MS&E 246: Lecture 15
Perfect Bayesian equilibrium

Ramesh Johari

Dynamic games

In this lecture, we begin a study of *dynamic games of incomplete information.*

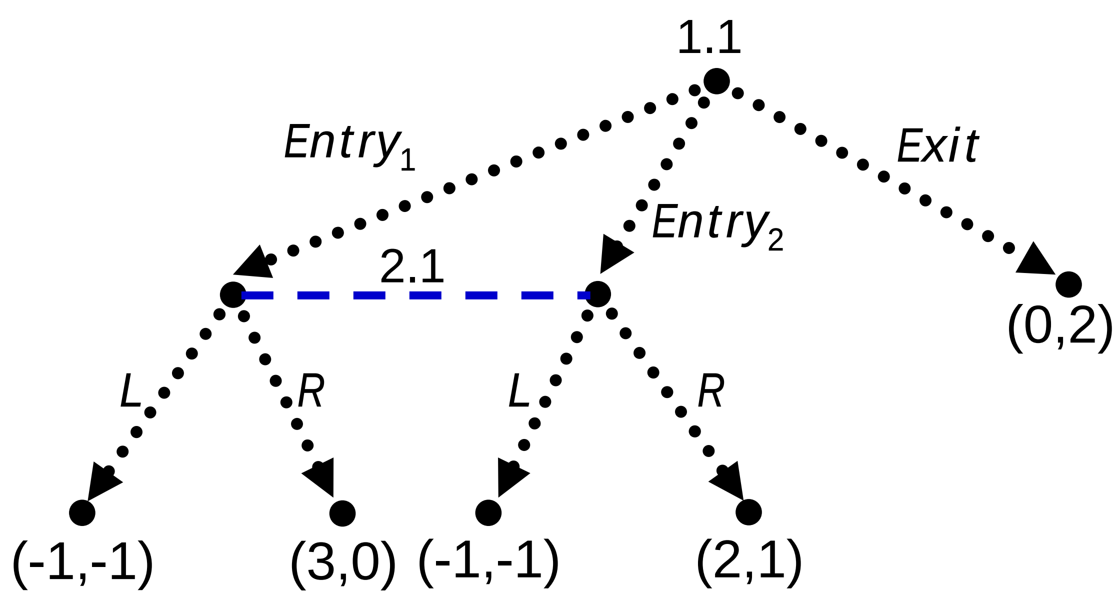
We will develop an analog of Bayesian equilibrium for this setting, called *perfect Bayesian equilibrium*.

Why do we need beliefs?

Recall in our study of subgame perfection that problems can occur if there are “not enough subgames” to rule out equilibria.

Entry example

- Two firms
- First firm decides if/how to enter
- Second firm can choose to “fight”



Entry example

Note that this game only has *one* subgame.

Thus SPNE are *any* NE of strategic form.

		Firm 2	
		L	*R*
	$Entry_1$	(-1,-1)	(3,0)
Firm 1	$Entry_2$	(-1,-1)	(2,1)
	Exit	(0,2)	(0,2)

Entry example

Two pure NE of strategic form:

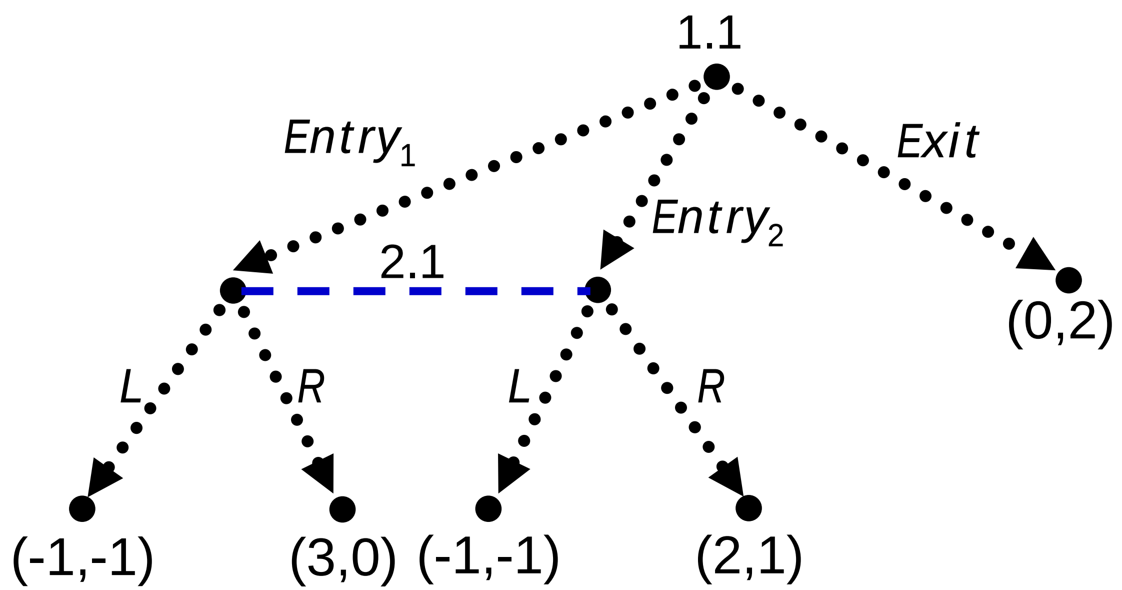
($Entry_1$, *R*) and (*Exit*, *L*)

