Distributed Max-flow algorithm

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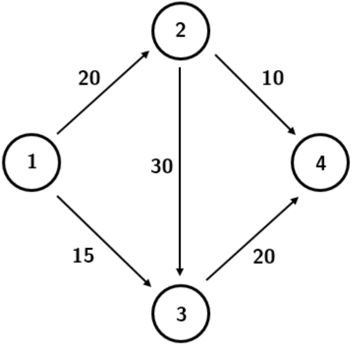
Overview

1. Edmonds-Karp algorithm for max-flow
   - Single Machine Algorithm
   - Distributed Algorithm
   - Details
2. Analysis
   - Communication cost
   - Runtime

Edmonds-Karp algorithm for max-flow

- We increment the flow from s to t by finding a flow-augmenting path
- We do this by finding a path in the *residual graph*
- The total flow is increased by the maximum capacity found on our path
- Maximal flow is found when there are no more flow-augmenting paths
- Note that we can lower the flow on a particular edge to favor another path

Residual graph toy example



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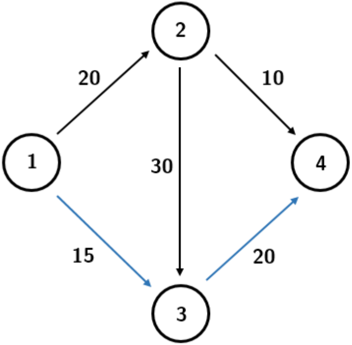

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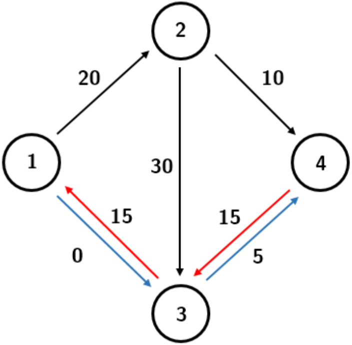

Assumptions and Methods

- n vertices: can fit on a single machine
- m edges: too large to fit
- Integer edge capacities
- Use Pregel and MapReduces to distribute

Distributed max-flow

Initialization:

- Set flows in all edges to 0
- Set residual graph R_G equal to initial graph

While there is a path from s to t in R_G:

- Find the shortest path P between s and t in R_G
- Find max flow f_{max} you can push along P
- Broadcast P
- Update flows
- Update R_G using P and f_{max}

Data structures

- We use the graph object provided by *GraphX* to build the residual graph
- Edges and flows are stored in a RDD which will be updated at each iteration (each time we find a path)
- The path found in the residual graph is stored in an array of size $\mathcal{O}(n)$ that will be *broadcasted*

Finding the shortest path in Pregel

- Vertex attribute: (d, c, id)
- d: distance from source s
- c: minimum capacity the node has seen so far
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- If two paths have the same length, we choose the one with maximum capacity (flow)

Finding the shortest path in Pregel

Communication cost

Because the state of a node is changed once at most, there will be at most one message sent per edge: $\mathcal{O}(m)$.

Runtime

Initializing vertices: $\mathcal{O}(n)$.
Pregel: #messages/#machines, i.e. $\mathcal{O}(\frac{m}{k})$.

Updating the residual graph

Algorithm 1 Updating the residual graph R_G

Each key value pair is of the form $((i,j) : c)$ **Map** (input: edge; output: edge):

- if P contains edge (i,j) in R_G:
  - emit $((i,j) : c - f_{max})$
  - emit $((j,i) : f_{max})$
- else: emit $((i,j) : c)$ (no changes)

Reduce: sum

Updating the residual graph

Shuffle size

Map operation emits at most 2 values per edge: $\mathcal{O}(m)$.

Runtime

Reduce sums at most 2 values for each edge along the path. But since no a priori knowledge of path: $\mathcal{O}(\frac{m}{k})$.

Communication cost

Step	**Cost**
Shortest path	$\mathcal{O}(m)$
Broadcast	$\mathcal{O}(nk)$
Residual graph update	$\mathcal{O}(m)$

Table: Communication cost

Runtime

Step	Sequential	Distributed
Shortest path	$\mathcal{O}(m)$	$\mathcal{O}(m/k)$
Path building	$\mathcal{O}(n)$	$\mathcal{O}(n)$
Broadcast	0	$\mathcal{O}(n\log(k))$
Residual graph update	$\mathcal{O}(m)$	$\mathcal{O}(m/k)$
Flow update	$\mathcal{O}(m)$	$\mathcal{O}(m/k)$

Table: Runtime

Comparison with sequential algorithm

Number of iterations

Algorithm terminates after $min(c, m(n-1))$ iterations where c is the max-flow. For large graphs usually $c \ll m(n-1)$

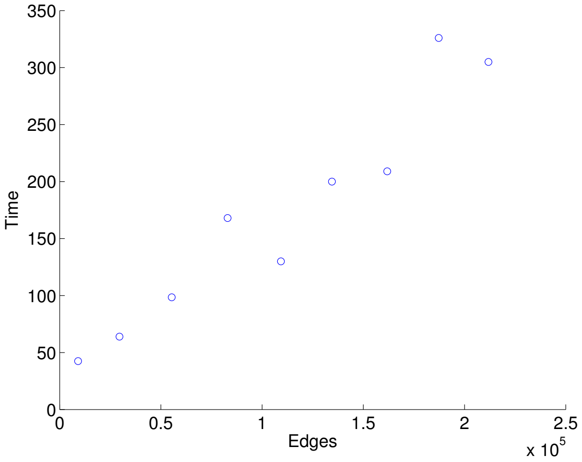
Sequential algorithm

- Runtime: $\mathcal{O}(cm)$

Distributed algorithm

- Runtime: $\mathcal{O}(cm/k) + \mathcal{O}(cn \log k)$
- Communication cost: $\mathcal{O}(cm) + \mathcal{O}(cnk)$

Some experimental results



Conclusion

- Problem scales on m (n has to fit on a single machine)
- Runtime optimal: $\mathcal{O}(cm) \to \mathcal{O}(\frac{cm}{k})$
- Communication cost potentially high, but not for vast majority of applications
- With optimal $k = m/n$. Runtime: $\mathcal{O}(cn\log(m/n))$. Communication cost: $\mathcal{O}(cm)$
- Largest graph tested: half a million edges