Discrete Mathematics and Algorithms Qualifying Exam
June 2013

1. (30 points) Consider the following problem: Given n items with sizes $a_1, a_2, \cdots a_n$ all in $(0, 1]$, find a packing in unit size bins that minimizes the number of bins used.

   (a) Prove that the following algorithm is a factor 2 approximation: Consider the items in an arbitrary order. In the i^{th} step, suppose you have a list of partially packed bins, say $B_1, B_2, \ldots, B_k$. If possible, put a_i into any one of them. If a_i does not fit into any of these bins, open a new bin B_{k+1} and put a_i in it.

   (b) Give an example on which the above algorithm does at least as bad as 5/3 of OPT, where OPT is the number of bins in the optimal packing.

   (c) Consider a modification of the algorithm in part (a). At the i^{th} step, suppose you have a list of partially packed bins, say $B_1, B_2, \ldots, B_k$. You may only put a_i into bin B_k. If a_i does not fit into bin B_k, open a new bin B_{k+1} and put a_i in it. Prove that this modified algorithm also gives a factor 2 approximation.

2. (20 points) A simple graph $G(V, E)$ is called Hamiltonian if it contains a cycle which visits all nodes exactly once. Prove that if every vertex has degree at least $|V|/2$, then G is Hamiltonian.