Firm 2

Firm 1		*L*	*R*
	$Entry_1$	(-1,-1)	(3,0)
	$Entry_2$	(-1,-1)	(2,1)
	Exit	(0,2)	(0,2)

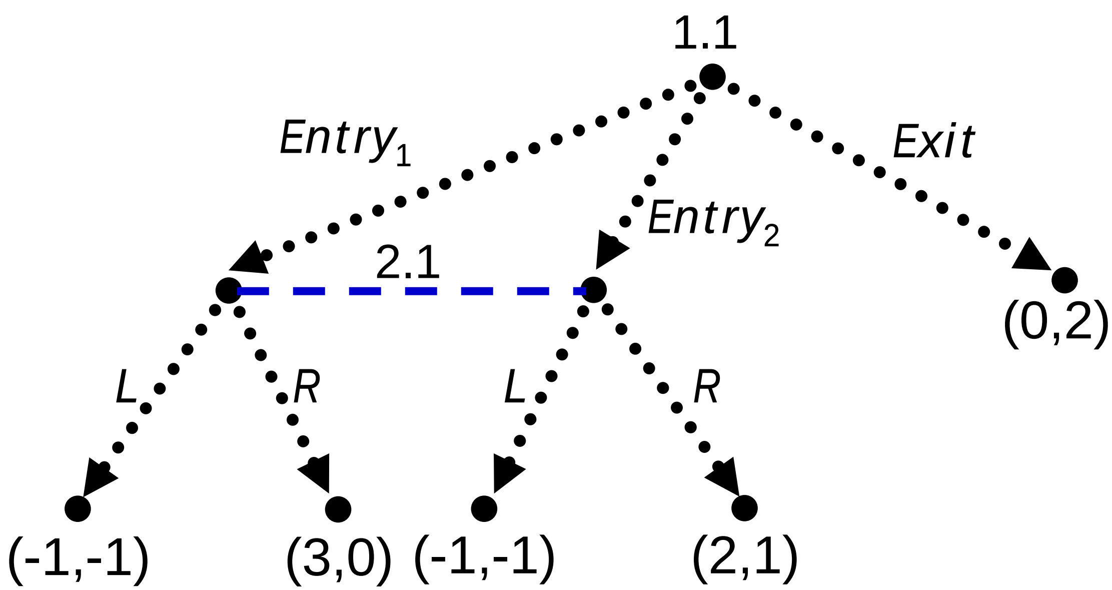
Entry example

But firm 1 should *"know"* that if
it chooses to enter,
firm 2 will never "fight."



Entry example

So in this situation,
there are too many SPNE.



Beliefs

A solution to the problem of the entry game is to include *beliefs* as part of the solution concept:

Firm 2 should never fight, regardless of what it believes firm 1 played.

Beliefs

In general, the beliefs of player i are:

a conditional distribution over everything player i *does not know*,

given everything that player i *does know.*

Beliefs

In general, the beliefs of player i are:

a conditional distribution over the nodes of the information set i is in,

given player i is at that information set.

(When player i is in information set h, denoted by $P_i(v \mid h)$, for $v \in h$)

Beliefs

One example of beliefs:
In static Bayesian games, player i's *belief* is $P(\theta_{-i} \mid \theta_i)$ (where θ_j is type of player j).

But *types* and *information sets* are in 1-to-1 correspondence in Bayesian games, so this matches the new definition.

