Lecture-17   CME 305   3/5/2015

Agenda:
⇒ Discrete Optimization:
1. Matroids.
2. Submodularity.

E: ground set
$f: 2^E \to R$
max/min of f
powersets.

MATROID: Given groundset E and a collection of subsets of E, $I \subseteq 2^E$, $M = (E, I)$
"Independent sets"

then M is a matroid iff
(1) If $A \in I \Rightarrow \forall B \subseteq A$, $B \in I$
(2) If $A, B \in I$ and $|B| > |A|$, then $\exists b \in B$ $\notin B$ such that $A \cup \{b\} \in I$

Power Set: $E = \{1, \dots, n\}$

2^E: $\{\{\}, \{1\}, \{2\}, \dots \{1,2\} \dots \{1, \dots n\}\}$

$|2^E| = 2^{|E|}$

$E = \{V_1, V_2, \dots, V_n\}$ $V_i \in R^d$

Independent sets: $I = \{A \mid \{\mathrm{rank}(A) = |A|\}\}$
(all vectors in A are linearly independent)

B, A both linearly independent

$|B| > |A|$

Assume otherwise (2) contradiction.

⇒ every vector in B can be written as linear combn. of vectors in A.

⇒ $\mathrm{span}\{A\} \supseteq \mathrm{span}\{B\}$
$\dim|A| > \dim|B|$

⇒ contradiction because $\dim|A| = |A| \neq \dim|B| = |B|$
↳ which gives the contradiction.

GRAPHIC MATROID

E: Edges of a graph.

$I = \{$Those subsets of E that are a forest$\}$

Notion of independence is the same as that of being a matroid.

Why Matroid?

(1) proof: any subgraph of a forest is a forest

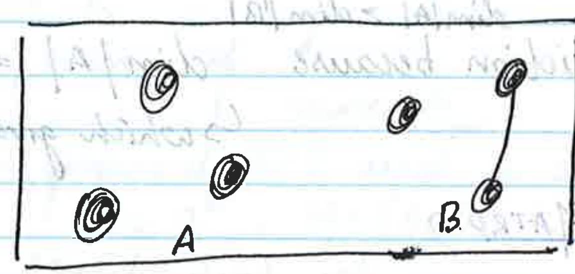
(2) Two forests, A, B, $|B| > |A|$.

$|A| < |B|$

⇒ $\#CC(B) \overset{<}{\cancel{>}} \#CC(A)$
connected components

∃ Two nodes u, v connected in B but not in A, consider the unique path between u & v, then for some edge e on that path must go between different components of A.

$A \cup \{e\}$ is acyclic $\Rightarrow$ A is independent.



MATROID OPTIMIZATION

~~edge el~~ Each element of E has some profit $p(E)$, $p(A = \{e_1, e_2, e \dots e_k\}) = \sum_{e \in A} p(e)$

can find $\max_{S \in I} p(S)$

Proof: sort descending $p(e_1) \geq p(e_2) \dots \geq p(e_n)$

$S_0 = \phi$, $k = 0$

for $j = 1$ to n

  if $S_k + \{e_j\} \in I$ (is independent) then

    $k = k + 1$

    $S_k \leftarrow S_{k-1} \cup \{e_j\}$

end

Output $S_1, \dots, S_k$.

Claim S_k is the largest profit independent set of size k.

Proof Suppose S_k is not ↑

then $\exists\ T = \{t_1, \dots, t_k\}$ $P(t_1) \geq \dots \geq P(t_k)$

big S

$S = \{S_1, \dots S_k\}$ $P(S_1) \geq \dots \geq P(S_k)$

∃ first index p s.t $p(t_p) > p(s_p)$

$\rightarrow$ $A = \{t_1, \dots, t_p\}$ $B = \{s_1, \dots, s_{p-1}\}$

supposed true best set.

size $|A| >$ (strictly) size $|B|$

$\exists\ t_i \notin B$ s.t $B \cup \{t_i\} \in I$

$p(t_i) \geq p(t_p) > p(s_p)$

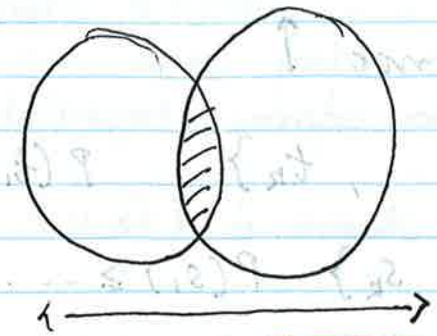
$p \geq i$ because first index p

$\Rightarrow$ t_i would have been considered at time i and its addition ~~would~~ to B would have retained independence.

$\therefore \Rightarrow$ contradiction.

this is the size → Two Sets A, B

$|A \cap B| + |A \cup B| = |A| + |B|$



Dfn: SUBMODULAR

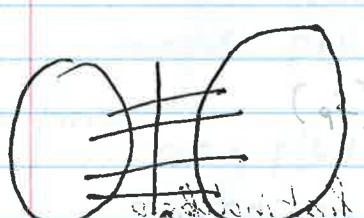
$f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$

Cut function is sub modular.

set $G(V,E)$

$S \subseteq V$ $\quad f(S) = \text{cut}(S) = \mathbb{1}_S^T L_G \mathbb{1}_S$

$\Rightarrow f(S)$ is submodular.



Straight up minimize submodular fn.

SUBMODULAR OBJ	N/O CONSTRAINTS.
min	✓ $O(n^5)$ calls to function (2000)
max	$\frac{1}{2}$ (2012) Vordak.

CONSTRAINED

max $f(S)$ $\quad$ f sub.
$S \in I$ $\quad$ $M = (E, I)$ is matroid.

$1 - \frac{1}{e}$ approx for it.