CME 305: Discrete Mathematics and Algorithms

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HW#1 – Due at the beginning of class Thursday 01/26/17

1. Prove that at least one of G and $\overline{G}$ is connected. Here, $\overline{G}$ is a graph on the vertices of G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

2. A vertex in G is *central* if its greatest distance from any other vertex is as small as possible. This distance is the *radius* of G.

   (a) Prove that for every graph G

   $$\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G$$

   (b) Prove that a graph G of radius at most k and maximum degree at most $d \geq 3$ has fewer than $\frac{d}{d-2}(d-1)^k$ vertices.

3. A random permutation π of the set $\{1, 2, \ldots, n\}$ can be represented by a directed graph on n vertices with a directed arc (i, π_i), where π_i is the i'th entry in the permutation. Observe that the resulting graph is just a collection of distinct cycles.

   (a) What is the expected length of the cycle containing vertex 1?

   (b) What is the expected number of cycles?

4. Let $v_1, v_2, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$. Prove that there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \{-1, 1\}$ such that

   $$||\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n||_2 \leq \sqrt{n}$$

5. Consider a graph G on $2n$ vertices where every vertex has degree at least n. Prove that G contains a perfect matching.

6. Let $G = (V, E)$ be a graph and $w : E \rightarrow R^+$ be an assignment of nonnegative weights to its edges. For $u, v \in V$ let $f(u, v)$ denote the weight of a minimum $u - v$ cut in G.

   (a) Let $u, v, w \in V$, and suppose $f(u, v) \leq f(u, w) \leq f(v, w)$. Show that $f(u, v) = f(u, w)$, i.e., the two smaller numbers are equal.

   (b) Show that among the $\binom{n}{2}$ values $f(u, v)$, for all pairs $u, v \in V$, there are at most $n - 1$ distinct values.

7. Let T be a spanning tree of a graph G with an edge cost function c. We say that T has the *cycle property* if for any edge $e' \notin T$, $c(e') \geq c(e)$ for all e in the cycle generated by adding e' to T. Also, T has the *cut property* if for any edge $e \in T$, $c(e) \leq c(e')$ for all e' in the cut defined by e. Show that the following three statements are equivalent:

   (a) T has the cycle property.

   (b) T has the cut property.

   (c) T is a minimum cost spanning tree.

Remark 1: Note that removing $e \in T$ creates two trees with vertex sets V_1 and V_2. A *cut* defined by $e \in T$ is the set of edges of G with one endpoint in V_1 and the other in V_2 (with the exception of e itself).

8. Given a graph $G = (V, E)$, a set of vertices $D \subseteq V$ is called a *dominating set* if every vertex in $V \setminus D$ is adjacent to a vertex in D. Suppose $|V| = n$ and the minimum degree of $G = \delta > 0$. Show that G contains a dominating set of size at most:

$$\frac{n(\log(1+\delta)+1)}{1+\delta}$$

9. Consider the following scenario. Due to large-scale flooding in a region, paramedics have identified a set of n injured people distributed across the region who need to be rushed to hospitals. There are k hospitals in the region, and each of the n people needs to be brought to a hospital that is within a half-hour's driving time of their current location (so different people will have different options for hospitals, depending on where they are right now).

   At the same time, one doesn't want to overload any one of the hospitals by sending too many patients its way. The paramedics are in touch by cell phone, and they want to collectively work out whether they can choose a hospital for each of the injured people in such a way that the load on the hospitals is *balanced*: Each hospital receives at most $\lceil n/k \rceil$ people.

   Create a polynomial time algorithm that outputs an assignment of people to hospitals if a valid assignment exists and outputs no otherwise.