Perfect Bayesian equilibrium

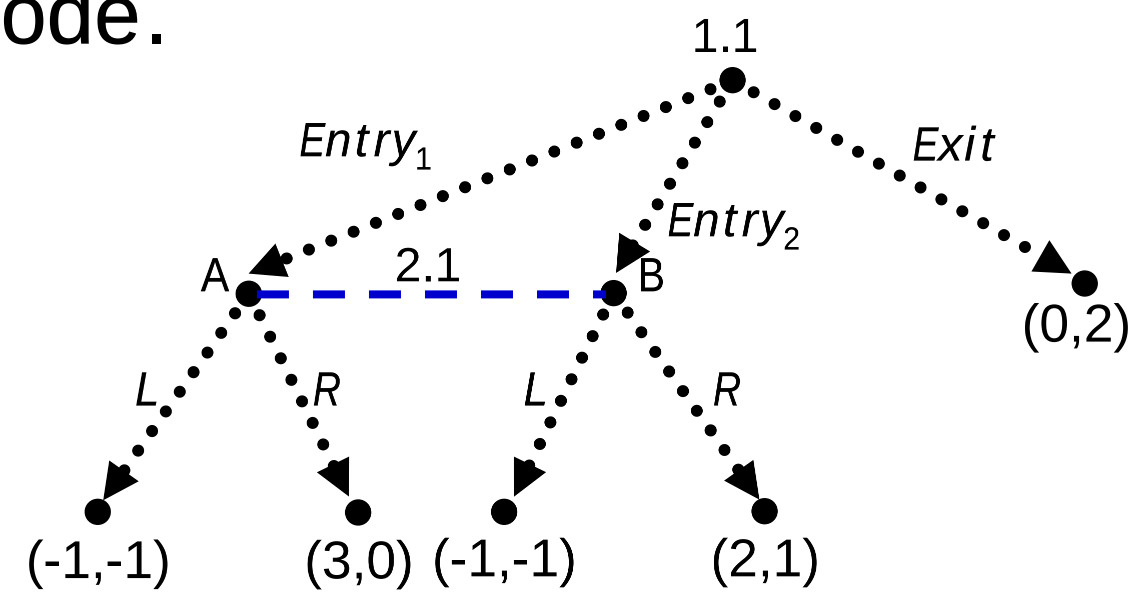
Perfect Bayesian equilibrium (PBE) strengthens subgame perfection by requiring two elements:

- a complete strategy for each player i (mapping from info. sets to mixed actions)
- beliefs for each player i ($P_i(v \mid h)$ for all information sets h of player i)

Entry example

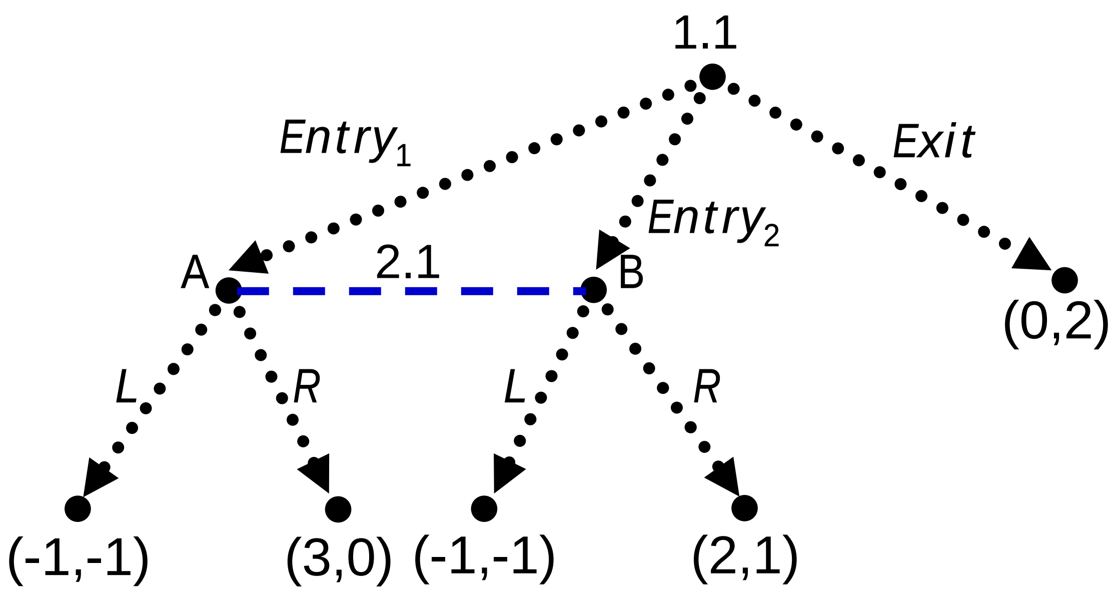
In our entry example, firm 1 has only one information set, containing one node.

