MS&E 246: Lecture 17
Network routing

Ramesh Johari

Network routing

- Basic definitions
- Wardrop equilibrium
- Braess’ paradox
- Implications

Network routing

- N users travel across a network
  - Transportation
  - Internet
- s = source, d = destination
- J : set of links in the network
- Path : chain of links from s to d
- What paths do users select?

Delay

- Assume users experience a *cost* when they cross a link
- Usually interpreted as *delay*
  - but could be any congestion measure (lost packets, etc.)
- $l_j(n_j)$:

  delay of link j when used by n_j users

The network routing game

Assume:

- *Strategy space of each user:*
  - Paths from s to d
- *Cost to each user:*
  - Total delay experienced on chosen path

The network routing game

Formally:

- P : set of paths from s to d

  (*Note*: $p \in P \Rightarrow p \subset J$)

- p_r : path chosen by user r
- $\mathbf{p} = (p_1, \ldots, p_N)$
- $n_j(\mathbf{p})$: number of users r with $j \in p_r$

The network routing game

Formally:

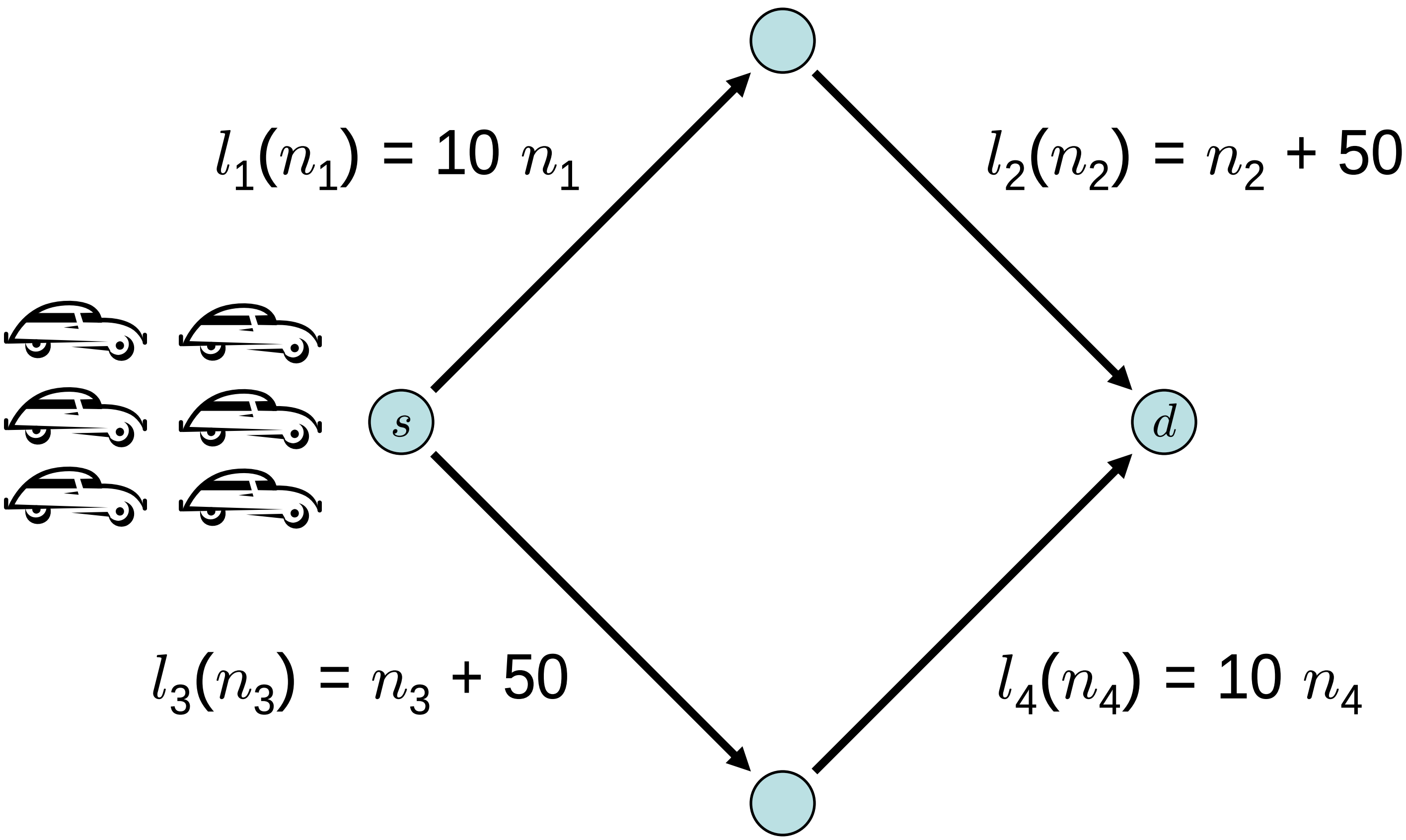
Cost to user r = total delay =

$$\Pi_r(\mathbf{p}) = \sum_{j \in p_r} l_j(n_j(\mathbf{p}))$$

Note: in this game, players *minimize* payoffs.

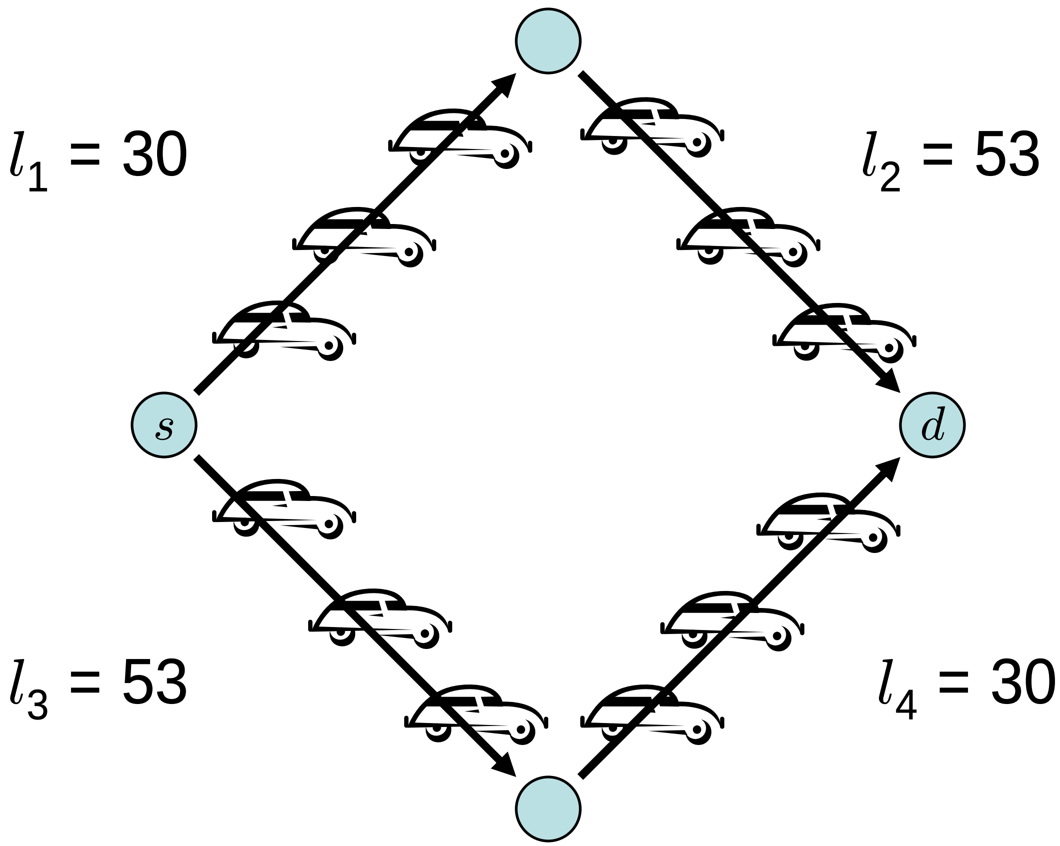
Example 1

N = 6; 4 links; 2 paths



Example 1

Nash equilibrium:



