Lecture - 14 2/24/2015

AGENDA.

→ ALGEBRAIC GRAPH THEORY

- Adjacency Matrix
- Incidence Matrix
- Laplacian
- Random Walk

→ SPARSIFICATION

- Naive
- Effective Resistance.

4 matrices connected with a Graph.

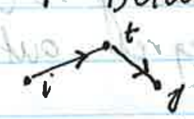
ADJACENCY MATRIX

- Given a graph undirected and unweighted

$[A]_{ij} = 1$ iff $\exists$ edge $(i \to j)$

Notice that, this means $\exists$ path of length 1 between i and j

$$[A^2]_{ij} = \sum_{t=1}^{n} A_{it} A_{tj}$$



$[A^k]_{ij}$ = exactly # paths of length k from i to j.

- path of lengths k or smaller: $A + A^2 + A^3 + \dots + A^k$.

- Diagonalize the matrix:

$A = U \Sigma U^T$

$A^2 = U \Sigma^2 U^T$ {only works when diagonalizable}.

INCIDENCE MATRIX

Directed, unweighted graph.

$$B_{m \times n}(e, v) = \begin{cases} 1 & v \text{ is the head of } e. \\ -1 & v \text{ is the tail of } e. \\ 0 & \text{otherwise.} \end{cases}$$

(For undirected graphs to define. B, arbitrarily orient edges).

(HW#4) Totally Unimodular.

LAPLACIAN $L : n \times n$ $L = B^T B$ where B is the incidence matrix.

counting cuts.

$L = B^T B$

cut : $S \subseteq V$ $X_s = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ iff $i \in S$ $\Big\} n$

$X_S^T L X_S = \text{cut}(S)$

$$x^T B^T B x = (Bx)^T Bx$$

$$= \|Bx\|_2^2 = \sum_{(i,j) \in E} (x_i - x_j)^2$$

just summing up entries of Bx vector.

indicator vector of 1's & 0's.

$L = D - A$ (Exercise)

A ← Adjacency Matrix

D ← Diagonal matrix of degrees.

$f : V \to \mathbb{R}$

$$f = \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ \vdots \\ f(v_n) \end{bmatrix} \qquad f^T L x = \sum_{(i,j) \in E} (f_i - f_j)^2$$

Cut size of directed graph, the edges going out.

* Pre-conditioners : Sparsifiers.

→ Laplacian is always singular

Singleton nodes?

→ For a connected graph, (undirected)

* $\boxed{\text{Span}\{\mathbb{1}_n\} = \text{Ker}\{L\} = \ker\{B\}}$

$$\begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \updownarrow n$$

any vector that is going to be destroyed by B, is going to get destroyed by L.

→ $x \in \ker[L]$

$Lx = 0$

$x^T L x = 0 \Rightarrow x^T B^T B x = 0 \Leftrightarrow \|Bx\|_2^2 \Leftrightarrow Bx = 0$

$\Rightarrow x \in \ker(B)$

$$\Rightarrow \sum_{(i,j) \in E} (x_i - x_j)^2 = 0 \Rightarrow x_i = x_j \Rightarrow x \in \text{span}\{\mathbb{1}_n\}$$

RANDOM WALK MATRIX. (Undirected graph)

$$\rightarrow [W]_{ij} = \begin{cases} \text{Prob. of transition from } i \text{ to } j \\ \\ \end{cases}$$

$$W = D^{-1} A$$

$$\begin{bmatrix} \deg(1) & \ddots & 0 \\ 0 & & \deg(n) \end{bmatrix}$$

← Throw away singleton nodes

$[W^k]_{ij}$ = probability of a random walk starting at i and finishing at j in exactly k steps.

SPARSIFICATION

Markov Inequality.

CHERNOFF BOUND

$X = \sum_{i=1}^{n} X_i$ $\qquad \cancel{X_i \in \{0,}$

$E[X] = \mu$

these r.v.'s don't have same prob.

$X_{i \ldots n}$ independence A.

$X_i \in \{0,1\}$ with prob. p

$$P_r[X \geq (1+\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{3}\right)$$

$$P_r[X \leq (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2}\right)$$

→ (SPECTRAL / CUT) SPARSIFICATION (for unweighted, undirected connected G).

$\forall S \subseteq V$

Cut $\quad f_G(S) = \text{cut}(S)$

Sparsifier H which is another graph such that H is on same

But: different edges & different weights.

$$\frac{|f_G(S) - f_H(S)|}{f_G(S)} \leq \varepsilon \qquad \forall S \subseteq V$$

→ How low can # edges in H be?

In 2009, Spielman - Srivastava (using effective resistances) $m = O\left(\frac{n \log(n)}{\varepsilon^2}\right)$

2011, Batson - Spielman - Srivastava.

$m = O\left(\frac{n}{\varepsilon^2}\right)$

→ Naive approach: set of probability p. Include each edge in H with probability p and weight $\frac{1}{p}$.

select or don't.

cut S

$$E\left[f_H(S)\right] = \sum_{\substack{edges\\ \in cut_G(S)}} p \cdot \frac{1}{p} = cut_G(S)$$
$$= f_G(S).$$

Lecture - 15 2/26/2015

- Agenda
  - → Sparsification
  - → Effective Resistances via Linear Algebra

SPARSIFICATION:

Find H s.t. $\forall S \subseteq V$ $\left|\frac{f_G(S) - f_H(S)}{f_G(S)}\right| \leq \varepsilon$

Try flipping coins with prob. p.

Give edges weight $\frac{1}{p}$ when adding them to H.

for fixed S

$$\Rightarrow E\left[f_H(S)\right] = \sum_{(i,j)\in cut_G(S)} p \cdot \frac{1}{p} = \sum_{(i,j)\in cut_G(S)} 1 = f_G(S).$$

$$m = \binom{n}{2} p$$

For chernoff bound + a little bit
fixed S

$$\Pr\left[\,|f_G(S) - f_H(S)| \geq \varepsilon f_G(S)\right] \leq \exp\left(-\frac{\varepsilon^2}{3} p f_G(S)\right)$$

If we set $p = \frac{10 \log(n)}{\varepsilon^2 \cancel{f_G(S)}\, c}$ $\leq \frac{1}{n^{3.3}}$

where c = global min cut

c ranges between $1 \leq c \leq n-1$