His belief just puts probability 1 on this node.



Entry example

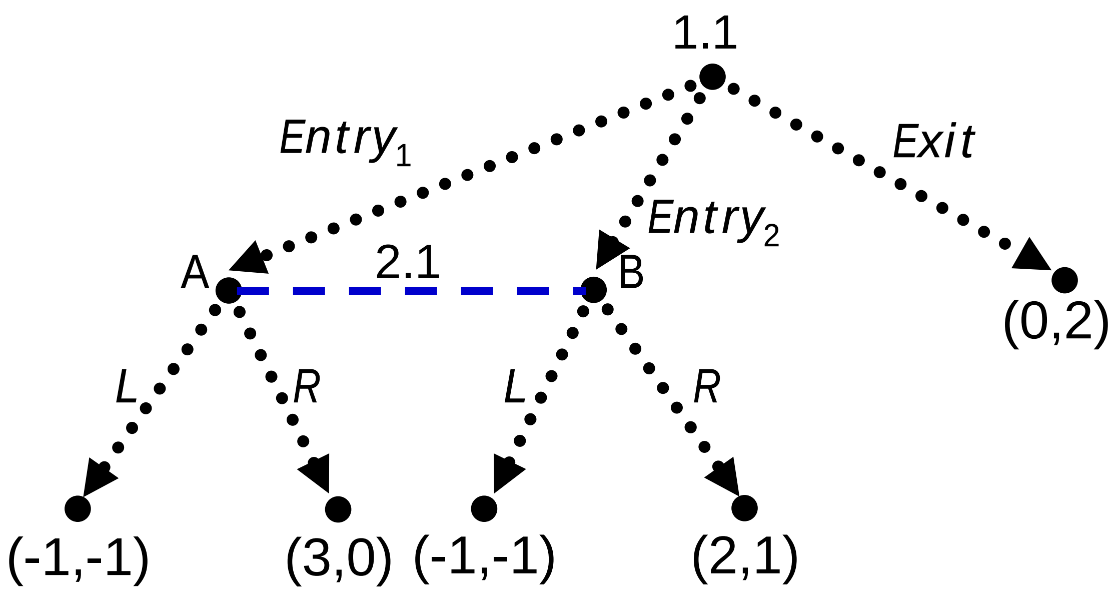
Suppose firm 1 plays a mixed action with probabilities $(p_{\text{Entry}_1}, p_{\text{Entry}_2}, p_{\text{Exit}})$, with $p_{\text{Exit}} < 1$.



Entry example

What are firm 2's *beliefs* in 2.1?

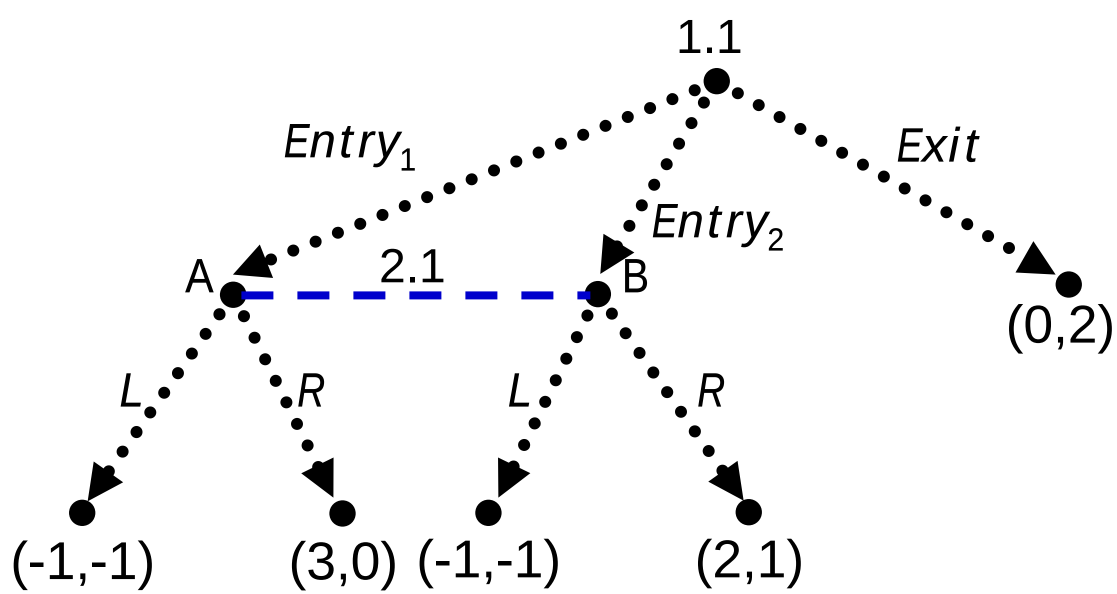
Computed using Bayes' Rule!