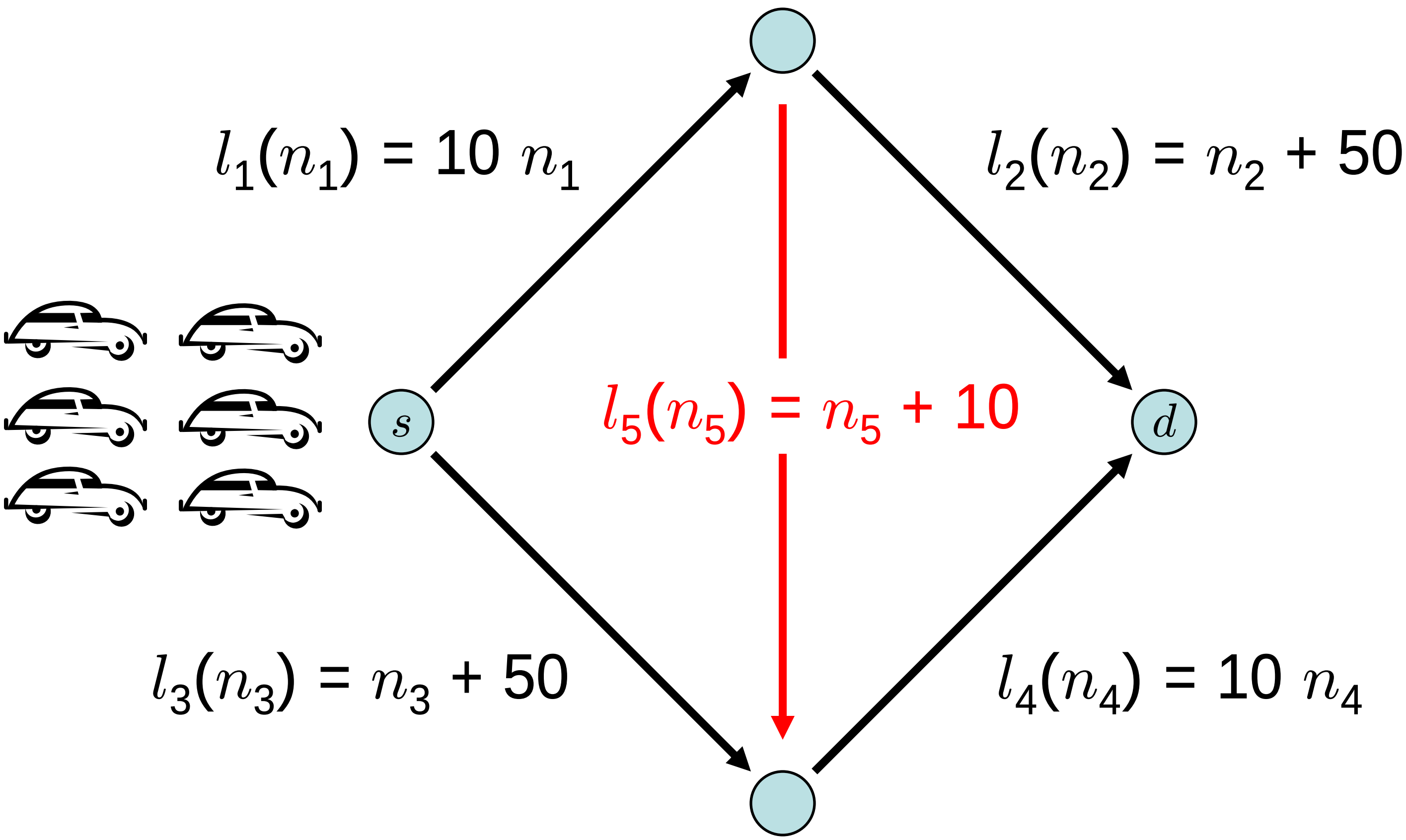
Example 1: summary

- This Nash equilibrium is unique. (Why?)
- This Nash equilibrium is *Pareto efficient.* (Why?)

