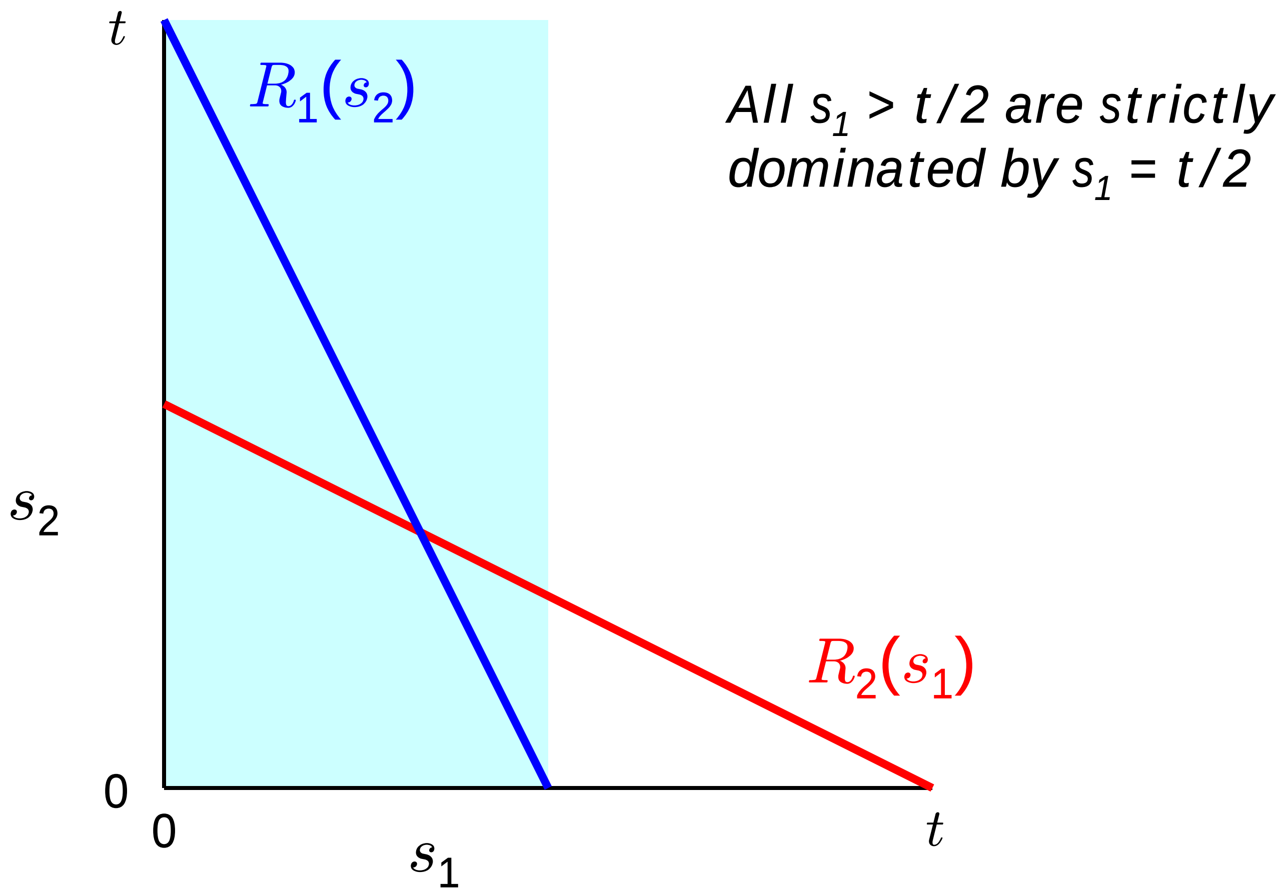
Example: Cournot duopoly

For simplicity, let $t = (a - c)/b$



