CME 305: Discrete Mathematics and Algorithms

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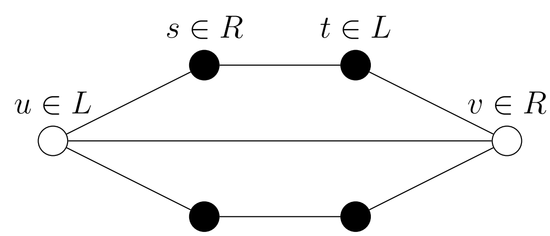
1. (10 points) A complete bipartite graph or biclique is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set. Prove an $O(n \log n)$ upper bound for the covering time of the biclique which has n nodes in each side of the bipartition (for a total of $2n$ nodes).

   Solution via Hitting Times: This is similar to covering two cliques of size n separately. Suppose our partitions denoted by L, R, and without loss of generality suppose we start from a vertex in L. Realize that after two steps, we return to one of the nodes in L uniformly at random. So, after we have visited k nodes in L, the probability of discovering a new node after two steps is $\frac{n-k}{n}$. So the time it takes to reach a new node described by $2 \cdot \text{Geom}(\frac{n-k}{n})$. So, the expected number of steps to visit all nodes in L is

$$\begin{aligned}\mathbb{E}\left[2\text{Geom}\left(\frac{n-1}{n}\right) + \ldots + 2\text{Geom}\left(\frac{1}{n}\right)\right] &= 2\left(\frac{n}{n-1} + \ldots + n\right) \\ &= 2n\left(1 + \ldots + \frac{1}{n-1}\right) \leq 2n \cdot c \log n.\end{aligned}$$

   To get an upper bound for the whole biclique, we simply double the above quantity, since we may always wait for our random walk to cover L, and then wait for it to cover R. Hence cover time bounded by $O(n \log n)$.

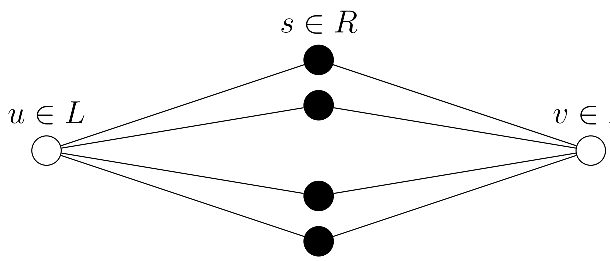
   Solution via Effective Resistances: Suppose $u \in L$ and $v \in R$. Consider paths from $u \rightsquigarrow v$ of length at most three:



   Notice that there are other paths between $u \rightsquigarrow v$ of length three or more, but by excluding them we are omitting non-negative terms from the denominator, hence we realize an upper bound on effective resistance between u and v:

$$R_{u,v} \leq \frac{1}{1 + \sum_{i=1}^{n-1} \frac{1}{3}} = \frac{1}{1 + \frac{n-1}{3}} = O\left(\frac{1}{n}\right).$$

   Now suppose that u, v are in the same partition L or R. Then,



By a similar argument (of ignoring paths longer than two thus omitting non-negative terms from denominator), we have that

$$R_{u,v} \leq \frac{1}{\sum_{i=1}^{n} \frac{1}{2}} = \frac{1}{\frac{n}{2}} = \frac{2}{n} = O\left(\frac{1}{n}\right)$$

Hence the effective resistance between any two nodes is $O(1/n)$, and therefore $R(G) = O(1/n)$. Whence we use our bound that $C(G) \leq 2e^3 m \ln n R(G) + n$ to see that

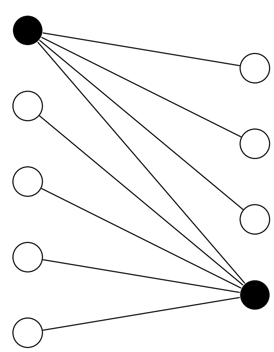
$$C(G) = O(n \log n).$$

2. (10 points) Given a weighted undirected graph G with distinct weights, for any cycle C in the graph, if the weight of an edge e of C is larger than the individual weight of all other edges of C, prove that this edge e cannot belong to any Minimum Spanning Tree of G.

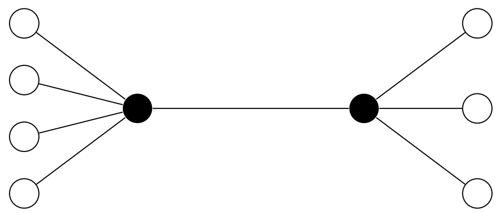
   Solution: Assume there is an MST T which has edge $e = (u, v) \in C$, where weight of e strictly larger than individual weights of all other edges of C. Consider the cut defined defined by removing e from spanning tree T. Since $e \in C$, there exists another edge $e' \in C$ where $e' \neq e$ which is also in the cut. When we removed e from T, we created two connected components, one containing vertex u and another vertex v. If we add back e', we recover a spanning tree with cost strictly less than cost of T, since weight(e') < weight(e) (we started with e the heaviest edge in a cycle). Hence T cannot be an MST, a contradiction, and we have shown e cannot be in any MST.

3. (10 points) Prove that the Minimum Vertex Cover problem can be solved in polynomial time, on trees.

   A note on picking the smaller partition of a bipartite graph: In class we discussed that all trees are bipartite. Many students argued that the smaller side of a bipartite graph forms the minimum vertex cover. It's certainly a cover, but it's not necessarily minimum. Consider the following counter example,



The left partition has 5 nodes, the right partition has 4 nodes.[1] Notice that the two black nodes form a vertex cover of size two, strictly smaller than choosing the vertex cover formed by the smaller partition. The above graph is isomorphic to the following, we simply "grab" the white nodes and "stretch" our graph apart.



