MS&E 246: Lecture 13
Auctions: Imperfect information

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Auctions: Theory

- Basic definitions
- Revelation principle
- Truthtelling lemma
- Payoff equivalence theorem
- Revenue equivalence theorem
- Symmetric BNE
- Examples next lecture

A basic auction model

- Assume two players want the same item

- Type of player i : valuation $v_i \geq 0$

  Assume: P($v_i \leq x_i$) = $\Phi_i(x_i)$

  F_i : continuous dist. on [0,V], with pdf ϕ_i

  e.g. uniform: $\phi_i(x_i)$ = 1/V, for $x_i \in$ [0,V]

A basic auction model

Payoffs depend on *winning* and *payment*

- Let w = i if player i wins
- Let p_i = payment of player i
- Payoff to player i of type v_i:

$$\Pi_i(w, p_i; v_i) = \begin{cases} v_i - p_i, & \text{if } w = i \\ 0, & \text{otherwise} \end{cases}$$

A basic auction model

An *auction mechanism* is:

- action set for each player, B_i
- mapping from actions to:

  winner: $w(b_1, b_2) \in \{ 1, 2 \}$

  payments: $p_i(b_1, b_2) \in [0, \infty), \quad i = 1, 2$

Payoff to i: $Q_i(b_1, b_2 ; v_i) =$

$$\Pi_i(w(\mathbf{b}), p_i(\mathbf{b}) ; v_i)$$

Example: Second price auction

- Action space (bids): $B_i = [0, \infty)$, $i = 1, 2$

- Winner: $w(b_1, b_2) = 1$ if $b_1 > b_2$;
  $= 2$ if $b_1 \leq b_2$

- Payments: $p_i(b_1, b_2) = b_{-i}$ if $w(b_1, b_2) = i$;
  $= 0$ otherwise

Bayes-Nash equilibrium

Strategy of i : $s_i : [0,\infty) \rightarrow B_i$

s_1 is a *Bayesian best response* to s_2 if:

$$\int_0^\infty Q_1(s_1(v_1), s_2(v_2); v_1)\ \phi_2(v_2)\ dv_2 \geq \int_0^\infty Q_1(b_1, s_2(v_2); v_1)\ \phi_2(v_2)\ dv_2$$

for all $b_1 \in B_1$, and $v_1 \geq 0$

(similar definition for player 2)

Bayes-Nash equilibrium

Strategy of i : s_i : $[0,\infty) \rightarrow B_i$

s_1 is a *Bayesian best response* to s_2 if:

$$\mathbf{E}[Q_1(s_1(v_1), s_2(v_2); v_1) | v_1] \geq \mathbf{E}_{v_2}[Q_1(b_1, s_2(v_2); v_1) | v_1]$$

for all $b_1 \in B_1$, and $v_1 \geq 0$

(Similarly for player 2)

Bayes-Nash equilibrium

(s_1, s_2) is a BNE if:

- s_1 is a Bayesian best response to s_2
- s_2 is a Bayesian best response to s_1

Second price auction and BNE

We start by finding a BNE for the second price auction.

Recall: Given type v_i, *truthtelling* is a *weak dominant action* for i:

$$d_i(v_i) = v_i$$

Dominant actions and BNE

Consider *any* Bayesian game with type spaces T_1, T_2.

Suppose for each type t_i, player i has a *(weakly) dominant action* $d_i(t_i)$:

$$Q_i(d_i(t_i), a_{-i} ; t_i) \geq Q_i(a_i, a_{-i} ; t_i)$$

for any other action a_i

Dominant actions and BNE

Then $(d_1(\cdot),\ d_2(\cdot))$ is a BNE.

We know that for each player i:

$$Q_i(d_i(t_i),\ d_{-i}(t_{-i})\ ;\ t_i)$$
$$\geq Q_i(a_i,\ d_{-i}(t_{-i})\ ;\ t_i)$$

for all types t_i, t_{-i}, and actions a_i.

Dominant actions and BNE

Then $(d_1(\cdot), d_2(\cdot))$ is a BNE.

Take expectations:

$$E[\ Q_i(d_i(t_i), d_{-i}(t_{-i})\ ;\ t_i)\ |\ t_i\ ]$$
$$\geq E[\ Q_i(a_i, d_{-i}(t_{-i})\ ;\ t_i)\ |\ t_i\ ]$$

for all types t_i, and actions a_i.