- *Total delay to each user:* 83 minutes

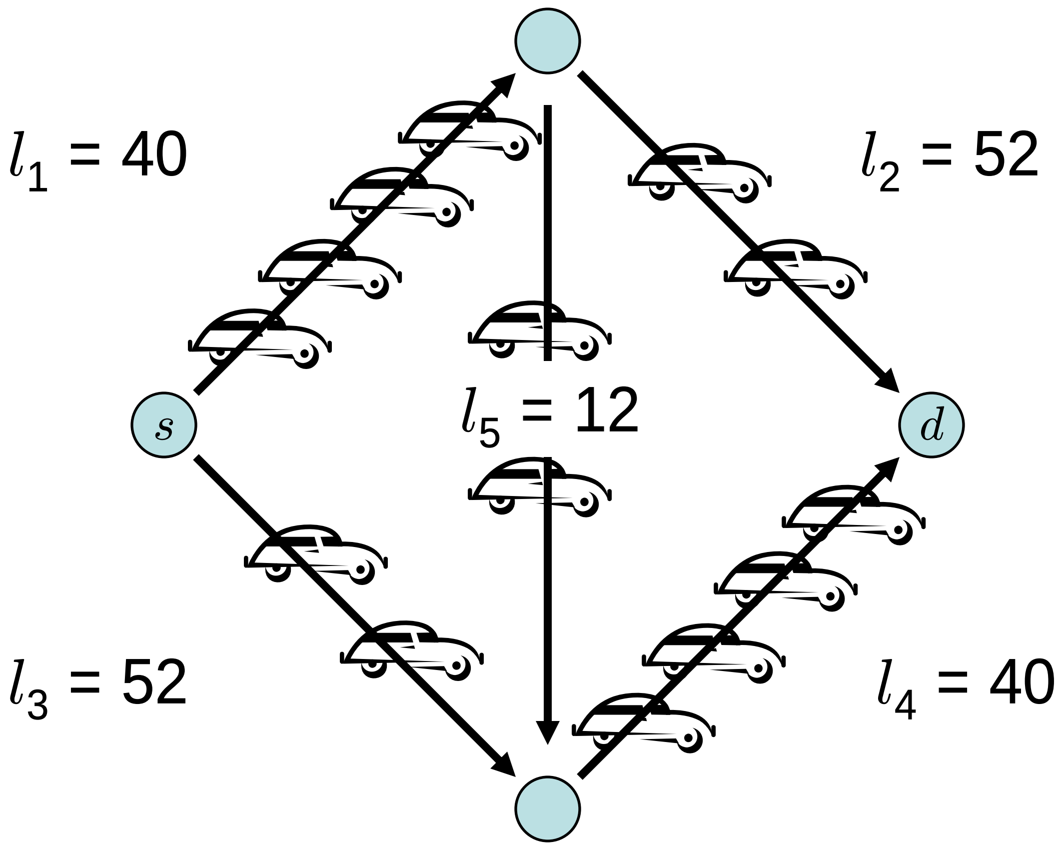
Example 2

$N = 6$; 5 links; 3 paths



Example 2

Nash equilibrium:



Example 2: summary

- The Nash equilibrium is unique. (Why?)
- *Total delay to each user:* 92 minutes
- Is the Nash equilibrium Pareto efficient?

Adding a link can increase delay for everyone!

This is called *Braess' paradox.*

Braess' paradox

Why does Braess' paradox occur?

A form of *tragedy of the commons:*

Players do not care about the negative externality they impose on each other.

Characterizing pure NE

Suppose the current path assignment is $\mathbf{p}$.

Player r considers switching to p_r'.

What is the change in player r's payoff?

$\Pi_r(p_r, \mathbf{p}_{-r}) = \sum_{j \in p_r} l_j(n_j(\mathbf{p}_{-r}) + 1)$

$\Pi_r(p_r', \mathbf{p}_{-r}) = \sum_{j \in p_r'} l_j(n_j(\mathbf{p}_{-r}) + 1)$

Characterizing pure NE

Suppose the current path assignment is $\mathbf{p}$.

Player r considers switching to p_r'.