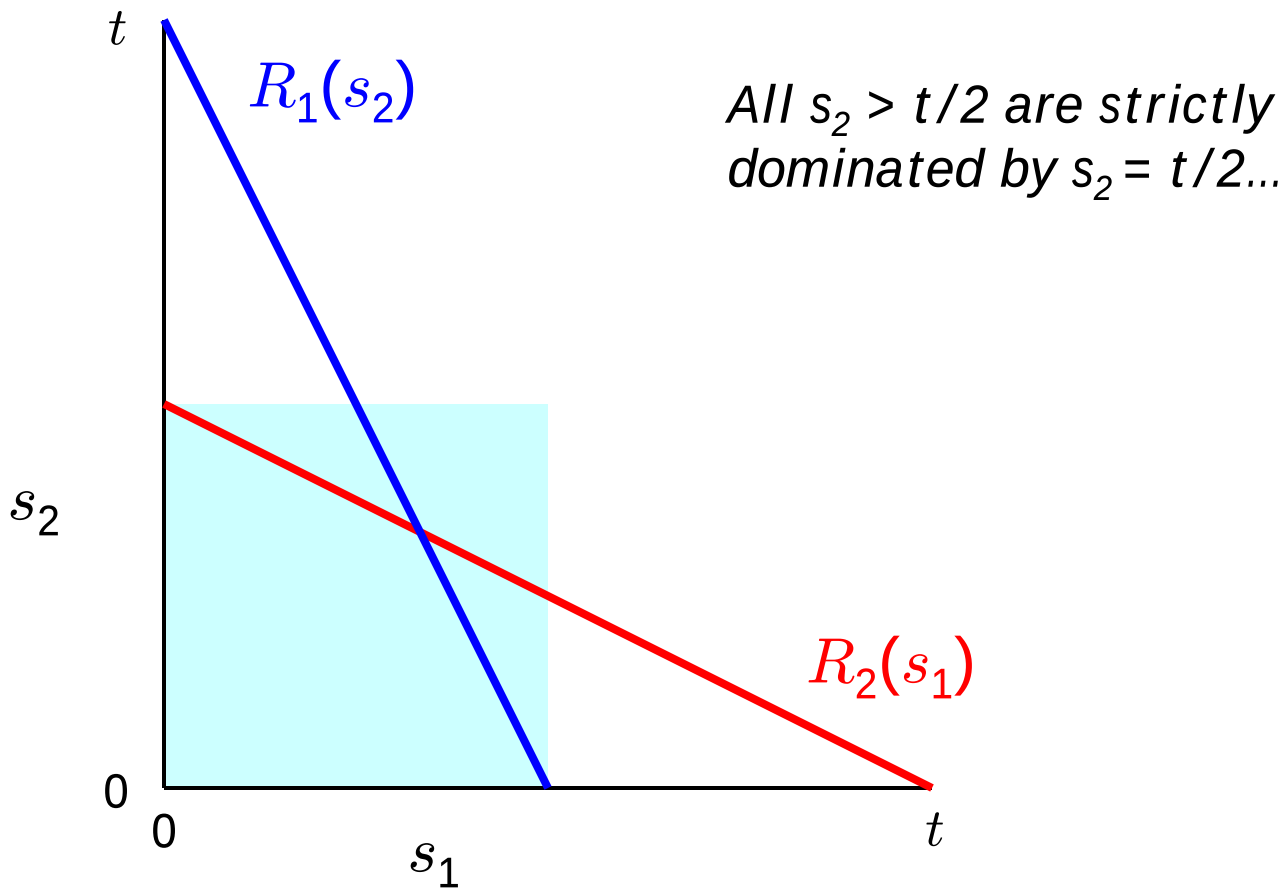
Example: Cournot duopoly

Step 1: Remove strictly dominated s_1.