This is exactly the condition for a BNE.

Second price auction and BNE

Conclusion:

In the second price auction, *truthtelling is a BNE :*

$$s_i(v_i) = d_i(v_i) = v_i$$

(Note that this requires a dominant action for *every* possible type!)

Incentive compatibility

Auctions where *truthtelling is a BNE*, i.e., where:

1. B_i = [0, ∞) for i = 1, 2, and
2. $s_i(v_i) = v_i$ for i = 1, 2 is a BNE

are called *incentive compatible.*

Revelation principle

The *revelation principle* shows how to create an incentive compatible auction from any auction with a BNE.

The revelation principle

Given: B_1, B_2, $w(\cdot)$, $p_1(\cdot)$, $p_2(\cdot)$
and a BNE $s_1(\cdot)$, $s_2(\cdot)$

Create a new auction with:

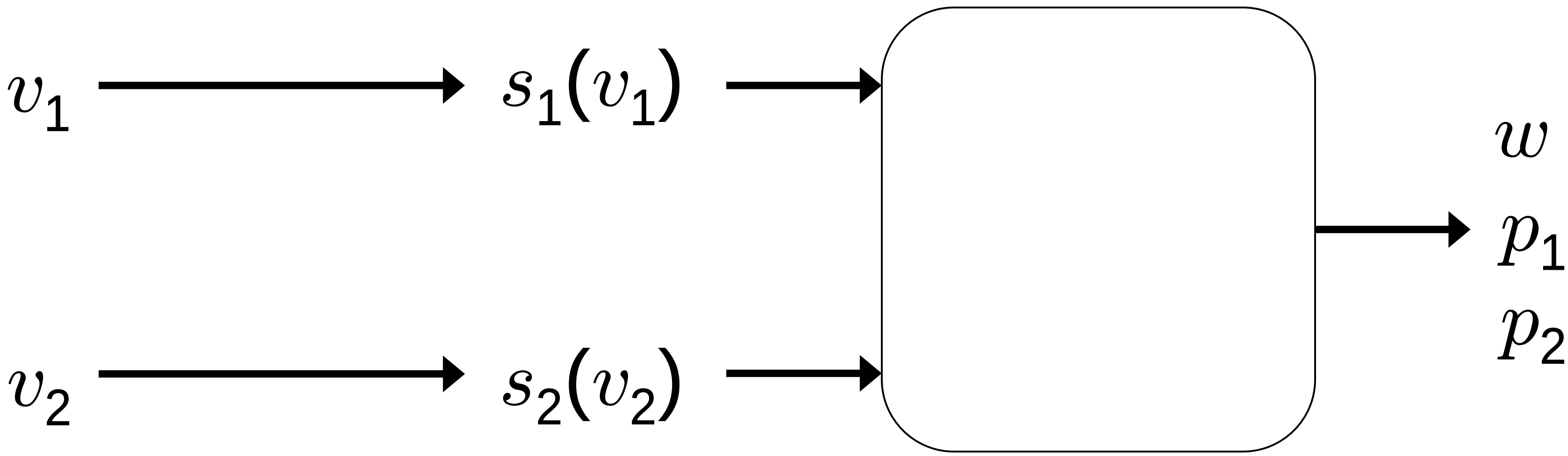
$\underline{B}_1 = \underline{B}_2 = [0, \infty)$

$\underline{w}(\underline{b}_1, \underline{b}_2) = w(s_1(\underline{b}_1), s_2(\underline{b}_2))$

$\underline{p}_i(\underline{b}_1, \underline{b}_2) = p_i(s_1(\underline{b}_1), s_2(\underline{b}_2)), \quad i = 1, 2$