What is the change in player r's payoff?

$$\Pi_r(p_r, \mathbf{p}_{-r}) - \Pi_r(p'_r, \mathbf{p}_{-r}) =$$
$$\sum_{j \in p_r - p'_r} l_j(n_j(\mathbf{p}_{-r}) + 1)$$
$$- \sum_{j \in p'_r - p_r} l_j(n_j(\mathbf{p}_{-r}) + 1)$$

Characterizing pure NE

Define the following function V:

$$V(\mathbf{p}) = \sum_{j \in J} \sum_{i=0}^{n_j(\mathbf{p})} l_j(i)$$

Characterizing pure NE

Observe that:

$$V(p_r, \mathbf{p}_{-r}) = \sum_{j \in J} \sum_{i=1}^{n_j(\mathbf{p}_{-r})} l_j(i) + \sum_{j \in p_r} l_j(n_j(\mathbf{p}_{-r}) + 1)$$

Characterizing pure NE

Observe that:

$$\begin{aligned} V(p_r, \mathbf{p}_{-r}) - V(p'_r, \mathbf{p}_{-r}) = \\ \sum_{j \in p_r - p'_r} l_j(n_j(\mathbf{p}_{-r}) + 1) \\ - \sum_{j \in p'_r - p_r} l_j(n_j(\mathbf{p}_{-r}) + 1) \end{aligned}$$

Characterizing pure NE

Observe that:

$$V(p_r, \mathbf{p}_{-r}) - V(p'_r, \mathbf{p}_{-r}) = \Pi_r(p_r, \mathbf{p}_{-r}) - \Pi_r(p'_r, \mathbf{p}_{-r})$$

So: a unilateral deviation is profitable if and only if V strictly decreases.

Characterizing pure NE

Definition:

A *local minimum* of V is a vector $\mathbf{p}$ such that $V(\mathbf{p})$ - $V(p_{r'}, \mathbf{p}_{-r}) \leq 0$ for all p_r'.

Conclude:

Any pure strategy Nash equilibrium is a local minimum of V.

Characterizing pure NE

Since V has a global minimum, at least one pure NE exists

If V has a unique local minimum, then the game has a unique pure NE

Best response dynamic

Suppose that at each time t = 1, 2, ...,
a player is randomly selected, and switches to a better path if one exists (otherwise continues on the same path).

This is called the *best response dynamic.*

Let $\mathbf{p}(1)$, $\mathbf{p}(2)$, ...
be the resulting path assignments.

Best response dynamic

At each stage:

If $\mathbf{p}(t+1) \neq \mathbf{p}(t)$, then
$V(\mathbf{p}(t+1)) < V(\mathbf{p}(t))$

Since V cannot decrease forever,

eventually we reach a pure NE.

i.e., *best response dynamic converges.*

Best response dynamic

The best response dynamic has two interpretations:

1) A way to find Nash equilibria (if it converges)
2) A model of *bounded rationality* of the players

Potential games

The function V is called a *potential.*

Games with functions V such that:

$$V(s_r, \mathbf{s}_{-r}) - V(s_r', \mathbf{s}_{-r}) = \Pi_r(s_r, \mathbf{s}_{-r}) - \Pi_r(s_r', \mathbf{s}_{-r}) \quad \text{for all } r$$

are called *exact potential games.*

Potential games

More generally, games with functions V such that

$V(s_r, \mathbf{s}_{-r}) - V(s_r', \mathbf{s}_{-r})$

has the same sign (+, -, or zero) as

$\Pi_r(s_r, \mathbf{s}_{-r}) - \Pi_r(s_r', \mathbf{s}_{-r})$ for all r

are called *ordinal potential games.*

Potential games

Assume strategy spaces are finite.

A potential game (ordinal or exact):

- has a pure strategy NE
- has convergent best response dynamic

Braess' paradox

In our games, the potential has a unique local minimum $\Rightarrow$ unique pure NE.

In other words, the NE achieves the minimum value of V.

Braess' paradox

Find one Pareto efficient point by minimizing *total delay:*

$$\sum_{r=1}^{N} \sum_{j \in p_r} l_j(n_j(\mathbf{p}))$$

This is the *utilitarian* solution.

Braess' paradox

Find one Pareto efficient point by minimizing *total delay:*

$$\sum_{j \in J} n_j(\mathbf{p}) l_j(n_j(\mathbf{p}))$$

Note that this is *not* the same as V!

Braess' paradox

In our network routing games, minimizing total delay makes everyone strictly better off.

Just like tragedy of the commons:

Individual optimization does not imply global efficiency.