$$E[f_H(S)] = \sum_{\substack{\text{edges} \\ \in \text{cut}_G(S)}} p \cdot \frac{1}{p} = \text{cut}_G(S)$$
$$= f_G(S).$$

Lecture - 15 2/26/2015

- Agenda:
  - → Sparsification
  - → Effective Resistances via Linear Algebra.

SPARSIFICATION:

$\forall S \subseteq V$

Find H s.t $\dfrac{|f_G(S) - f_H(S)|}{f_G(S)} \le \varepsilon$

Try flipping coins with prob. p.

Give edges weight $\frac{1}{p}$ when adding them to H.

for fixed S

$\Rightarrow E[f_H(S)] = \sum_{(i,j) \in \text{cut}_G(S)} p \cdot \frac{1}{p} = \sum_{(i,j) \in \text{cut}_G(S)} 1 = f_G(S).$

$m = \binom{n}{2} p.$

For ~~chernoff~~ chernoff bound + a little bit
fixed S

$\Pr[\,|f_G(S) - f_H(S)| \ge \varepsilon f_G(S)\,] \le \exp\left(-\frac{\varepsilon^2}{3} p f_G(S)\right)$

If we set $p = \dfrac{10 \log(n)}{\varepsilon^2 \cancel{f_G(S)}\, c}$ $\le \frac{\varepsilon}{n^3 \cdot 3}$

where c = global min cut

c ranges between $1 \le c \le n-1$

1. need union bound
2. using global min cut
$\Big\} \Rightarrow$ HW 4

$$m = O\left(\binom{n}{2}\frac{\log n}{\varepsilon^2 c}\right) = O\left(\frac{n^2 \log(n)}{\varepsilon^2 c}\right)$$

SPIELMAN SRIVASTAVA 2009.

→ change the sampling scheme, draw from distn. an edge.

For $i = 1:q$

pick edge e from distn. that has $p_e \propto \frac{R_e^G}{n-1}$

add e to H with additional weight $\frac{1}{p_e q}$

like $\rightarrow (+=)$

end

$$q = O\left(\frac{n \log(n)}{\varepsilon^2}\right)$$

Linear Algebra starts here:

Build a matrix $\Pi = B L^{\dagger} B^T$: $m \times m$ (↙ pseudo inverse)

$\Pi \rightarrow$ diagonal entries effective resistance for edge E.

$$\Pi_{e,e} = R_e$$

$$R(e,v) = \begin{cases} -1 & \text{if } v \text{ is head of } e. \\ 1 & \text{"} \quad \text{"} \quad \text{tail of "} \\ 0 & \text{otherwise.} \end{cases}$$

Pseudo Inverse: SVD with non singular values inverted.

$L = B^T B$ ↙ positive semidefinite.

{arbitrarily orient edges and construct B}

$$L = \sum_{\lambda_i \neq 0} \lambda_i v_i v_i^T$$

defn. pseudo inverse ☆ ↷

$$L^{+} \equiv \sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} v_i v_i^T$$

LL^T = proj. onto range(L)

$$= \sum_{\lambda_i = 0} v_i v_i^T \ldots$$

SPAN $\{\mathbb{1}_n\}$ = ker{L} = ker{B}

→ Write electrical laws using matrix notation.

Conservation of current i_{ext} : $n \times 1$ where we insert / extract current.

$B^T i = i_{ext}$ $\quad i$: $m \times 1$ current flowing in edge i.

Ohm's Law: $I_m i = BV$

$i_{ext} = B^T B v = L v$ (linear system)

$v = L^{+} i_{ext}$ (a particular soln. to system given by pseudoinverse)

(wiki : pseudoinverse → solving linear systems)

$$\chi_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i$$

$e = (i, j)$ $i_{ext} = \chi_i - \chi_j$

Voltages at particular node.

$V = L^{+}(\chi_i - \chi_j)$ $V = iR$, ΔV

$R_e = V_i - V_j$

$R_e = (\chi_i - \chi_j)^T L^{+} (\chi_i - \chi_j)$

$R_e = (B L^{+} B^T)_{e,e} \equiv \Pi_{e,e}$

To prove projector : $\Pi^2 = \Pi$.

$$\Pi\Pi = B L^{+} B^T B L^{+} B$$
$$= B \underbrace{L^{+} L L^{+}}_{L^{+}} B^T = B L^{+} B^T$$

$\boxed{Img(B) = Img(\Pi)}$

prove other way.

$y = Bx$ $x \perp span\{1_n\}$

$\Pi y = \Pi B x$

$= B L^{+} B^T B x$

$= B L^{+} L x$ → projection onto range of image of L.

x lives in this space that $L^{+}L$ projects into.

⇒ $L^{+}L$ is identity on x.

Projection Matrices :

1. Have 0 or 1 e-values.
2. Trace $(\Pi) = \sum_{i=1}^{n} \lambda_i$
3. # non zero e-values of a matrix = Rank.
4. $Tr(\Pi) = Rank(\Pi) \to n-1$

$= Rank(B)$ Range of img. of B is (n-1) dimensio

$= n-1$

$= \sum_{e \in E} R_e$ → effective resistance.

now go to Spielman/Sri first para