The revelation principle

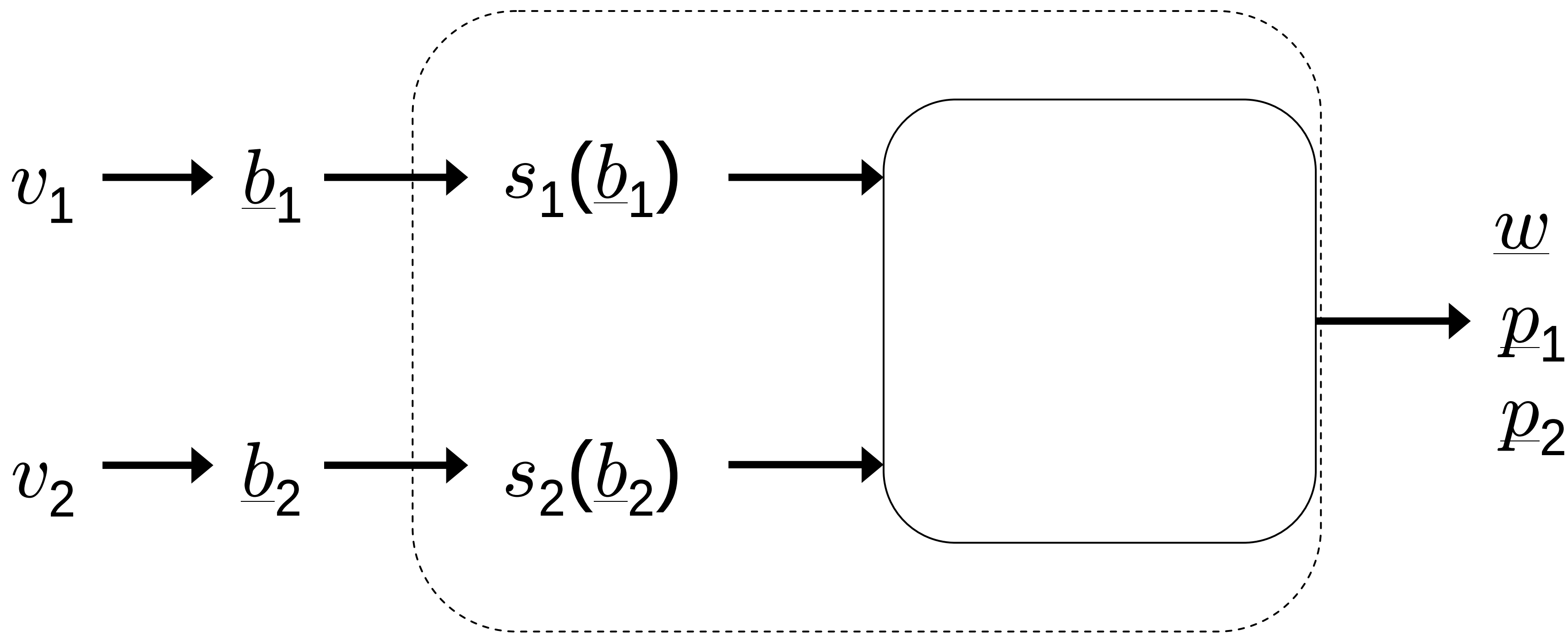
At the BNE of the original auction:



The revelation principle

In the new auction:

Ask players to *declare valuation*.



The revelation principle

Theorem:

The new auction is incentive compatible.

Further, the truthtelling strategies
in the new auction give exactly
the same outcomes as
the BNE of the original auction.

The revelation principle: Proof

For all v_1,

$$E[\, Q_1(v_1, v_2 \,;\, v_1) \mid v_1 \,]$$

$$= \; E[\, Q_1(s_1(v_1), s_2(v_2) \,;\, v_1) \mid v_1 \,]$$

$$\geq \; E[\, Q_1(b_1, s_2(v_2) \,;\, v_1) \mid v_1 \,]$$

for all $b_1 \in B_1$

The revelation principle: Proof

For all v_1,

$$E[\, \underline{Q}_1(v_1, v_2 \,;\, v_1) \mid v_1 \,]$$

$$= \quad E[\, Q_1(s_1(v_1), s_2(v_2) \,;\, v_1) \mid v_1 \,]$$

$$\geq \quad E[\, Q_1(s_1(\underline{b}_1), s_2(v_2) \,;\, v_1) \mid v_1 \,]$$

for all $\underline{b}_1 \in [0, \infty)$

The revelation principle: Proof

For all v_1,

$$E[\, \underline{Q}_1(v_1, v_2 \,;\, v_1) \mid v_1 \,]$$

$$= \quad E[\, Q_1(s_1(v_1), s_2(v_2) \,;\, v_1) \mid v_1 \,]$$

$$\geq \quad E[\, \underline{Q}_1(\underline{b}_1, v_2 \,;\, v_1) \mid v_1 \,]$$

for all $\underline{b}_1 \in [0, \infty)$

(Similarly for player 2)

The revelation principle: Proof

Given v_1, v_2 :

Outcome at truthtelling strategies

= $(\underline{w}(v_1, v_2), \underline{p}_1(v_1, v_2), \underline{p}_2(v_1, v_2))$

= $(w(s_1(v_1), s_2(v_2)),$
$p_1(s_1(v_1), s_2(v_2)),$
$p_2(s_1(v_1), s_2(v_2)))$

= outcome at original BNE

The revelation principle

The new auction is called a

direct revelation mechanism (DRM).