Entry example

What are firm 2's *beliefs* in 2.1?

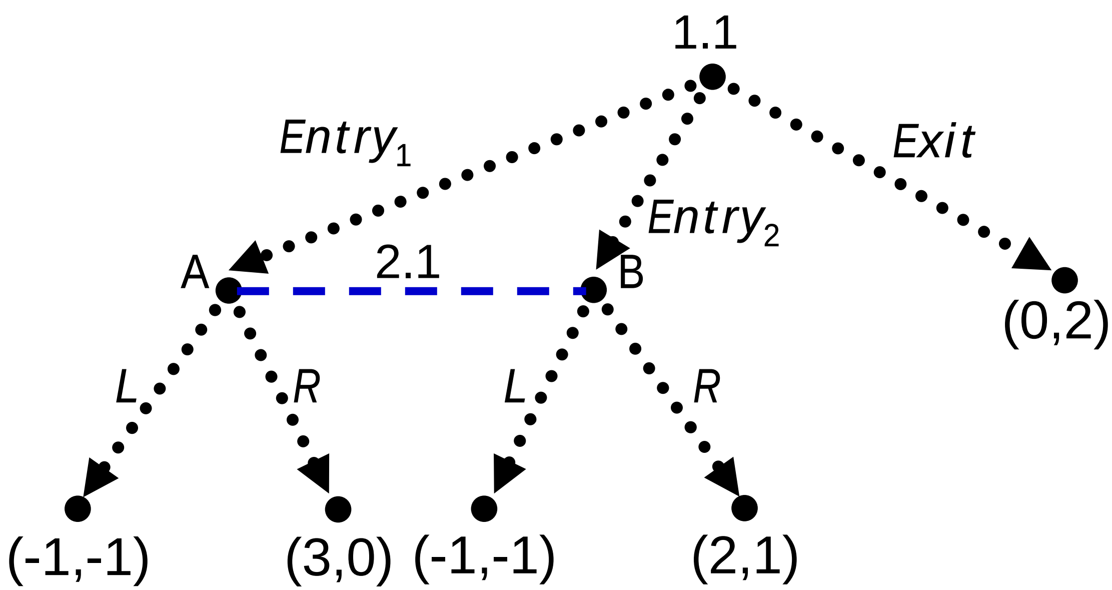
$$P_2(A \mid 2.1) = p_{\text{Entry}_1} / (p_{\text{Entry}_1} + p_{\text{Entry}_2})$$



Entry example

What are firm 2's *beliefs* in 2.1?

$$P_2(B \mid 2.1) = p_{\text{Entry}_2} / (p_{\text{Entry}_1} + p_{\text{Entry}_2})$$



Beliefs

In a perfect Bayesian equilibrium,

"*wherever possible*",

beliefs must be computed
using Bayes' rule and
the strategies of the players.

(At the very least, this ensures information sets that can be reached with positive probability have beliefs assigned using Bayes' rule.)