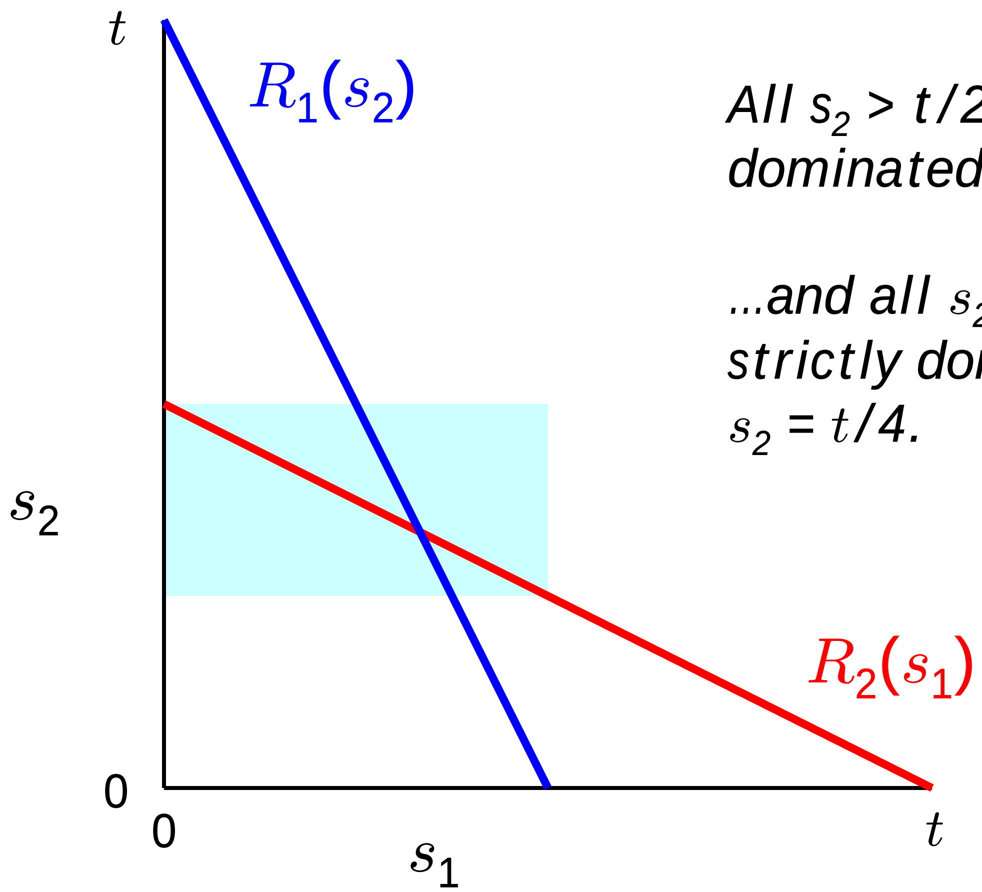
Example: Cournot duopoly

Step 2: Remove strictly dominated s_2.



Example: Cournot duopoly

Step 2: Remove strictly dominated s_2.

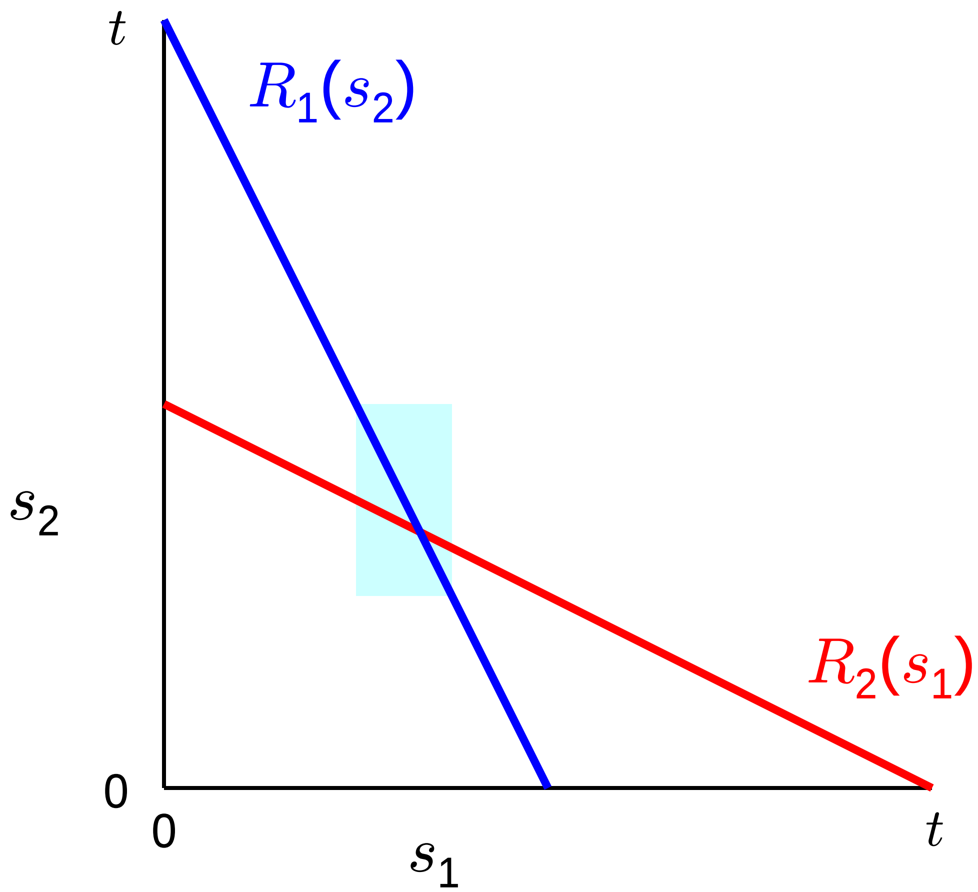


All $s_2 > t/2$ are strictly dominated by $s_2 = t/2$...

...and all $s_2 < t/4$ are strictly dominated by $s_2 = t/4$.