Note:

It may have other, undesirable equilibria!!

Two useful results

For a wide range of auctions (including first and second price),

we will show
payoff equivalence and
revenue equivalence:

These auctions all have the *same* payoffs and auctioneer revenue at BNE.

Definitions

Suppose we are given a DRM.

Truthtelling expected payoff to player 1:

$S_1(v_1) = \mathrm{E}[\ Q_1(\ v_1,\ v_2\ ;\ v_1)\ |\ v_1\ ]$

Truthtelling expected probability of winning for player 1:

$P_1(v_1) = \int_0^\infty I\{\ w(v_1,\ v_2) = 1\}\ \phi_2(v_2)\ dv_2$

The truthtelling lemma

Lemma: Truthtelling is a BNE if and only if for i = 1, 2:

(1) $S_i(v_i) = S_i(0) + \int_0^{v_i} \mathrm{P}_i(z)\, \mathrm{d}z$

(2) P_i is nondecreasing:

$$v_i \geq v_i' \Rightarrow \mathrm{P}_i(v_i) \geq \mathrm{P}_i(v_i')$$

The truthtelling lemma: Proof

Truthtelling is a BNE if and only if:

$$S_1(v_1) \geq \text{E}[\ Q_1(\ v_1',\ v_2\ ;\ v_1)\ |\ v_1\ ] \quad \text{for all } v_1' \geq 0$$

(Similarly for player 2)

The truthtelling lemma: Proof

Truthtelling is a BNE if and only if:

$S_1(v_1) \geq$ E[Q_1 (v_1', v_2 ; v_1) | v_1]
for all $v_1' \geq 0$

E[Q_1 (v_1', v_2 ; v_1) | v_1] =

$\int_0^\infty [v_1 - p_1(v_1', v_2)] \, I\{ w(v_1', v_2) = 1\} \, \phi_2(v_2) \, dv_2$

The truthtelling lemma: Proof

Truthtelling is a BNE if and only if:

$S_1(v_1) \geq$ E[Q_1 (v_1', v_2 ; v_1) | v_1]
for all $v_1' \geq 0$

E[Q_1 (v_1', v_2 ; v_1) | v_1] =

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The truthtelling lemma: Proof

Truthtelling is a BNE if and only if:

$S_1(v_1) \geq$ E[Q_1 (v_1', v_2 ; v_1) | v_1]

for all $v_1' \geq 0$

E[Q_1 (v_1', v_2 ; v_1) | v_1] =

v_1 P$_1$ (v_1') -

$\int_0^\infty [p_1(v_1', v_2)]\, I\{ w(v_1', v_2) = 1\}\, \phi_2(v_2)\, dv_2$

The truthtelling lemma: Proof

Truthtelling is a BNE if and only if:

$S_1(v_1) \geq$ E[Q_1 (v_1', v_2 ; v_1) | v_1]

for all $v_1' \geq 0$

E[Q_1 (v_1', v_2 ; v_1) | v_1] =

v_1 P$_1$ (v_1') - v_1' P$_1$ (v_1')

$\int_0^\infty [v_1' - p_1(v_1', v_2)]\, I\{ w(v_1', v_2) = 1\}\, \phi_2(v_2)\, dv_2$

The truthtelling lemma: Proof

Truthtelling is a BNE if and only if:

$S_1(v_1) \geq$ E[Q_1 (v_1', v_2 ; v_1) | v_1]
for all $v_1' \geq 0$

E[Q_1 (v_1', v_2 ; v_1) | v_1] =

v_1 P_1 (v_1') - v_1' P_1 (v_1') + S_1 (v_1')

The truthtelling lemma: Proof

Conclude:

Truthtelling is a BNE if and only if for i = 1, 2, and for all v_i , v_i' $\geq$ 0:

$S_i(v_i) \geq S_i(v_i') + P_i(v_i')(v_i - v_i')$

i.e., S_i is *convex*.