Rationality

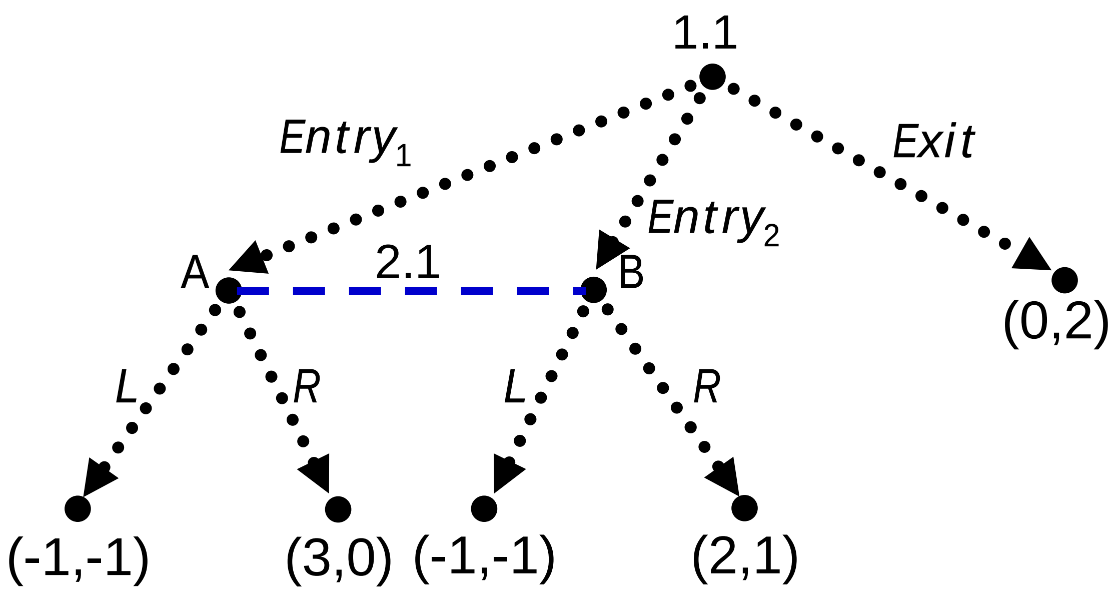
How do player's choose strategies?

As always, they do so to
maximize payoff.

Formally:
Player i's strategy $s_i(\cdot)$ is such that
in any information set h of player i,
$s_i(h)$ maximizes player i's expected payoff,
given his beliefs and others' strategies.

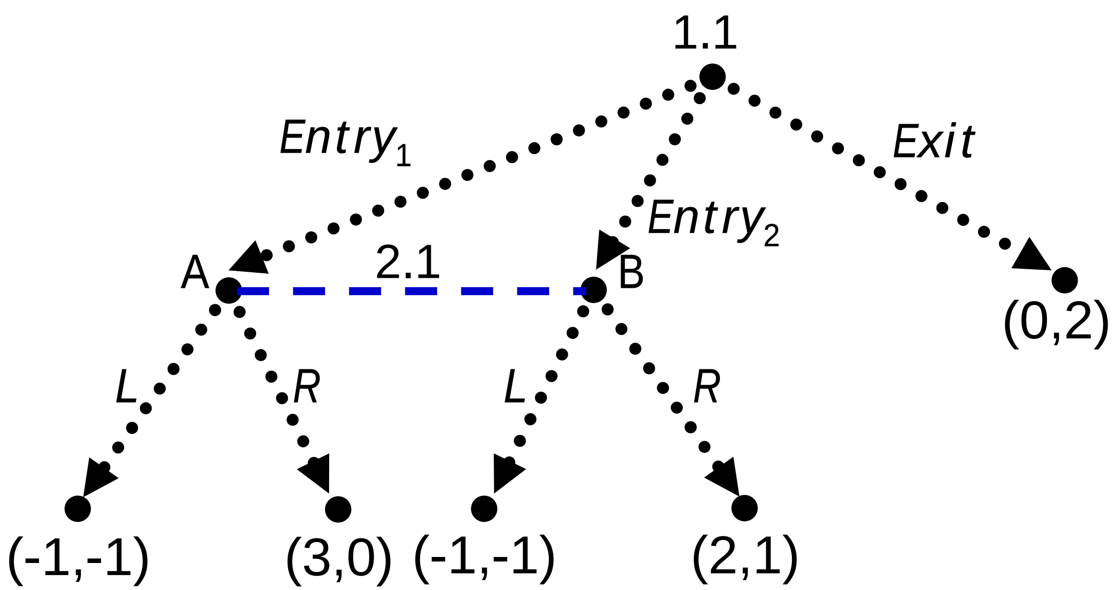
Entry example

For *any* beliefs player 2 has in 2.1,
he maximizes expected payoff by playing *R*.



