MS&E 246: Lecture 14
Auctions: Examples

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Example: Second price auction

- Action space (bids): $B_i = [0, \infty)$, $i = 1, 2$
- Winner: $w(b_1, b_2) = 1$ if $b_1 > b_2$;
  $= 2$ if $b_1 \leq b_2$
- Payments: $p_i(b_1, b_2) = b_{-i}$ if $w(b_1, b_2) = i$;
  $= 0$ otherwise
- Assume: $\phi_1 = \phi_2 = \phi$, continuous, positive everywhere on its domain, with distribution F

Example: Second price auction

Since $d_i(v_i) = v_i$ is a dominant action for each player, $s_i(v_i) = v_i$ for i = 1, 2 is a BNE.

It is a *symmetric* and *truthtelling* BNE.

So the second price auction is *incentive compatible.*

Example: Second price auction

The truthtelling lemma tells us:

$S_1(v_1) = S_1(0) + \int_0^{v_1} P_1(z)\, dz$

But: $S_1(0) = 0$

$P_1(v_1) = P(v_1 > v_2 \mid v_1) = \int_0^{v_1} \phi(v_2)\, dv_2$

So $S_1(v_1) = \int_0^{v_1} [\int_0^{z} \phi(v_2)\, dv_2]\, dz$

(Similarly for player 2)

Example: Second price auction

The truthtelling lemma tells us:

$S_1(v_1) = S_1(0) + \int_0^{v_1} \mathrm{P}_1(z)\, \mathrm{d}z$

But: $S_1(0) = 0$

$\mathrm{P}_1(v_1) = \mathrm{P}(v_1 > v_2 \mid v_1) = \int_0^{v_1} \phi(v_2)\, \mathrm{d}v_2$

So $S_1(v_1) = \int_0^{v_1} F(z)\, \mathrm{d}z$

(Similarly for player 2)

Example: Second price auction

Expected payoff to player 1, given type v_1

$= S_1(v_1) = \int_0^{v_1} F(z)\,\mathrm{d}z$

(Similarly for player 2)

Example: Second price auction

Are there any *other* symmetric BNE of the second price auction?

Suppose $s_1 = s_2 = s$ is such a BNE.

Example: Second price auction

By symmetric BNE theorem,

s is strictly increasing, and

player 1 wins if and only if $v_1 > v_2$

Expected payoff to player 1 of type v_1:

$\underline{S}_1(v_1) = \int_0^{v_1} (v_1 - s(v_2))\, \phi(v_2)\, dv_2$

Example: Second price auction

Observe that:

- $\underline{S}_i(0) = 0$ for $i = 1, 2$
- Highest valuation player always wins

So by payoff equivalence theorem:

$$S_1(v_1) = \underline{S}_1(v_1)$$

Example: Second price auction

By payoff equivalence:

$$S_1(v_1) = \underline{S}_1(v_1)$$

Example: Second price auction

By payoff equivalence:

$$\int_0^{v_1} F(z)\,\mathrm{d}z = \int_0^{v_1} (v_1 - s(v_2))\,\phi(v_2)\,\mathrm{d}v_2$$

Example: Second price auction

By payoff equivalence:

$$\int_0^{v_1} F(z)\, \mathrm{d}z = v_1 F(v_1) - \int_0^{v_1} s(v_2))\, \phi(v_2)\, \mathrm{d}v_2$$

Differentiate:

$$F(v_1) = F(v_1) + v_1\, \phi(v_1) - s(v_1)\, \phi(v_1)$$

Example: Second price auction

By payoff equivalence:

$$\int_0^{v_1} F(z)\,\mathrm{d}z = v_1 F(v_1) - \int_0^{v_1} s(v_2))\,\phi(v_2)\,\mathrm{d}v_2$$

Differentiate:

$$0 = v_1\,\phi(v_1) - s(v_1)\,\phi(v_1)$$

Example: Second price auction

By payoff equivalence:

$$\int_0^{v_1} F(z)\, \mathrm{d}z = v_1 F(v_1) - \int_0^{v_1} s(v_2))\, \phi(v_2)\, \mathrm{d}v_2$$

Differentiate:

$$0 = (\ v_1 - s(v_1)\ )\ \phi(v_1)$$

So: $v_1 = s(v_1)$

Example: Second price auction

Conclude: truthtelling is unique symmetric BNE.

Expected revenue to auctioneer:

$\mathbf{E}$[second highest valuation]

Example: First price auction

- Action space (bids): $B_i = [0, \infty)$, $i = 1, 2$
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  $= 0$ otherwise
- Assume: $\phi_1 = \phi_2 = \phi$, continuous, positive everywhere on its domain, with distribution F

Example: First price auction

What are the symmetric BNE of the first price auction?

Suppose $s_1 = s_2 = s$ is a symmetric BNE.

Example: First price auction

By symmetric BNE theorem,

s is strictly increasing, and

player 1 wins if and only if $v_1 > v_2$

Expected payoff to player 1 of type v_1:

$$S_1^{FP}(v_1) = \int_0^{v_1} (v_1 - s(v_1))\,\phi(v_2)\,dv_2$$
$$= (v_1 - s(v_1))\,F(v_1)$$

Example: First price auction

Observe that:

- $S_i(0) = 0$ for $i = 1, 2$
- Highest valuation player always wins

So by payoff equivalence theorem,

$S_1^{FP}(v_1)$ = expected payoff to type v_1
player in second price auction

Example: First price auction

By payoff equivalence:

$$S_1^{\mathrm{FP}}(v_1) = \int_0^{v_1} F(z)\,\mathrm{d}z$$

Example: First price auction

By payoff equivalence:

$$(v_1 - s(v_1))\, F(v_1) = \int_0^{v_1} F(z)\, \mathrm{d}z$$

So:

$$s(v) = v - \frac{\int_0^v F(z) dz}{F(v)}$$

(e.g., when Φ is uniform: $s(v) = v/2$)

Example: First price auction

$$s(v) = v - \frac{\int_0^v F(z)dz}{F(v)}$$

Observe that $s(v) < v$.

This practice is called *bid shading*.

Example: First price auction

Revenue equivalence also holds, so:

Expected revenue to auctioneer =

expected revenue under
second price auction =

$\mathbf{E}$[second highest valuation]

< $\mathbf{E}[\max\{ v_1, v_2 \}]$

Revenue

The shortfall between $\mathbf{E}[\ \max\{v_1, v_2\}\ ]$ and expected revenue is called an *information rent:*

At an equilibrium the buyers must make a profit if they reveal their private valuation.

Revenue

However, this relies on *independent private valuations.*

If valuations are correlated, the auctioneer can get expected revenue = $\mathbf{E}[\max\{v_1, v_2\}]$

See Problem Set 6.

(Theorem: Cremer and McLean, 1985)