The truthtelling lemma: Proof

Assume v_i' > v_i. Then:

$S_i(v_i) \geq S_i(v_i') + P_i(v_i')(v_i - v_i')$

$S_i(v_i') \geq S_i(v_i) + P_i(v_i)(v_i' - v_i)$

$\Rightarrow P_i(v_i)(v_i' - v_i) \leq P_i(v_i')(v_i' - v_i)$

So $P_i(v_i) \leq P_i(v_i') \Rightarrow P_i$ is *nondecreasing*

The truthtelling lemma: Proof

If $v_i > v_i'$:

$$\frac{S_i(v_i) - S_i(v_i')}{v_i - v_i'} \geq \mathsf{P}_i(v_i')$$

If $v_i < v_i'$:

$$\frac{S_i(v_i) - S_i(v_i')}{v_i - v_i'} \leq \mathsf{P}_i(v_i')$$

Take $v_i' \uparrow v_i$, $v_i' \downarrow v_i$ $\Rightarrow$ $S_i'(v_i) = \mathsf{P}_i(v_i)$

The truthtelling lemma

How to use the truthtelling lemma:

(1) Use a BNE of an auction
to create an incentive compatible DRM

(2) Apply the truthtelling lemma
to characterize the original BNE

Payoff equivalence

Given two auctions with BNE such that:

-in each BNE, if v_i = 0 then player i gets zero payoff; and

-in each BNE, item *always* goes to highest valuation player

Theorem: Both BNE yield the same *expected payoff* to each player.

Payoff equivalence: Proof

- Fix given BNE (s_1, s_2) of one of the auctions
- Construct incentive compatible DRM using revelation principle
- For this DRM:

  $\underline{S}_i(0) = 0$, and

  $\underline{P}_i(v_i) = \int_0^{v_i} \phi_{-i}(v_{-i}) \, dv_{-i}$

Payoff equivalence: Proof

Expected payoff to player 1 of type v_1:

depends only on $S_1(0)$ and $P_1(\cdot)$,

by the truthtelling lemma

(Similarly for player 2)

Payoff equivalence: Proof

Expected payoff to player 1 of type v_1:

E[$Q_1(s_1(v_1), s_2(v_2) ; v_1) \mid v_1$]

= E[$\underline{Q}_1 (v_1, v_2 ; v_1) \mid v_1$]

= $\underline{S}_1 (v_1) = \int_0^{v_1} [\int_0^{v_1'} \phi_2(v_2) \, dv_2] \, dv_1'$

by the truthtelling lemma

(Similarly for player 2)

Payoff equivalence: Proof

So at given BNE of *either auction*,
expected payoff to player 1 of type v_1 is:

$$\int_0^{v_1} \left[\int_0^{v_1'} \phi_2(v_2)\, dv_2 \right] dv_1'$$

This does not depend on the BNE!

(Similarly for player 2)

Revenue equivalence

Given two auctions with BNE such that:

-in each BNE, if v_i = 0 then player i gets zero payoff; and

-in each BNE, item *always* goes to highest valuation player

Theorem: Both BNE yield the same *expected revenue* to the auctioneer.

Revenue equivalence: Proof

Fix BNE (s_1, s_2) of one of the auctions

Note:

$$\sum^{2}_{i=1} Q_i\,(s_1(v_1),\ s_2(v_2)\ ;\ v_i)$$

$$= v_{w(s_1(v_1),\ s_2(v_2))} - p_{w(s_1(v_1),\ s_2(v_2))}$$

Revenue equivalence: Proof

Taking expectations:

Sum of expected payoffs to players

$= \mathbf{E}$ [max $\{v_1, v_2\}$] -

Expected revenue to auctioneer

Revenue equivalence: Proof

Taking expected values:

Expected revenue to auctioneer

= E[max $\{v_1, v_2\}$] -

Sum of expected payoffs to players

Revenue equivalence: Proof

Taking expected values:

Expected revenue to auctioneer

= E[max $\{v_1, v_2\}$] -

Sum of expected payoffs to players

Right hand side is same for BNE of both auctions (by payoff equivalence)

Revenue equivalence

Note that at BNE,
expected payoffs to players are ≥ 0.

So:

Revenue to auctioneer $\leq$ E[max $\{v_1, v_2\}$]

[Aside: optimal auction theory]

The problem of maximizing the equilibrium revenue to the auctioneer is called *optimal auction design.*

For this problem to be well-defined, an additional constraint is needed, *individual rationality*:

E[$Q_i(s_i(v_i)$, $\mathbf{s}_{\text{-}i}(\mathbf{v}_{\text{-}i})$; $v_i)$ | v_i] $\geq$ 0 for all i.