Solution via Dynamic Programming: Examine a leaf on the tree (every tree has at least two leaves), and specifically its incident edge. This edge can only be covered by either the leaf node or its parent. Since at least one node must be picked, and since it's possible for the parent node to cover more than just one edge, we can't do worse by choosing the parent instead of the leaf for inclusion in the vertex cover. So for each leaf in the tree, take its parent for inclusion in the vertex cover. Then delete from our tree the edge we have covered, the leaf, and the parent we have included in our cover. In general, we are left with a forest. Repeating the same argument on each of the trees remaining, we have our algorithm.

Solution via Ford Fulkerson: We first show that if M a matching and C a vertex cover, that $|M| \leq |C|$. Then, using the fact that trees are bipartite, we use FF to get a min-cut, which we will then use to generate a vertex cover. We will show this vertex cover the size of a maximum matching and therefore minimal.

First, given a matching M, we know that each vertex in our graph incident to at most one edge. If the matching is a maximum matching, each vertex incident to at least one edge. Hence each edge in M must contain at least one incident node in any cover C. Hence $|M| \leq |C|$.

Now, suppose we have set up our tree into a bipartite graph with partitions L, R. We attach source node s to each node in L with a directed arc of unit capacity, and each node in R to sink node t with another directed arc, also of unit capacity. We then draw directed edges from nodes in L to nodes in R if and only if the nodes are neighbors in our input tree, each with infinite capacity. Without loss of generality, suppose $|L| \leq |R|$.

We now compute the reach of s in the residual graph after termination of FF to find the min s-t cut (A, B).[2] Notice the edges leaving the min-cut $\delta^+(A)$ must all be edges of the form (s, u) where $u \in L$ or (w, t) where $w \in R$ since no edge from L to R may be

[1] In general, we may construct a family of graphs which exhibit this behavior: construct a bipartite graph where exactly one node in each partition is connected to all other nodes in the other partition. Another way to look at it, is to take two star graphs whose centers are joined by a common edge.

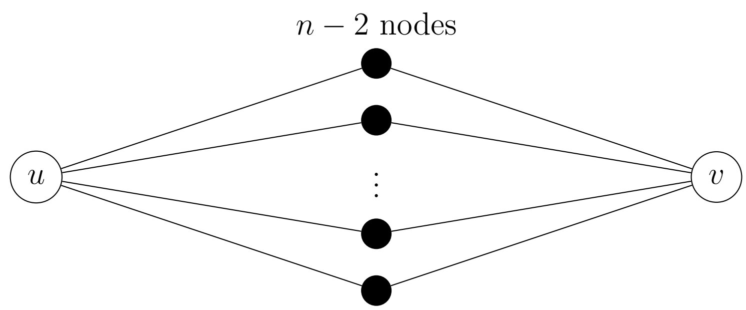
[2] Note that the ∞ edge-capacities are no problem since the cut $A = \{s\}$ has size $|L| < \infty$, hence the min s-t cut must be finite as well.

included (infinite cost). So, for our vertex cover C, we just pick all nodes (excluding s and t) from all edges in $\delta^+(A)$. We are guaranteed this is a vertex cover for tree T.[3]

Is this cut minimal? Yes. Since FF outputs a maximum matching of L and R, it has the size of the max-flow and hence the size of the min-cut, which is the size of C by our construction. Thus, our vertex cover C has size $|C| = |M|$ for a matching M, hence it's minimal. Since FF runs in polynomial time, and computing reach of s is linear time, we're done.

4. (15 points) Let $G(n, 1/2)$ be the Erods-Renyi graph on n nodes where each edge is present with probability $1/2$. Prove that $G(n, 1/2)$ has diameter $O(1)$ with probability approaching 1 as $n \to \infty$.

Solution: Select two vertices $u, v \in V$ and consider the following.



For each intermediate node w, there are four disjoint equally likely events: both of (u, w) and (w, v) are present, neither present, or exactly one present. Notice that in any case but the first, no length-2 path exists between u, v. Denote the event that there is no 2-path between u, v by $\bar{P}^2_{u,v}$. Then, since the $n - 2$ possible nodes w distinct,

$$\Pr(\bar{P}^2_{u,v}) = \left(\frac{3}{4}\right)^{n-2}.$$

By union bound, we have that the probability there exists a pair of nodes u, v such that there is no 2-path between them given by

$$\Pr(\cup_{u,v \in V} \bar{P}^2_{u,v}) \leq \sum_{u,v \in V} \Pr(\bar{P}^2_{u,v}) = \binom{n}{2} \left(\frac{3}{4}\right)^{n-2}.$$

As $n \to \infty$, the geometric term shrinks faster than does the second degree polynomial grow, whence $\Pr(\cup_{u,v \in V} \bar{P}^2_{u,v}) \to 0$. This means the probability that there exists some 2-path goes to 1, by complementarity. Finally, note that if each pair of vertices has a 2-path, then the diameter of our graph is $O(1)$.

[3] Suppose not, then there exists an edge (u, w) such that both $u \notin C$ (our cover) and $w \notin C$. Then (s, u) and (w, t) do not cross the cut. But neither does (u, w) (with infinite capacity). Hence $s \rightsquigarrow u \rightsquigarrow w \rightsquigarrow t$ is an s-t path which never crosses the s-t cut; a contradiction.