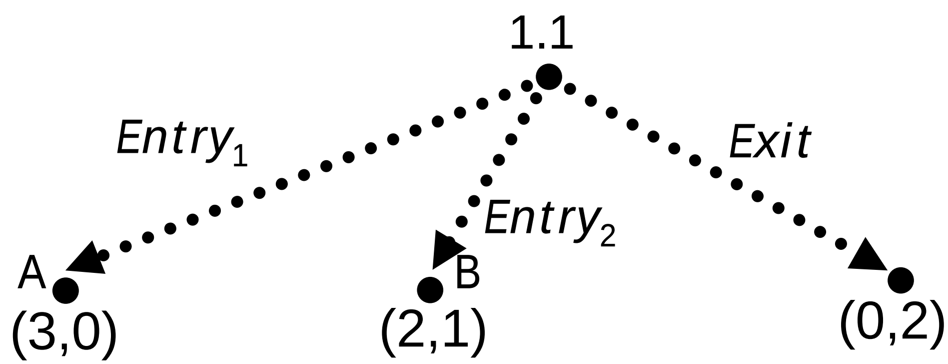
Entry example

Thus, in any PBE, player 2 must play *R* in 2.1.



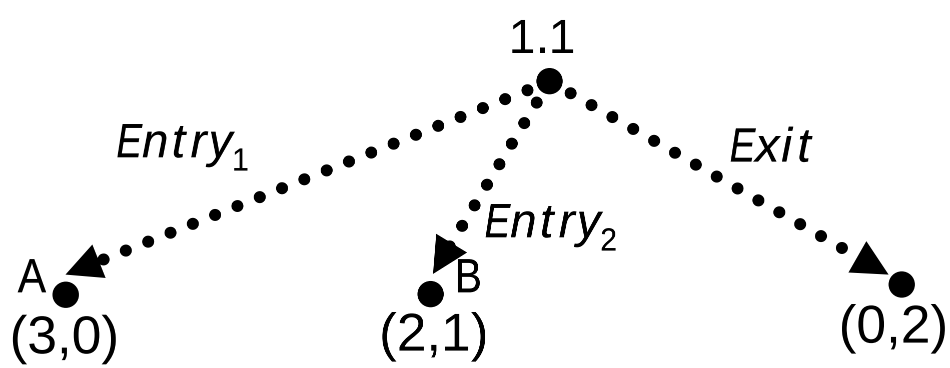
Entry example

Thus, in any PBE, player 2 must play *R* in 2.1.



Entry example

So in a PBE, player 1 will play $Entry_1$ in 1.1.



Entry example

Conclusion: unique PBE is ($Entry\_1$, R).

We have eliminated the NE ($Exit$, L).

PBE vs. SPNE

Note that a PBE is *equivalent to* SPNE for dynamic games of complete and perfect information:

All information sets are singletons, so beliefs are trivial.

In general, PBE is *stronger* than SPNE for dynamic games of complete and imperfect information.

Summary

- Beliefs: conditional distribution at every information set of a player
- Perfect Bayesian equilibrium:
    1. Beliefs computed using Bayes’ rule and strategies (when possible)
    2. Actions maximize expected payoff, given beliefs and strategies

An ante game

- Let t_1, t_2 be uniform[0,1], independent.
- Player i observes t_i;
  each player puts $1 in the pot.
- Player 1 can force a “showdown”,
  or player 1 can “raise” (and add $1 to the pot).
- In case of a showdown, both players show t_i;
  the highest t_i wins the entire pot.
- In case of a raise, Player 2 can “fold” (so player 1 wins) or “match” (and add $1 to the pot).
- If Player 2 matches, there is a showdown.

An ante game

To find the perfect Bayesian equilibria of this game:

Must provide strategies $s_1(\cdot)$, $s_2(\cdot)$; and beliefs $P_1(\cdot \mid \cdot)$, $P_2(\cdot \mid \cdot)$.

An ante game

Information sets of player 1:

t_1 : His *type*.

Information sets of player 2:

(t_2, a_1) : type t_2,
and action a_1 played by player 1.

An ante game

Represent the beliefs by *densities*.

Beliefs of player 1:

$p_1(t_2 \mid t_1) = t_2$ (as types are independent)

Beliefs of player 2:

$p_2(t_1 \mid t_2, a_1)$ = density of player 1's type, *conditional* on having played a_1
= $p_2(t_1 \mid a_1)$ (as types are indep.)

An ante game

Using this representation, can you find a perfect Bayesian equilibrium of the game?