(Otherwise bidder i would not participate.)

[Aside: optimal auction theory]

The framework defined here can be used to characterize the optimal auction design for any distribution of players' valuations.

(See Myerson 1979)

Symmetric BNE

From now on, assume:

- $B_1 = B_2 = [0, \infty)$
- $\phi_1 = \phi_2 = \phi$ *(same distribution)*

  (Assume ϕ is positive on its entire domain)

A BNE is *symmetric* if:

$s_1(v) = s_2(v)$ for all $v \geq 0$

$s(v) = s_i(v)$ is called the *bid function*.

Symmetric BNE theorem

Theorem:

If highest bidder wins,

then in a symmetric BNE with bid function s,

s is strictly increasing,

so the winning bidder also has *the highest valuation*.

(In case of tie, assume player 2 wins)

Symmetric BNE: Proof

Apply revelation principle to build new incentive compatible auction:

$\underline{w}(\underline{b}_1, \underline{b}_2) = w(s(\underline{b}_1), s(\underline{b}_2))$

$\underline{p}_i(\underline{b}_1, \underline{b}_2) = p_i(s(\underline{b}_1), s(\underline{b}_2)), \quad i = 1, 2$

In this auction:

$\underline{P}_1(v_1) = \int_0^\infty I\{\, \underline{w}(v_1, v_2) = 1\}\, \phi(v_2)\, dv_2$

Symmetric BNE: Proof

Apply revelation principle to build new incentive compatible auction:

$\underline{w}(\underline{b}_1, \underline{b}_2) = w(s(\underline{b}_1), s(\underline{b}_2))$

$\underline{p}_i(\underline{b}_1, \underline{b}_2) = p_i(s(\underline{b}_1), s(\underline{b}_2)), \quad i = 1, 2$

In this auction:

$\underline{P}_1(v_1) = \int_0^\infty I\{ w(s(v_1), s(v_2)) = 1\} \phi(v_2) \, dv_2$

Symmetric BNE: Proof

Apply revelation principle to build new incentive compatible auction:

$\underline{w}(\underline{b}_1, \underline{b}_2) = w(s(\underline{b}_1), s(\underline{b}_2))$

$\underline{p}_i(\underline{b}_1, \underline{b}_2) = p_i(s(\underline{b}_1), s(\underline{b}_2)), \quad i = 1, 2$

In this auction:

$\underline{P}_1(v_1) = \int_0^\infty I\{ s(v_1) > s(v_2) \} \phi(v_2)\, dv_2$

Symmetric BNE: Proof

By truthtelling lemma,

$$\int_0^\infty I\{\ s(v_1) > s(v_2)\ \}\ \phi(v_2)\ \mathrm{d}v_2$$

is nondecreasing in v_1.

Only possible if $s(v)$ is nondecreasing in v.

We only need to show s is *strictly increasing.*

Symmetric BNE: Proof

We will show s is *strictly increasing* in the special case of the *first price auction:*

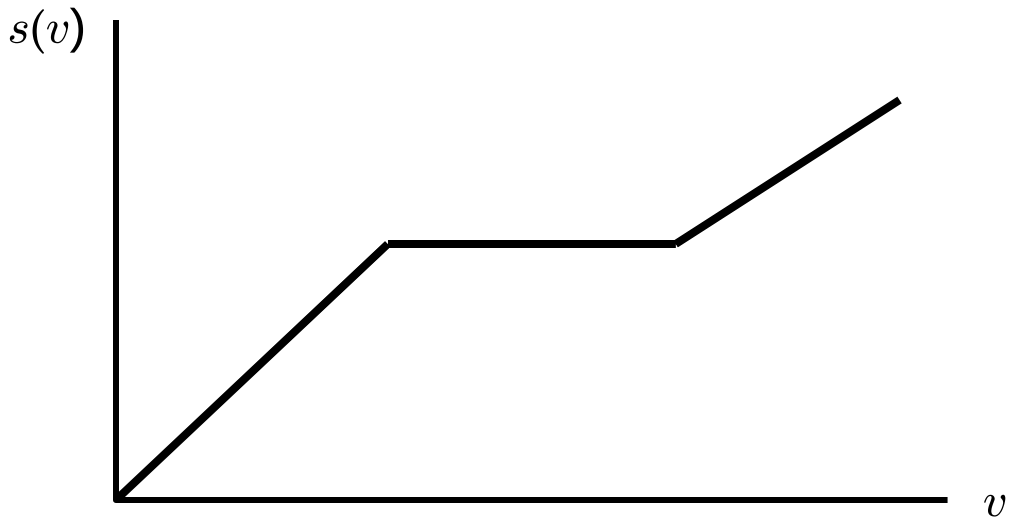
$p_i(b_1, b_2) = b_i$ if $w(b_1, b_2) = i$;

$= 0$ otherwise

However, the result holds more generally for the other auctions we consider.

