1. Consider a model of a nonbipartite undirected graph in which two particles (starting at arbitrary positions) follow a random walk i.e. with each time step both particles uniform randomly move to one of the neighbors. Prove that the expected time until they collide is $O(n^6)$. A collision is when both particles are on the same node at the same time step.

2. Let $A$ be a $n \times n$ matrix, $B$ a $n \times n$ matrix and $C$ a $n \times n$ matrix. We want to design an algorithm that checks whether $AB = C$ without calculating the product $AB$. Provide a randomized algorithm that accomplishes this in $O(n^2)$ time with high probability.

3. Given a connected, undirected graph $G = (V, E)$ and a set of terminals $S = \{s_1, s_2, \ldots, s_k\} \subseteq V$, a multiway cut is a set of edges whose removal disconnects the terminals from each other. The multiway cut problem asks for the minimum weight such set. The problem of finding a minimum weight multiway cut is NP-hard for any fixed $k \geq 3$. Observe that the case $k = 2$ is precisely the minimum $s-t$ cut problem.

Define an isolating cut for $s_i$ to be a set of edges whose removal disconnects $s_i$ from the rest of the terminals. Consider the following algorithm

- For each $i = 1, \ldots, k$, compute a minimum weight isolating cut for $s_i$, say $C_i$.
- Discard the heaviest of these cuts, and output the union of the rest, say $C$.

Each computation in Step 1 can be accomplished by identifying the terminals in $S - \{s_i\}$ into a single node, and finding a minimum cut separating this node from $s_i$; this takes one max-flow computation. Clearly, removing $C$ from the graph disconnects every pair of terminals, and so is a multiway cut.

(a) Prove that this algorithm achieves a $2 - 2/k$ approximation.

(b) Prove that this analysis is tight by providing an example graph where the approximation bound is exactly achieved.

4. Consider variants on the metric TSP problem in which the object is to find a simple path containing all the vertices of the graph. Three different problems arise, depending on the number (0, 1, or 2) of endpoints of the path that are specified. If zero or one endpoints are specified, obtain a $3/2$ factor algorithm.

**Hint.** Consider modifying Christofides algorithm for metric TSP.

5. Recall the minimum vertex cover problem: given a graph $G(V, E)$ find a subset $S^* \subseteq V$ with minimum cardinality such that every edge in $E$ has at least one endpoint in $S^*$. Consider the following greedy algorithm. Find the highest degree vertex, add it to the vertex $S$ and remove it along with all incident edges. Repeat iteratively. Prove that this algorithm has an unbounded approximation factor i.e. for any $c$ there exists an input graph $G$ such that $|S| \geq c \text{ OPT}$. 

6. A dominating set of a graph is a subset of vertices such that every node in the graph is either in the set or adjacent to a member of the set. The DOMINATING-SET problem is as follows: given a graph $G$ and a number $k$, determine if $G$ contains a dominating set of size $k$ or less.

(a) Show the DOMINATING-SET problem is NP-complete.
(b) Obtain a $\ln(n)$-approximation to the DOMINATING-SET problem.

7. An oriented incidence matrix $B$ of a directed graph $G(V, E)$ is a matrix with $n = |V|$ rows and $m = |E|$ columns with entry $B_{ve}$ equal to 1 if edge $e$ enters vertex $v$ and $-1$ if it leaves vertex $v$. Let $M = BB^T$.

(a) Prove that $\text{rank}(M) = n - w$ where $w$ is the number of connected components of $G$.
(b) Show that for any $i \in \{1, \ldots, n\}$,
$$\det M_{ii} = \sum_{N} (\det N)^2,$$
where $M_{ii} = M \setminus \{i^{th} \text{ row and column}\}$, and $N$ runs over all $(n - 1) \times (n - 1)$ submatrices of $B \setminus \{i^{th} \text{ row}\}$. Note that each submatrix $N$ corresponds to a choice of $n - 1$ edges of $G$.
(c) Show that
$$\det N = \begin{cases} 
\pm 1 & \text{if edges form a tree} \\
0 & \text{otherwise}
\end{cases}$$
This implies that $t(G) = \det M_{ii}$, where $t(G)$ is the number of spanning trees of $G$. In this definition of a tree, we treat a directed edge as an undirected one.
(d) Show that for the complete graph on $n$ vertices $K_n$,
$$\det M_{ii} = n^{n-2}.$$

8. Given a sequence $p_i$ of stock prices on $n$ days, we need to find the best pair of days to buy and sell, i.e. find $i$ and $j$ that maximizes $p_j - p_i$ subject to $j \geq i$. Give an $O(n)$ dynamic programming solution.

9. Let $G$ be a graph with minimum degree 3. Show $G$ contains a cycle of even length.