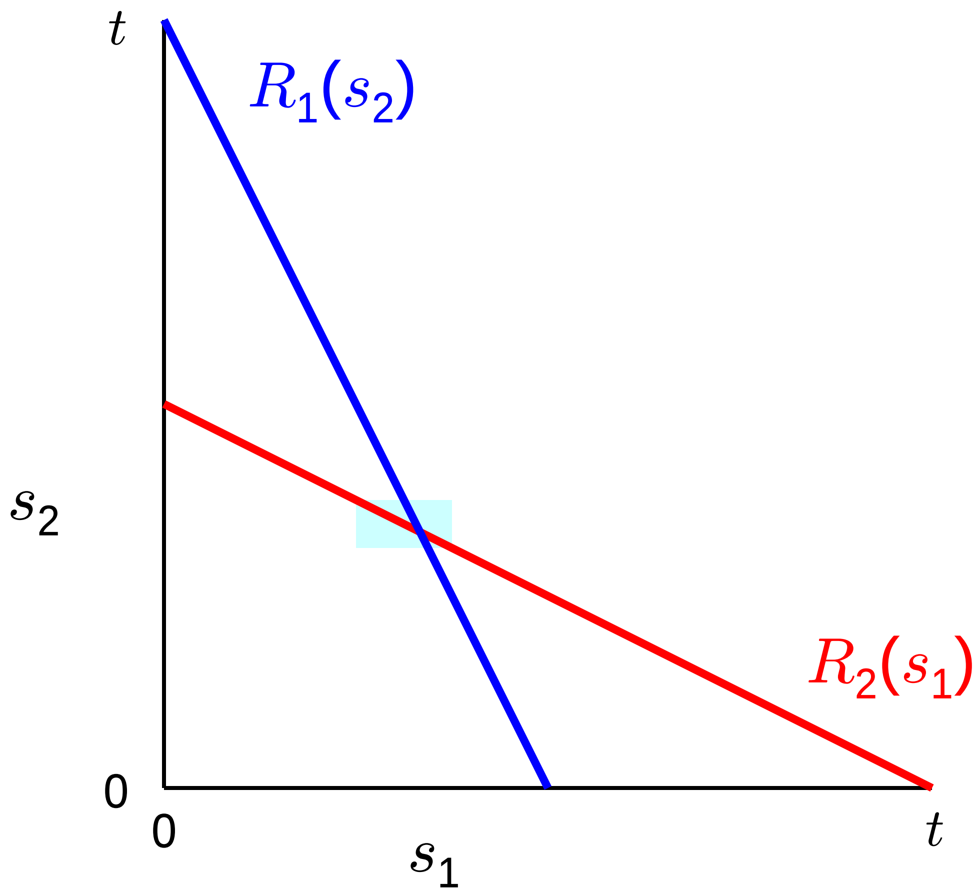
Example: Cournot duopoly

Step 3: Remove strictly dominated s_1.



Example: Cournot duopoly

Step 4: Remove strictly dominated s_2.



Example: Cournot duopoly

The process converges to the intersection point: $s_1 = t/3$, $s_2 = t/3$

Step #	Undominated s_1
1	$[0, t/2]$
3	$[t/4, 3t/8]$
5	$[5t/16, 11t/32]$
7	$[21t/64, 43t/128]$

Example: Cournot duopoly

Lower bound =

$$t \sum_{k=1}^{\infty} (1/2)^{2k} = t/3.$$

Upper bound =

$$t \left(1 - \sum_{k=1}^{\infty} (1/2)^{2k-1} \right) = t/3.$$

Dominance solvability: comments

- Order of elimination doesn’t matter
- Just as most games don’t have DSE, most games are not dominance solvable

Rationalizable strategies

Given a game:

- For each player n, remove strategies from each S_n that are not best responses for *any* choice of other players' strategies.
- Repeat this procedure.

Strategies that survive this process are called *rationalizable strategies.*

Rationalizable strategies

In a two player game, a strategy s_1 is rationalizable for player 1 if there exists a *chain of justification*

$$s_1 \to s_2 \to s_1' \to s_2' \to \ldots \to s_1$$

where each is a best response to the one before.

Rationalizable strategies

- If s_n is rationalizable, it also survives iterated strict dominance. (Why?)

$\Rightarrow$ For a dominance solvable game, there is a unique rationalizable strategy, and it is the one given by iterated strict dominance.

Rationalizability: example

Note that M is not *rationalizable*,
but it survives iterated strict dominance.

Player 2

		L	M	R
Player 1	T	(1,0)	(1,1)	(1,5)
	B	(1,5)	(1,1)	(1,0)

Rationalizable strategies

Note for later:

When "mixed" strategies are allowed,

rationalizability = iterated strict dominance

for two player games.