Symmetric BNE: Proof

Suppose s is not strictly increasing:



Symmetric BNE: Proof

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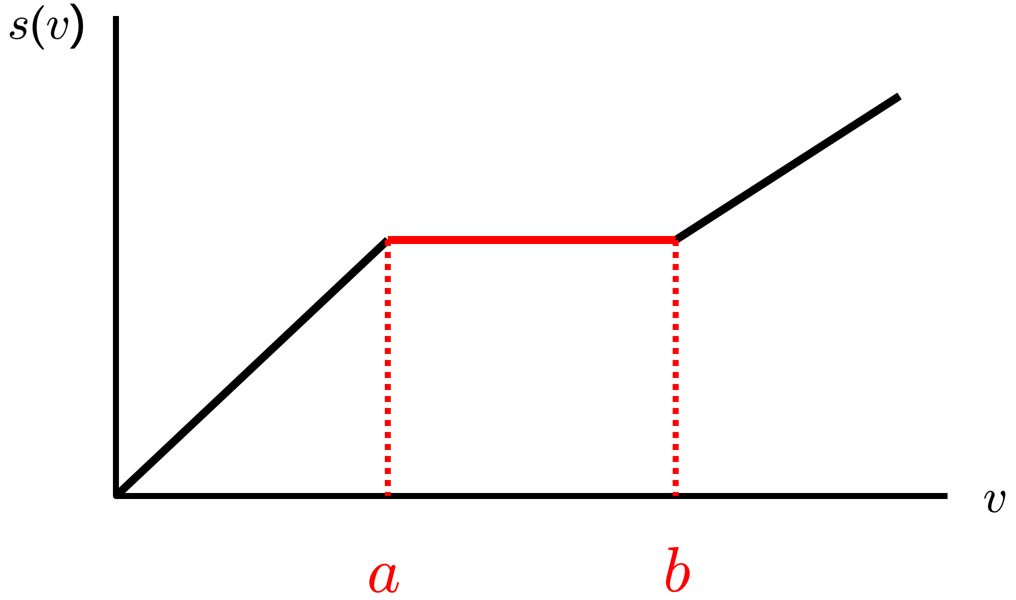

Symmetric BNE: Proof

Suppose s is not strictly increasing.

Fix $a < b$ such that:

$$s(v) = s(a), \quad a \leq v \leq b$$

We can assume: $a > s(a)$.

(If not, just increase a slightly.)

Symmetric BNE: Proof

Given player 2 is using $s_2 = s$,
suppose player 1 bids

$$b_1 = s(a) + \varepsilon \text{ when } v_1 = a.$$

Then when $v_1 = a$:

- Expected *payment* by player 1 increases by at most ε
- Player 1 wins if $v_2 \in [a,b]$

Symmetric BNE: Proof

Given player 2 is using $s_2 = s$,

suppose player 1 bids

$$b_1 = s(a) + \varepsilon \text{ when } v_1 = a.$$

Player 1's change in expected payoff

$$\geq (a - s(a) - \varepsilon)(F(b) - F(a)) - \varepsilon$$

$$> 0 \quad \text{for small enough } \varepsilon$$

Profitable deviation!

Symmetric BNE: Proof

Conclude:

s is strictly increasing

So:

Winner must have highest valuation

Symmetric BNE

In general, can show the same result if:

(1) $p_i(b_1, b_2) \geq 0$ for all b_1, b_2;

(2) the winner's payment is positive when at least one of b_1, b_2 is positive; and

(3) $p_1(b_1, b_2) = p_2(b_2, b_1)$ for all b_1, b_2 *(permutation invariance)*

Moral

Symmetric BNE of "standard auctions"

(first price, second price, etc.)

have the *same* expected payoffs and auctioneer revenue.

In particular,

Expected revenue = E[second highest bid]

(Why?)